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Finite Lifetime Effects in Top Quark Pair Production at Threshold

# Finite Lifetime Effects in Top Quark Pair Production at Threshold 

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#### Abstract

In this work electroweak effects to the $e^{+} e^{-}$top pair production cross section at threshold are determined that are associated to the finite top lifetime. A new effective theory approach is developed that involves complex Wilson coefficients and anomalous dimensions and that allows to treat these effects systematically at NNLL order in a non-relativistic expansion. The imaginary parts are associated to experimental cuts that are imposed to define the cross section in analogy to the optical theory in an absorptive medium. In addition electroweak effects at NNLL order not related to instability are computed. The results are important for an extraction of Standard Model parameters at a future Linear Collider.


## Zusammenfassung

In dieser Arbeit werden elektroschwache Effekte zum $e^{+} e^{-}$-Topquark-Paar-Produktionsquerschnitt an der Schwelle bestimmt, die verknüpft sind mit der endlichen Lebensdauer des Topquarks. Ein neuer Zugang basierend auf einer effektiven Feldtheorie wird entwickelt, der komplexe Wilson-Koeffizienten und anomale Dimensionen einbezieht und der erlaubt, diese Effekte systematisch auf der Ordnung NNLL in einer nicht-relativistischen Entwicklung zu behandeln. Die Imaginärteile sind verknüpft mit experimentellen Schnitten, die notwendig sind, um den Wirkungsquerschnitt zu definieren. Dies ist in Analogie zur Beschreibung eines absorptiven Mediums in der optischen Theorie. Zusätzlich werden elektroschwache Effekte auf der Ordnung NNLL berechnet, welche nicht mit der Instabilität zusammenhängen. Die Resultate sind wichtig für eine Extraktion von Parametern des Standard-Modells an einem zukünftigen Linearbeschleuniger.

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## Introduction

The discovery of the top quark by the CDF and DØ collaborations [1, 2] at the Tevatron in 1995 was a major success of both experimental and theoretical elementary particle physics. It accomplished the quest for the complete set of matter particles predicted by the Standard Model of particles physics. The top quark takes an exposed location within this model. The current world average of the top mass measured at the CDF and $\mathrm{D} \varnothing$ experiments is $170.9 \pm 1.8 \mathrm{GeV}$ [3]. Therefore the top is about 35 times heavier than the next lighter fermion, the bottom quark. Due to this large value, the lifetime of the top quark is extremely short according to the Standard Model, $\tau_{t} \sim 10^{-24} s$, in particular it is shorter than the hadronization time $\tau_{\text {had }} \sim 10^{-23} s$. Therefore the top quark does not even live long enough to hadronize into bound states [4]. The top is the only quark with this property. The exceedingly short lifetime makes high demands on an accurate theoretical treatment of decay effects in the case of the top quark.

Due to its large mass, the top quark plays a key role in the precise determination of several Standard Model parameters. For instance the shift of the $W$ boson mass $M_{W}$ by the top quark through quantum corrections is proportional to the top mass squared, $\delta M_{W} \propto m_{t}^{2}$. At the same time the Higgs boson, which is the only Standard Model particle that eludes experimental detection to this day, generates another shift of $M_{W}$ through virtual quantum loops, $\delta M_{W} \propto \ln \left(M_{H} / M_{W}\right)$. By the knowledge of the precise $W$ boson and, in particular, the top quark mass it is therefore possible to make predictions for the Higgs mass even before this hypothetical particle has been observed in experiment. An accurate top mass is also important to constrain the parameter space of extensions of the Standard Model, e.g. the Minimal Supersymmetric Standard Model (MSSM). After the discovery of a Higgs boson a precise top mass will still be valuable for consistency checks of the Standard Model (or the MSSM). In the case of failure or if no Standard Model or MSSM Higgs boson will be found at all, the top quark could be a guidepost in the exploration of the untrodden territory of new physics, because it is the heaviest existing elementary particle known up to now. In any case it is desirable to know the top mass as precisely as possible.

An improvement of the current $1 \%$ top mass accuracy is expected from the experiments that will be conducted at the Large Hadron Collider (LHC) at CERN in the near future, where top-antitop pairs will be produced in proton-proton
collisions. The proposed International Linear Collider (ILC) can provide an accuracy of roughly one order of magnitude better than the one obtained from LHC measurements that are based on reconstruction [5-7]. Since it uses electron and positron beams, which do not suffer from hadronization effects, the energy of the initial state particles can be tuned. This allows for a top mass determination using a threshold scan, which relies on the measurement of the top-antitop production rate for several center-of-mass (c.m.) energy points in the resonance region of two times the top mass. A threshold scan is the appropriate tool because the line-shape of the cross section as a function of the c.m. energy is expected to be a smooth function due to the fast top quark decay. Simulations have shown that, provided with an integrated luminosity of $300 \mathrm{fb}^{-1}$, it can lead to an experimental top mass uncertainty at the order of only $50 \mathrm{MeV}[8]$, which corresponds to a relative uncertainty of $0.03 \%$. In addition a threshold scan allows for a determination of the strong coupling $\alpha_{s}\left(M_{Z}\right)$, the total top quark decay width $\Gamma_{t}$ and, if the Higgs boson is light, the top Yukawa coupling $g_{\text {tth }}[8] .{ }^{1}$ Because the observable cross section is a convolution of the theory prediction with the partly machine-dependent luminosity spectrum arising from QED effects [8, 11], high demands are imposed on both theoretical predictions and experimental analyses to make these measurements possible. In particular, theoretical predictions need to have a precision at the level of $3 \%$.

This goal has come into reach by the development of a number of theoretical tools within the past few years. The use of effective theory methods based of non-relativistic quantum chromodynamics (NRQCD) guarantees a systematic summation of Coulomb singular terms $\propto\left(\alpha_{s} / v\right)^{n}$ arising in the threshold region and employs a double expansion in the strong coupling $\alpha_{s}$ and the velocity $v$ of the top and the antitop quark in the c.m. frame [12, 13]. By now the next-to-next-to-leading order (NNLO) corrections to the total cross section are known [14] and also results at the next-to-next-to-next-to-leading order ( $\mathrm{N}^{3} \mathrm{LO}$ ) are available [15-20]. A disadvantage of this "fixed-order" approach is the fact that sizable logarithms $\left(\alpha_{s} \ln v\right)^{n}$ in the higher order matrix elements that are not taken into account entail large uncertainties in the normalization of the threshold cross section and jeopardize the required precision. At NNLO these uncertainties are at the level of $20 \%$ [14]. On the other hand, a summation of such logarithms can be achieved by means of renormalization group methods and leads to a significant reduction of these uncertainties [21-23]. Concerning QCD effects the renormalization group improved computations for the leading-logarithmic (LL) and next-to-leading logarithmic (NLL) order prediction of the threshold cross section are completed [24-28]. At the next-to-next-to-leading logarithmic (NNLL) order the QCD evolution of almost all required couplings is known [25, 29-33] except for missing subleading mixing effects in the running of the top quark pair

[^0]production current. Taking into account all known QCD effects up to NNLL order (except for some recently obtained results), the uncertainty in the normalization of the threshold cross section prediction is at best $6 \%$ [34].

Most of recent computations of corrections to the top-antitop threshold cross section was concentrated on QCD effects at NNLL order. Electroweak effects, in particular effects originating from the finite top quark lifetime, on the other hand, did not attract so much attention in the literature although they play an equally important role for the improvement of the cross section prediction. The electroweak interaction is responsible for various physical effects. At leading order the three basic electroweak effects are the electron-positron annihilation process leading to top pair production by virtual photon and $Z$ boson exchange, the finite top quark lifetime, which is described by a shift of the c.m. energy into the complex plane by the amount of the top decay width, $\sqrt{s} \rightarrow \sqrt{s}+i \Gamma_{t}[4,35,36]$, and initial state QED beam effects. At leading order these three effects can be treated independently. While the first two effects are included in the theoretical cross section prediction, QED beam effects are taken into account by a convolution of the theoretical cross section with the luminosity spectrum obtained from experimental simulations $[8,11,37]$. Since the QED beam effects consist of the machine-dependent beam energy spread, beamstrahlung and conventionally also initial state radiation, the latter is excluded from the theoretical cross section prediction. Concerning electroweak effects at the subleading level, a coherent treatment has not yet been achieved. Previous partial analyses have indicated that they can reach the level of a few percent [38-40]. Because a systematic treatment is based on the consistent separation of off-shell (non-resonant) and close-to-mass-shell (resonant) fluctuations, the concept of effective theories provides an efficient tool for the computation of electroweak and in particular effects of the top decay $[41,42]$.

In this work we develop a systematic procedure within the framework of NRQCD for the treatment of electroweak effects beyond LL order that are associated with the finite top quark lifetime. In addition, NNLL order electroweak effects not related to the top decay are presented, excluding pure QED effects.

The fundamental idea of our effective theory description of the unstable top quarks is to not consider the decay products as dynamical degrees of freedom but to include the instability effects in the couplings of the effective Lagrangian by an on-shell (hard-scale) matching. This is possible since top-antitop dynamics occurs in the low-energy regime whereas the top quark decay constitutes a hard scale effect and therefore the dynamical degrees of freedom of the decay products can be integrated out. As a result the effective theory Wilson coefficients and anomalous dimensions receive imaginary parts that contain quantities associated with the instability such as the decay rate $\Gamma_{t}$. Apart from that the imaginary parts depend on an external parameter related to experimental cuts on the kinematical properties of the top/antitop decay products. This parameter specifies what kind of events are described by the effective theory and therefore defines the effective
absorption process, in analogy to the treatment of absorptive processes in the optical theory where complex coefficients appear in the Maxwell's equations.

At LL order the well-known replacement $\sqrt{s} \rightarrow \sqrt{s}+i \Gamma_{t}$ corresponds to an imaginary matching condition for quark bilinear operators. Beyond LL order, there is a variety of sources for absorptive coefficients of different effective theory operators as will be shown in this work. For example corrections to the top quark lifetime that are $v^{2}$-suppressed lead to imaginary matching conditions which contribute at NNLL order. At the same order, interference effects contribute that involve underlying theory diagrams which contain the top decay products but lack one top or antitop line. These interferences are not directly related to the top decay width, but nevertheless constitute instability effects and are described by imaginary matching conditions for the top-antitop production currents.

By the inclusion of imaginary parts in the NRQCD couplings conceptually new aspects arise, since these imaginary parts render the effective theory topantitop phase space infinite. One implication is the appearance of ultraviolet phase space divergences. These divergences are connected to the instability of the top quark and would vanish if the top quark were considered as a stable particle. The according phase space logarithms are summed up into the coefficients of forward scattering operators. The coefficients are again imaginary and contribute to the cross section already at NLL order. Another implication is that the imaginary NRQCD matching conditions are in general associated with an external parameter $\Lambda$. These $\Lambda$-dependent matching conditions have to be imposed on the effective theory to specify the scope of the effective absorption process. The parameter $\Lambda$ corresponds to experimental cuts on the invariant masses of the top/antitop decay products. Therefore it is related to the allowed phase space of the top-antitop pair or rather its decay products. Technically, the $\Lambda$-dependent (hard scale) matching conditions for operators, such as forward scattering operators as well as production and annihilation currents, are determined systematically by an operator product expansion of time-ordered products of top-antitop production and annihilation currents that contain a phase space cut.

Concerning the numerical impact of the instability effects on the cross section prediction, we find that NNLL order corrections not only shift the normalization of the cross section, but even change its line-shape. Hence, they are able to shift the peak position and therefore the top mass value by $30-50 \mathrm{MeV}$, which is already at the level of the above mentioned experimental top mass uncertainty. The phase space effects contribute at NLO and give rise to an additive, for the most part c.m.-independent shift of the cross section at the level of -50 fb . They compensate the unphysical behaviour of previous NRQCD predictions for the top-antitop cross section below threshold that do not treat the phase space correctly. The absolute shift of -50 fb corresponds to a relative change at the five- or ten-percent level (the exact value depending on the c.m. energy point that is considered) and is therefore important for a cross section prediction with
a precision of $3 \%$.
Apart from electroweak effects originating from the finite top quark lifetime we determine NNLL order (usual) electroweak effects (not including pure QED effects). They are described by matching conditions for the top-antitop production and annihilation currents and are obtained from one-loop electroweak theory corrections to the process $e^{+} e^{-} \rightarrow t \bar{t}$. These corrections have already been computed in Ref. [38]. We will give a comparison of our results and those in Ref. [38] and point out discrepancies. The NNLL order electroweak corrections feature a rather strong Higgs mass dependence if the Higgs mass is small and therefore motivate a determination of Higgs boson properties from the top-antitop threshold cross section.

The program of this work is as follows. In Chap. 1 effective theory methods that are capable of describing the known QCD and electroweak effects are introduced. In addition the basic arrangements which allow for the formal inclusion of the new electroweak effects are provided. In Sec. 1.1 a theoretical motivation for the use of the effective theory is given. In Sec. 1.2 the basic principles of the effective theory treatment are discussed. Sec. 1.3 contains an overview of the ultrasoft, soft and potential pieces of the Lagrangian, whereas in Sec. 1.4 current and forward scattering operators are introduced. The cross section determination by means of the optical theorem using the forward scattering amplitude is discussed in Sec. 1.5. In Sec. 1.6 the replacements that have to be made to include initial state polarization effects are given. Sec. 1.7 contains an overview of electroweak effects that have to be taken into account for a cross section prediction up to NNLL order. In Chap. 2 the NNLL order matching conditions for the topantitop production currents describing usual electroweak effects are determined. The basic components of the electroweak standard model are introduced that is employed for the required calculations. In Sec. 2.1 the on-shell renormalization schemes which are used are discussed and compared to each other. In Sec. 2.2 we present the results of our computation of one-loop electroweak diagrams and make a comparison to earlier computations in the literature. In Sec. 2.3 we introduce an $\overline{\mathrm{MS}}$ definition for the Wilson coefficients of the dominant production currents which shifts the fermionic vacuum polarization effects to the effective QED coupling evaluated at the top mass scale. Sec. 2.4 contains a numerical analysis of the NNLL usual electroweak effects. In Chap. 3 we determine top quark instability effects at LL, NLL and NNLL order, excluding effects related to phase space cuts. In Sec. 3.1 the LL absorptive matching condition $i \Gamma_{t} / 2$ for the top quark bilinear operators is derived. NNLL absorptive matching conditions for the production operators, which originate from interference effects, are determined in Sec. 3.2. In Sec. 3.3 the summation of NLL phase space logarithms by means of renormalization group techniques is performed. Sec. 3.4 contains a brief numerical analysis of the considered NLL and NNLL instability effects. In Chap. 4 the phase space matching procedure is developed, which is a systematic approach to the computation of the NRQCD phase space. In Sec. 4.1 the ba-
sic idea is presented. Sec. 4.2 contains a comparison of the numerical inclusion of phase space effects to Green functions on the one hand and the systematic treatment by means of an operator product expansion on the other hand. In Sec. 4.3 we introduce the techniques used for analytic phase space integrations. In Sec. 4.4 NLO matching conditions for the forward scattering operators are determined by the calculation of cut effective theory one-loop diagrams, whereas NNLO matching conditions are obtained from cut effective theory two-loop diagrams in Sec. 4.5. In Sec. 4.6 we give a detailed numerical analysis of the phase space effects. In Sec. 4.7 we give the explicit form of the phase space cut imposed on the top/antitop decay products and determine relativistic corrections to the cut. Sec. 4.8 contains a comparison of our $\mathcal{O}\left(\alpha_{s}^{0}\right)$ results and those obtained using the programs MadGraph and MadEvent. In Chap. 5 we summarize the numerical impact of all determined electroweak effects on the cross section prediction. Finally, Chap. 6 contains our conclusions and outlook.

At this point we note that parts of this work have already been published, see Refs. [43-45].

## Chapter 1

## Non-Relativistic Top-Antitop Dynamics and the Effective Theory

### 1.1 Theoretical Motivation

The collision of electrons and positrons with a c.m. energy around two times the top quark mass, $\sqrt{s} \approx 344 \pm 5 \mathrm{GeV}$, allows for a threshold production of top-antitop ( $t t$ ) pairs, i. e. heavy quarks of non-relativistic velocity

$$
v=\sqrt{1-\frac{4 m_{t}^{2}}{s}} \ll 1
$$

in the c.m. frame. In Fig. 1.1 the tree-level electroweak Standard Model process of $e^{+} e^{-}$annihilation and $t \bar{t}$ production via virtual photon or $Z$ boson exchange is shown. In the non-relativistic energy regime, usual quantum chromodynamics (QCD) perturbation theory, in which the strong coupling $\alpha_{s}$ is the only expansion parameter, breaks down due to Coulomb singularities related to the binding of quark-antiquark pairs via the strong interaction: These arise as $\left(\alpha_{s} / v\right)^{n}$ terms in the loop expansion of the $t \bar{t}$ production current indicating that the instantaneous


Figure 1.1: Standard Model tree-level $e^{+} e^{-} \rightarrow t \bar{t}$ diagrams.


Figure 1.2: Loop expansion of the $t \bar{t}$ production current.
exchange of $n$ time-like gluons between top and antitop, which come with a suppression of $\alpha_{s}^{n}$, is associated with an enhancement factor of $(1 / v)^{n}$, see Fig. 1.2. As a consequence it is not sufficient to cut the perturbation series at a finite order in $\alpha_{s}$. Besides the large $\left(\alpha_{s} / v\right)^{n}$ terms, at the same time there are large logarithms of the velocity $\left(\alpha_{s} \ln v\right)^{m}$, which have to be accounted for in the theoretical description. By means of renormalization group methods it is possible to sum up such logarithms.

Another important feature of the non-relativistic top quark pair dynamics is the decay of the top and the antitop via the electroweak interaction. Because the Cabibbo-Kobayashi-Maskawa (CKM) matrix element $V_{t b}$ is approximately one in the Standard Model, the top quark decays predominantly into a bottom quark and a $W$ boson, $t \rightarrow b W$. The total decay width of the top, $\Gamma_{t} \approx 1.5 \mathrm{GeV}$, is extraordinarily large compared to the other quarks and, in particular, larger than the hadronization scale, $\Gamma_{t}>\Lambda_{\mathrm{QCD}}$. Therefore top and antitop quarks decay even before hadronization takes place, so that toponium bound states cannot be formed. As a consequence, the $e^{+} e^{-} \rightarrow t \bar{t}$ threshold cross section as a function of the c.m. energy does not exhibit many individual bound state resonances, but rises smoothly when $t \bar{t}$ production becomes kinematically allowed. Another welcome implication of the decay being faster than hadronization is that nonperturbative effects can be assumed to be suppressed $[4,36]$.

The solution to the above mentioned problems of Coulomb singularities and large logarithms is to go from usual QCD perturbation theory to an effective theory which respects and carries the characteristics of the non-relativistic energy regime and hence contains the relevant degrees of freedom and the proper power counting. Apart from that also the fast top quark decay is taken into account by the effective theory. Such a theory is generically called non-relativistic QCD (NRQCD). In this work we use a particular version of NRQCD called velocity NRQCD (vNRQCD) [24, 25].

### 1.2 Energy Scales, Power Counting, Degrees of Freedom

Non-relativistic $t \bar{t}$ dynamics is governed by different energy scales, namely the hard scale of the top mass $m_{t} \approx 172 \mathrm{GeV}$, the soft scale of the top 3-momentum $\mathrm{p} \sim m_{t} v \approx 25 \mathrm{GeV}$ and the ultrasoft scale of the kinetic energy $E \sim m v^{2} \approx$ $3-4 \mathrm{GeV}$ of the $t \bar{t}$ system, where $E=\sqrt{s}-2 m_{t}$. The decay width is at the same order of magnitude as the kinetic energy and therefore also of the ultrasoft scale. Finally, the QCD hadronization scale $\Lambda_{\mathrm{QCD}}$ is below the ultrasoft scale. Due to the small velocity we have the separation

$$
\begin{equation*}
m_{t} \gg m_{t} v \gg \Gamma_{t} \sim m_{t} v^{2}>\Lambda_{\mathrm{QCD}} \tag{1.1}
\end{equation*}
$$

This ordering holds even at threshold where the quarks are at rest. This is a consequence of the replacement $E \rightarrow E+i \Gamma_{t}[36]$ leading to $m_{t} v^{2}=E+i \Gamma_{t}$ and therefore the absolute value of the effective velocity is bounded below. Thus, $\Gamma_{t}$ serves as an infrared cutoff which prohibits non-perturbative dynamics that take place at $\Lambda_{\mathrm{QCD}}$.

From the comparison of these energy scales, one immediately obtains a power counting, which allows for a classification of the different contributions (e.g. loop diagrams) to a quantity (e. g. operator coefficients or the cross section) according to their expected numerical importance. In analogy to the Hydrogen atom one can assume a balance of kinetic energy $E \sim m_{t} v^{2}$ and potential energy $\sim m_{t} \alpha_{s}^{2}$ in the $t \bar{t}$ system. This leads to a relation between the velocity and the strong coupling. The counting of electroweak effects results from the numerical relation $\Gamma_{t} \sim m_{t} \alpha \approx E \sim m_{t} v^{2}$, where $\alpha$ is the fine structure constant. Altogether we obtain

$$
\begin{equation*}
E / m_{t} \sim v^{2}, \quad \alpha_{s} \sim v, \quad \Gamma_{t} / m_{t} \sim \alpha \sim v^{2} \tag{1.2}
\end{equation*}
$$

and use $v$ as the counting parameter. As an example we consider the QCD contributions to the threshold cross section. They can be written in the schematic form

$$
\begin{aligned}
R=\frac{\sigma_{\mathrm{thr}}}{\sigma_{\mu^{+} \mu^{-}}}= & v \sum_{n, m}\left(\frac{\alpha_{s}}{v}\right)^{n}\left(\alpha_{s} \ln v\right)^{m} \\
& \times\left\{1(\mathrm{LL}) ; v, \alpha_{s}(\mathrm{NLL}) ; v^{2}, \alpha_{s} v, \alpha_{s}^{2}(\mathrm{NNLL})\right\},
\end{aligned}
$$

where $\left(\alpha_{s} / v\right)^{n}$ and $\left(\alpha_{s} \ln v\right)^{m}$ terms are summed. We do not show terms beyond NNLL order here. Electroweak effects including phase space effects are also not written down at this point and will be discussed later.

The important momentum regions that follow from the physical energy scales
are [46]

$$
\begin{array}{ll}
\text { hard: } & \quad\left(p^{0}, \mathbf{p}\right) \sim\left(m_{t}, m_{t}\right), \\
\text { soft: } & \left(p^{0}, \mathbf{p}\right) \sim\left(m_{t} v, m_{t} v\right), \\
\text { potential: } & \left(p^{0}, \mathbf{p}\right) \sim\left(m_{t} v^{2}, m_{t} v\right), \\
\text { ultrasoft: } & \left(p^{0}, \mathbf{p}\right) \sim\left(m_{t} v^{2}, m_{t} v^{2}\right)
\end{array}
$$

with the notation $\left(p^{0}, \mathbf{p}\right)=($ energy, 3 -momentum $)$. The effective theory contains only those quark and gluonic degrees of freedom that can become on-shell for scales below the hard scale. These on-shell degrees of freedom are gluons and massless quarks in the soft and ultrasoft momentum region, and top quarks in the potential momentum region, referred to also as heavy quarks. Although soft gluons cannot be produced in the non-relativistic $t \bar{t}$ system of a kinetic energy $E \sim m_{t} v^{2}$, they are needed as relevant degrees of freedom since they are involved in the renormalization of effective theory operators. All off-shell fluctuations such as hard quarks and gluons, potential gluons and soft quarks in the QCD case are accounted for by on-shell matching of vNRQCD to full QCD at the hard scale. Electroweak effects are treated in the same way by on-shell matching to the full electroweak theory, as we will see in the following chapters. This matching fixes the Wilson coefficients of the effective theory operators at the hard scale.

The vNRQCD soft and ultrasoft fluctuations are separated from each other by means of a multipole expansion $[47,48]$ in analogy to heavy quark effective theory [49]. Ultrasoft momenta are continuous variables, whereas soft momenta appear as discrete labels for potential quarks and soft gluons, so that there exists an individual operator for each soft momentum. $\mathrm{SU}(3)$ gauge invariance, which is still present for ultrasoft energies and momenta, must be recovered for the soft energies and momenta through reparametrization invariance [24].

The regularization of loop integrals appearing in vNRQCD matrix elements is achieved by dimensional regularization [50-52] in $d=4-2 \epsilon$ dimensions. Renormalization is done in the modified minimal subtraction scheme $\overline{\mathrm{MS}}$ [53-57]. The subtlety in applying dimensional regularization to vNRQCD diagrams is that for the ultrasoft integrals an ultrasoft regularization parameter $\mu_{U}$ has to be used, whereas for the soft and potential integrals one has to use a soft parameter $\mu_{S}$ [24]. These parameters are not independent, because the soft and the ultrasoft scales are related, and thus $\mu_{S}^{2}=m_{t} \mu_{U}$. Therefore one can define $\mu_{S}=m_{t} \nu$ and $\mu_{U}=m_{t} \nu^{2}$, where $\nu$ is called the subtraction point velocity. By means of a renormalization group equation for $\nu$ it is possible to scale the Wilson coefficients of the effective theory from the hard matching scale $\mu_{S}=\mu_{U}=m_{t}$ down to $\mu_{S}=m_{t} v, \mu_{U}=m_{t} v^{2}$, i. e. from $\nu=1$ to $\nu=v$. In the calculation of matrix elements involving gluon loops one encounters logarithms of the form

$$
\ln \frac{\mu_{U}}{E}=\ln \frac{m_{t} \nu^{2}}{m_{t} v^{2}}, \quad \ln \frac{\mu_{S}}{\sqrt{m_{t} E}}=\ln \frac{m_{t} \nu}{m_{t} v},
$$

which are large at the hard scale $\nu=1$. By choosing instead the scale $\nu \sim v$ these logarithms of the ultrasoft and the soft energy are rendered small simultaneously, their contribution shifted from the matrix elements to the Wilson coefficients. In this way the use of an effective theory has solved the above mentioned problem of large logarithms.

### 1.3 Effective QCD Lagrangian

The vNRQCD Lagrangian contains ultrasoft, soft and potential QCD components, $\mathcal{L}_{\mathrm{QCD}}=\mathcal{L}_{u}+\mathcal{L}_{s}+\mathcal{L}_{p}$, given in Refs. [24-27, 48]. The ultrasoft piece reads

$$
\begin{gather*}
\mathcal{L}_{u}=\sum_{\mathbf{p}} \psi_{\mathbf{p}}^{\dagger}\left\{i D^{0}-\frac{(\mathbf{p}-i \mathbf{D})^{2}}{2 m_{t}}+\frac{\mathbf{p}^{4}}{8 m_{t}^{3}}+\frac{i}{2} \Gamma_{t}\left(1-\frac{\mathbf{p}^{2}}{2 m_{t}^{2}}\right)-\delta m_{t}+\ldots\right\} \psi_{\mathbf{p}} \\
+\left(\psi_{\mathbf{p}} \rightarrow \chi_{\mathbf{p}}\right)-\frac{1}{4} G_{u}^{\mu \nu} G_{\mu \nu}^{u}+\ldots \tag{1.3}
\end{gather*}
$$

where $G_{u}^{\mu \nu}$ is the ultrasoft field strength tensor and $\psi_{\mathbf{p}}, \chi_{\mathbf{p}}$ are heavy quark and antiquark two-component spinor fields of the soft momentum $\mathbf{p}$, respectively. The covariant derivative has the form $D^{\mu}=\partial^{\mu}+i g_{u} A^{\mu}$ with the renormalized ultrasoft QCD coupling $g_{u}=g_{u}\left(\mu_{U}\right)$ and the ultrasoft gluon field $A^{\mu}$. Therefore the ultrasoft gluon interaction is provided by ultrasoft gauge invariance. Powers of $\mu_{U}^{\epsilon}$ (as well as of $\mu_{S}^{\epsilon}$ ) are assigned to every effective theory operator unambiguously by considering its $v$-scaling. We do not write them down in the Lagrangian explicitly. Apart from that the terms for the ultrasoft ghost and massless quark fields are not shown explicitly.

The term $\delta m_{t}$ is a residual mass term, present if a threshold mass scheme is used. It reduces the intrinsic ambiguity [58,59] of order $\Lambda_{\mathrm{QCD}}$ of the pole mass $\left(m_{t}\right)$ definition to a size that is parametrically below $\Lambda_{\mathrm{QCD}}$. Threshold mass definitions that have been used in the literature are e.g. the kinetic mass [60], the potential subtracted mass [61] (see also [62]) and the $1 S$ mass [40,63]. We will use the $1 S$ mass scheme in the numerical analysis in Chap. 4.

In Eq. (1.3) we added for convenience to the ultrasoft QCD Lagrangian the electroweak matching condition $\propto i \Gamma_{t}$, which at LL effectively shifts the kinetic energy into the complex plane, $E \rightarrow E+i \Gamma_{t}$.

The soft piece of the Lagrangian reads

$$
\begin{aligned}
\mathcal{L}_{s}= & \sum_{q}\left\{\left|q^{\mu} A_{q}^{\nu}-q^{\nu} A_{q}^{\mu}\right|^{2}+\bar{\varphi}_{q} q \varphi_{q}+\bar{c}_{q} q^{2} c_{q}\right\} \\
& -g_{s}^{2} \sum_{\mathbf{p}, \mathbf{p}^{\prime}, q, q^{\prime}, \sigma}\left\{\frac{1}{2} \psi_{\mathbf{p}^{\prime}}^{\dagger}\left[A_{q^{\prime}}^{\mu}, A_{q}^{\nu}\right] U_{\mu \nu}^{(\sigma)} \psi_{\mathbf{p}}+\frac{1}{2} \psi_{\mathbf{p}^{\prime}}\left\{A_{q^{\prime}}^{\mu}, A_{q}^{\nu}\right\} W_{\mu \nu}^{(\sigma)} \psi_{\mathbf{p}}\right. \\
& \left.+\psi_{\mathbf{p}^{\prime}}^{\dagger}\left[\bar{c}_{q^{\prime}}, c_{q}\right] Y^{(\sigma)} \psi_{\mathbf{p}}+\left(\psi_{\mathbf{p}^{\prime}}^{\dagger} T^{B} Z_{\mu}^{(\sigma)} \psi_{\mathbf{p}}\right)\left(\bar{\varphi}_{q^{\prime}} \gamma^{\mu} T^{B} \varphi_{q}\right)\right\}+(\psi \rightarrow \chi, T \rightarrow \bar{T})
\end{aligned}
$$



Figure 1.3: Compton scattering diagrams ( $\mathrm{a}, \mathrm{b}, \mathrm{c}$ ) in full QCD generate the soft interaction indicated in diagram (d). The zigzag lines denote soft gluons.
where $A_{q}^{\mu}, c_{q}$ and $\varphi_{q}$ are gluon, ghost and massless quark fields of the soft 4momentum $q$, respectively, and $g_{u}=g_{u}\left(\mu_{S}\right)$ is the soft QCD coupling. The matrices $T^{A}$ denote the $\mathrm{SU}(3)$ generators, $\bar{T}^{A}$ is defined as $\bar{T}^{A}=-\left(T^{A}\right)^{*}$ and we use the symbol $f^{A B C}$ for the $\mathrm{SU}(3)$ structure constants. The soft interaction terms generate diagrams of the form shown in Fig. 1.3 (d). The tensors $U_{\mu \nu}^{(\sigma)}, W_{\mu \nu}^{(\sigma)}$, $Z_{\mu \nu}^{(\sigma)}$ and $Y_{\mu \nu}^{(\sigma)}$, where the index $\sigma$ denotes the relative order in the $v$ expansion, are functions of $\mathbf{p}^{\prime}, \mathbf{p}, q^{\prime}, q$ and matrices in spin. They are fixed at the hard scale by integrating out the off-shell quarks and gluons in the full QCD diagrams (a, b, c) in Fig. 1.3 and the analogous ones with external ghosts and massless quarks. Their explicit form is given in Ref. [26].

Finally, the potential part of the Lagrangian reads

$$
\begin{align*}
\mathcal{L}_{p}= & -\sum_{\mathbf{p}, \mathbf{p}^{\prime}} V_{\alpha \beta \lambda \tau}\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \psi_{\mathbf{p}^{\prime} \alpha}^{\dagger} \psi_{\mathbf{p} \beta} \chi_{-\mathbf{p}^{\prime} \lambda}^{\dagger} \chi_{-\mathbf{p} \tau} \\
& +\sum_{\mathbf{p}, \mathbf{p}^{\prime}} F_{j}^{A B C}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)\left(g_{u} \mathbf{A}_{j}^{C}\right)\left[\psi_{\mathbf{p}^{\prime}}^{\dagger} T^{A} \psi_{\mathbf{p}} \chi_{-\mathbf{p}^{\prime}}^{\dagger} \bar{T}^{B} \chi_{-\mathbf{p}}\right]+\ldots \tag{1.4}
\end{align*}
$$

with $\left(\mathbf{k}=\mathbf{p}^{\prime}-\mathbf{p}\right)$

$$
\begin{aligned}
V\left(\mathbf{p}, \mathbf{p}^{\prime}\right)=\left(T^{A} \otimes\right. & \left.\bar{T}^{A}\right)\left[\frac{\mathcal{V}_{c}^{(T)}}{\mathbf{k}^{2}}+\frac{\mathcal{V}_{k}^{(T)} \pi^{2}}{m_{t}|\mathbf{k}|}+\frac{\mathcal{V}_{r}^{(T)}\left(\mathbf{p}^{2}+\mathbf{p}^{\prime 2}\right)}{2 m_{t}^{2} \mathbf{k}^{2}}+\frac{\mathcal{V}_{2}^{(T)}}{m_{t}^{2}}\right. \\
& \left.+\frac{\mathcal{V}_{s}^{(T)}}{m_{t}^{2}} \mathbf{S}^{2}+\frac{\mathcal{V}_{\Lambda}^{(T)}}{m_{t}^{2}} \Lambda\left(\mathbf{p}^{\prime}, \mathbf{p}\right)+\frac{\mathcal{V}_{t}^{(T)}}{m_{t}^{2}} T(\mathbf{k})+\ldots\right] \\
& +(1 \otimes 1)\left[\frac{\mathcal{V}_{c}^{(1)}}{\mathbf{k}^{2}}+\frac{\mathcal{V}_{k}^{(1)} \pi^{2}}{m_{t}|\mathbf{k}|}+\frac{\mathcal{V}_{2}^{(1)}}{m_{t}^{2}}+\frac{\mathcal{V}_{s}^{(1)}}{m_{t}^{2}} \mathbf{S}^{2}+\ldots\right]
\end{aligned}
$$



Figure 1.4: QCD diagrams that generate vNRQCD potentials.

$$
\mathbf{S}=\frac{\boldsymbol{\sigma}_{1}+\boldsymbol{\sigma}_{2}}{2}, \quad \Lambda\left(\mathbf{p}^{\prime}, \mathbf{p}\right)=-i \frac{\mathbf{S} \cdot\left(\mathbf{p}^{\prime} \times \mathbf{p}\right)}{\mathbf{k}^{2}}, \quad T(\mathbf{k})=\boldsymbol{\sigma}_{1} \cdot \boldsymbol{\sigma}_{2}-\frac{3 \mathbf{k} \cdot \boldsymbol{\sigma}_{1} \mathbf{k} \cdot \boldsymbol{\sigma}_{2}}{\mathbf{k}^{2}}
$$

and

$$
F_{j}^{A B C}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)=\frac{2 i \mathcal{V}_{c}^{(T)} \mathbf{k}_{j}}{\mathbf{k}^{4}} f^{A B C}
$$

The operators $\boldsymbol{\sigma}_{1} / 2$ and $\boldsymbol{\sigma}_{2} / 2$ are the spin operators on top and antitop. The terms in $V\left(\mathbf{p}, \mathbf{p}^{\prime}\right)$ are the leading order Coulomb potential $\mathcal{V}_{c}$, the $v^{2}$-suppressed potentials contributing at $\mathcal{O}\left(\alpha_{s}\right)$ and the $v$-suppressed potential $\mathcal{V}_{k}$. The latter contributes at $\mathcal{O}\left(\alpha_{s}^{2}\right)$ since its tree level matching condition is zero, and hence it is also $v^{2}$-suppressed according to the $v$-counting $v \sim \alpha_{s}$. Matching to full QCD one-loop diagrams gives [64]

$$
\begin{equation*}
\mathcal{V}_{k}^{(T)}=\alpha_{s}^{2}\left(m_{t}\right)\left(\frac{7}{8} C_{A}-\frac{1}{8} C_{d}\right), \quad \mathcal{V}_{k}^{(1)}=\alpha_{s}^{2}\left(m_{t}\right) \frac{1}{2} C_{1} \tag{1.5}
\end{equation*}
$$

where $C_{F}$ and $C_{A}$ are the Casimir operators of the fundamental and the adjoint $\mathrm{SU}(3)$ representation, respectively. These and the constants $C_{1}$ and $C_{d}$ are given in App. F. Apart from their dependence on the momenta, the potentials $\mathcal{V}_{s}, \mathcal{V}_{\Lambda}$ and $\mathcal{V}_{t}$ have a dependence on the spin of top and antitop.

The various coefficients are fixed at the hard scale $m_{t}$ at order $\alpha_{s}$ by matching to full QCD tree level diagrams shown in Fig. 1.4. This leads to [26]

$$
\begin{align*}
& \mathcal{V}_{c}^{(T)}=4 \pi \alpha_{s}\left(m_{t}\right), \quad \mathcal{V}_{r}^{(T)}=4 \pi \alpha_{s}\left(m_{t}\right), \quad \mathcal{V}_{s}^{(T)}=-\frac{4 \pi \alpha_{s}\left(m_{t}\right)}{3}+\frac{1}{N_{c}} \pi \alpha_{s}\left(m_{t}\right), \\
& \mathcal{V}_{\Lambda}^{(T)}=-6 \pi \alpha_{s}\left(m_{t}\right), \quad \mathcal{V}_{t}^{(T)}=-\frac{\pi \alpha_{s}\left(m_{t}\right)}{3}, \quad \mathcal{V}_{s}^{(1)}=\frac{\left(N_{c}^{2}-1\right)}{2 N_{c}^{2}} \pi \alpha_{s}\left(m_{t}\right) \\
& \mathcal{V}_{c}^{(1)}=0, \quad \mathcal{V}_{2}^{(T)}=0, \quad \mathcal{V}_{2}^{(1)}=0 \tag{1.6}
\end{align*}
$$

The basis in terms of $(1 \otimes 1)$ and $\left(T^{A} \otimes \bar{T}^{A}\right)$ can be converted to the color singlet and octet basis by the linear transformation

$$
\left[\begin{array}{c}
V_{\text {singlet }}  \tag{1.7}\\
V_{\text {octet }}
\end{array}\right]=\left[\begin{array}{cc}
1 & -C_{F} \\
1 & \frac{1}{2} C_{A}-C_{F}
\end{array}\right]\left[\begin{array}{c}
V_{1 \otimes 1} \\
V_{T \otimes T}
\end{array}\right] .
$$

Since in our case the $t \bar{t}$ pair is produced from the intermediate photon or $Z$ boson, we will concentrate only on the color singlet channel in this work.

At this point we give the expression for the LL evolution of $\mathcal{V}_{r}^{(s)}$ from Ref. [22] since it will contribute to the NLL running of forward scattering operators as we will see in Sec. 3.3,

$$
\begin{align*}
\mathcal{V}_{r}^{(s)}(\nu) & =-4 \pi C_{F} \alpha_{s}\left(m_{t}\right) z\left[1+\frac{8 C_{A}}{3 \beta_{0}} \ln (2-z)\right] \\
z & \equiv \frac{\alpha_{s}\left(m_{t} \nu\right)}{\alpha_{s}\left(m_{t}\right)} \tag{1.8}
\end{align*}
$$

where $\beta_{0}$ is the $\mathcal{O}\left(\alpha_{s}^{2}\right)$ coefficient in the QCD $\beta$-function given in App. F and $n_{f}=5$ is the number of light fermions.

The velocity power counting of the fields is derived from demanding that the action for the kinetic terms is of order $v^{0}$. One obtains

$$
\begin{align*}
A_{q}^{\mu} & \sim v, \\
\psi_{\mathbf{p}} & \sim \chi_{\mathbf{p}} \sim v^{3 / 2}, \\
D^{\mu} & \sim A^{\mu} \sim v^{2} \tag{1.9}
\end{align*}
$$

for soft gluon fields, quark and antiquark fields, the (ultrasoft) covariant derivative and the ultrasoft gluon field, respectively.

### 1.4 Currents and Forward Scattering Operators

The components $\mathcal{L}_{u}, \mathcal{L}_{s}$ and $\mathcal{L}_{p}$ describe the non-relativistic $t \bar{t}$ dynamics due to the strong interaction once the heavy quark pair is produced. For the $t \bar{t}$ production itself one needs additional operators in the Lagrangian. For the treatment up to NNLL order in the cross section these are the dominant ${ }^{3} S_{1}$ current $\mathcal{O}_{\mathbf{p}, 1}^{j}$, the subleading ( $\mathbf{p}^{2} / m_{t}^{2}$-suppressed) ${ }^{3} S_{1}$ current $\mathcal{O}_{\mathbf{p}, 2}^{j}$ and the leading ${ }^{3} P_{1}$ current $\mathcal{O}_{\mathbf{p}, 3}^{j}$ [22], which is $\mathbf{p} / m_{t}$-suppressed compared to the dominant $S$-wave current. They have the form

$$
\begin{align*}
& \mathcal{O}_{\mathbf{p}, 1}^{j}=\psi_{\mathbf{p}}^{\dagger} \sigma^{j}\left(i \sigma^{2}\right) \chi_{-\mathbf{p}}^{*}, \quad \mathcal{O}_{\mathbf{p}, 2}^{j}=\frac{1}{m_{t}^{2}} \psi_{\mathbf{p}}^{\dagger} \mathbf{p}^{2} \sigma^{j}\left(i \sigma^{2}\right) \chi_{-\mathbf{p}}^{*} \\
& \mathcal{O}_{\mathbf{p}, 3}^{j}=\frac{-i}{2 m_{t}} \psi_{\mathbf{p}}^{\dagger}\left[\sigma^{j}, \boldsymbol{\sigma} \cdot \mathbf{p}\right]\left(i \sigma^{2}\right) \chi_{-\mathbf{p}}^{*} \tag{1.10}
\end{align*}
$$

$\mathbf{p}$ being the soft momentum label. In this basis of operators there is the additional $D$-wave current

$$
\mathcal{O}_{\mathbf{p}, 4}^{j}=\frac{1}{m_{t}^{2}} \psi_{\mathbf{p}}^{\dagger}\left(p^{j}(\boldsymbol{\sigma} \cdot \mathbf{p})-\sigma^{j} \mathbf{p}^{2} / 3\right)\left(i \sigma^{2}\right) \chi_{-\mathbf{p}}^{*}
$$

which is not needed here because it generates only contributions beyond NNLL order. The operators are obtained from a non-relativistic $v$-expansion of the full electroweak theory vector and axial-vector currents.

To ensure electroweak gauge invariance at subleading order it is necessary to include the initial $e^{+} e^{-}$fields, leading to the $t \bar{t}$ production operators

$$
\begin{align*}
\mathcal{O}_{V, \mathbf{p}, \sigma} & =\left[\bar{e}_{+} \gamma_{j} e_{-}\right] \mathcal{O}_{\mathbf{p}, \sigma}^{j} \\
\mathcal{O}_{A, \mathbf{p}, \sigma} & =\left[\bar{e}_{+} \gamma_{j} \gamma_{5} e_{-}\right] \mathcal{O}_{\mathbf{p}, \sigma}^{j} \tag{1.11}
\end{align*}
$$

where the index $j=1,2,3$ is summed and the index $\sigma=1,2,3$ distinguishes between the different currents in Eq. (1.10). Because the effective theory is constructed such that it describes $t \bar{t}$ production only in the threshold region and in the c.m. frame, only those initial $e^{+} e^{-}$states $^{1} a_{\tau^{\prime}}^{c \dagger}\left(\mathbf{k}^{\prime}\right) a_{\tau}^{\dagger}(\mathbf{k})|0\rangle$ are allowed that fulfill $s \equiv\left(k+k^{\prime}\right)^{2} \approx 4 m_{t}^{2}$ and $\mathbf{k}=-\mathbf{k}^{\prime}$. For simplicity we assume electron and positron to travel along the $z$-direction, therefore the explicit form of their 4 -momenta is

$$
\begin{align*}
k^{\mu} & =\left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2} \hat{\mathbf{e}}_{z}\right) \\
k^{\prime \mu} & =\left(\frac{\sqrt{s}}{2},-\frac{\sqrt{s}}{2} \hat{\mathbf{e}}_{z}\right), \tag{1.12}
\end{align*}
$$

$\hat{\mathbf{e}}_{z}$ being the unit vector in $z$-direction. The fields $e_{-}$and $e_{+}$in Eqs. (1.11) are now defined as

$$
\begin{align*}
& e_{-}(x)=\sum_{\tau, \sqrt{s}} a_{\tau}(\mathbf{k}) u_{\tau}(\mathbf{k}) e^{-i \hat{k} \cdot x} \\
& e_{+}(x)=\sum_{\tau, \sqrt{s}} a_{\tau}^{c \dagger}\left(\mathbf{k}^{\prime}\right) v_{\tau}\left(\mathbf{k}^{\prime}\right) e^{i \hat{k}^{\prime} \cdot x} \tag{1.13}
\end{align*}
$$

where $u_{\tau}(\mathbf{k})$ and $v_{\tau}\left(\mathbf{k}^{\prime}\right)$ denote Dirac spinors for electron and positron, respectively, and the momenta $\mathbf{k}$ and $\mathbf{k}^{\prime}$ refer to Eqs. (1.12). The sum over the c.m. energy $\sqrt{s}$ is restricted to the threshold region. For simplicity we did not include an integral over the angles of the electron and positron momenta. The phase factor in Eqs. (1.13) is defined such that it describes only the $t \bar{t}$ low-energy fluctuations,

$$
\begin{aligned}
\hat{k}^{\mu} & =\left(\frac{\sqrt{s}}{2}-m_{t}, \frac{\sqrt{s}}{2} \hat{\mathbf{e}}_{z}\right) \\
\hat{k}^{\prime \mu} & =\left(\frac{\sqrt{s}}{2}-m_{t},-\frac{\sqrt{s}}{2} \hat{\mathbf{e}}_{z}\right) .
\end{aligned}
$$

[^1]We note that the 3-momenta cancel after the operators $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$ have been applied to the initial $e^{+} e^{-}$state. The operators for $t \bar{t}$ annihilation are obtained from $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$ by Hermitian conjugation.

Due to the dependence of the intermediate photon and $Z$ boson propagator on the $\mathrm{c} . \mathrm{m}$. energy in the process $e^{+} e^{-} \rightarrow \gamma^{*}, Z^{*} \rightarrow t \bar{t}$ we introduce the additional operators

$$
\begin{align*}
\mathcal{O}_{V, \mathbf{p}, 1}^{(1)} & =\left[\bar{e}_{+} \gamma_{j}\left(\hat{E} / m_{t}\right) e_{-}\right] \mathcal{O}_{\mathbf{p}, 1}^{j} \\
\mathcal{O}_{A, \mathbf{p}, 1}^{(1)} & =\left[\bar{e}_{+} \gamma_{j} \gamma_{5}\left(\hat{E} / m_{t}\right) e_{-}\right] \mathcal{O}_{\mathbf{p}, 1}^{j} \tag{1.14}
\end{align*}
$$

for the formal treatment of electroweak effects at NNLL order. Here, $\hat{E}$ denotes the operator $\hat{E}=i \partial_{0}$ acting on the fields on the right and on the left and thus picks up the kinetic energy $E \equiv \sqrt{s}-2 m_{t}$ from the initial $e^{+} e^{-}$state. The operators $\mathcal{O}_{V / A, \mathbf{p}, 1}^{(1)}$ have the same QCD behaviour (such as QCD matching conditions and renormalization group running) as $\mathcal{O}_{V / A, \mathbf{p}, 1}$ in Eqs. (1.11). Similar additional operators related to $\mathcal{O}_{V / A, \mathbf{p}, 2}$ and $\mathcal{O}_{V / A, \mathbf{p}, 3}$ are not needed since they would give contributions beyond NNLL order.

The contribution of the currents to the Lagrangian reads

$$
\begin{align*}
\mathcal{L}_{\mathrm{cur}}=\sum_{\mathbf{p}}[ & C_{V, 1} \mathcal{O}_{V, \mathbf{p}, 1}+C_{A, 1} \mathcal{O}_{A, \mathbf{p}, 1} \\
& +C_{V, 1}^{(1)} \mathcal{O}_{V, \mathbf{p}, 1}^{(1)}+C_{A, 1}^{(1)} \mathcal{O}_{A, \mathbf{p}, 1}^{(1)} \\
& +C_{V, 2} \mathcal{O}_{V, \mathbf{p}, 2}+C_{A, 2} \mathcal{O}_{A, \mathbf{p}, 2} \\
& \left.+C_{V, 3} \mathcal{O}_{V, \mathbf{p}, 3}+C_{A, 3} \mathcal{O}_{A, \mathbf{p}, 3}\right]+ \text { H.c. } \tag{1.15}
\end{align*}
$$

Since we neglect QED radiative corrections, the electron and positron fields in the current operators act like classic fields and do not contribute to the nonrelativistic $t \bar{t}$ dynamics. The Hermitian conjugation (H.c.) refers to the operators only. Their Wilson coefficients, which are complex in our treatment of the unstable top quark undertaken in Chaps. 3 and 4, are not conjugated. They are fixed at the hard scale $\nu=1$ at lowest order by tree level matching to the full electroweak theory graphs shown in Fig. 1.1. We obtain

$$
\begin{align*}
C_{V, 1}^{\text {born }}(1) & =-4 \pi \alpha\left[\frac{Q_{t}}{4 m_{t}^{2}}-\frac{v_{e} v_{t}}{4 m_{t}^{2}-M_{Z}^{2}}\right], & C_{A, 1}^{\text {born }}(1)=-4 \pi \alpha \frac{a_{e} v_{t}}{4 m_{t}^{2}-M_{Z}^{2}}, \\
C_{V, 1}^{(1), \text {,brn }}(1) & =\pi \alpha\left[\frac{Q_{t}}{m_{t}^{2}}-\frac{16 v_{e} v_{t} m_{t}^{2}}{\left(4 m_{t}^{2}-M_{Z}^{2}\right)^{2}}\right], & C_{A, 1}^{(1), \text { born }}(1)=16 \pi \alpha \frac{a_{e} v_{t} m_{t}^{2}}{\left(4 m_{t}^{2}-M_{Z}^{2}\right)^{2}}, \\
C_{V, 3}^{\text {born }}(1) & =4 \pi \alpha \frac{v_{e} a_{t}}{4 m_{t}^{2}-M_{Z}^{2}}, & C_{A, 3}^{\text {born }}(1)=-4 \pi \alpha \frac{a_{e} a_{t}}{4 m_{t}^{2}-M_{Z}^{2}} \tag{1.16}
\end{align*}
$$

and $C_{V, 2}^{\text {born }}(1)=-1 / 6 C_{V, 1}^{\text {born }}(1), C_{A, 2}^{\text {born }}(1)=-1 / 6 C_{A, 1}^{\text {born }}(1)$, where

$$
v_{f}=\frac{t_{3}^{f}-2 Q_{f} s_{w}^{2}}{2 s_{w} c_{w}}, \quad a_{f}=\frac{t_{3}^{f}}{2 s_{w} c_{w}}
$$

the symbol $Q_{f}$ being the electric charge and $t_{3}^{f}$ the third component of the weak isospin of fermion $f$. The abbreviations $s_{w}$ and $c_{w}$ denote sine and cosine of the weak mixing angle, respectively. For simplicity we also use the notation $C_{V / A, \sigma}$ instead of $C_{V / A, \sigma}(1)$ in the following. Including the electroweak effects up to NNLL order that will be examined in Chaps. 2, 3 and 4 the Wilson coefficients assume the form

$$
\begin{align*}
C_{V / A, 1}(\nu ; \Lambda) & =C_{V / A, 1}^{\mathrm{born}} c_{1}(\nu)\left(1+i \delta \tilde{c}_{1}(\Lambda)\right)+i C_{V / A, 1}^{\mathrm{int}}\left(1+\delta \tilde{c}_{1}^{\mathrm{int}}(\Lambda)\right)+C_{V / A, 1}^{\mathrm{ew}} \\
C_{V / A, 1}^{(1)}(\nu) & =C_{V / A, 1}^{(1), \text { born }} c_{1}(\nu) \\
C_{V / A, 2}(\nu) & =C_{V / A, 1}^{\mathrm{born}} c_{2}(\nu) \\
C_{V / A, 3}(\nu) & =C_{V / A, 3}^{\mathrm{born}} c_{3}(\nu) \tag{1.17}
\end{align*}
$$

where $c_{\sigma}(\nu)$ contain the QCD hard scale matching conditions and the QCD running. The matching condition for $c_{1}$ is needed up to $\mathcal{O}\left(\alpha_{s}^{2}\right)$ and is well known [21, 22, 65-68]. Because all effective theory one-loop vertex diagrams that contribute to the renormalization of the operator $\mathcal{O}_{V / A, \mathbf{p}, 1}$ are ultraviolet finite, the LL running of $c_{1}$ is zero. For the same reason the coefficients $C_{V / A, 1}^{\text {int }}$ and $C_{V / A, 1}^{\mathrm{ew}}$ do not run at this level, and since they are already suppressed by $\alpha \sim v^{2}$, their scale-dependence is not written down explicitly here. This applies also to $C_{V / A, 1}^{(1)}$ because it contributes to the cross section with a $E / m_{t} \sim v^{2}$ suppression, therefore we could also have avoided writing down $c_{1}(\nu)$ in the second line. The NLL running of $c_{1}$ is well known [24-28]. At NNLL order the evolution is not fully known. The non-mixing contributions arising from threeloop vertex diagrams, which affect the evolution of $c_{1}$ directly, were calculated in Ref. [31]. Mixing contributions originate from the subleading evolution of the potential Wilson coefficients appearing in the NLL order renormalization group equation. This subleading evolution is known for the coefficients of the Coulomb potential $\mathcal{V}_{c}^{(s)}[25,29,30]$ and spin-dependent potential $\mathcal{V}_{s}^{(s)}$ [32,69]. For the spinindependent potentials $\mathcal{V}_{2}^{(s)}$ and $\mathcal{V}_{r}^{(s)}$ the evolution arising from two-loop ultrasoft contributions was computed in Ref. [33], whereas the soft contributions are unknown at the moment. For the $\mathcal{V}_{k}^{(s)}$ potential the evolution arising from two-loop ultrasoft contributions is also unknown at the moment.

For the matching conditions for $c_{2}$ and $c_{3}$, no QCD corrections need to be taken into account for a cross section prediction up to NNLL order, and therefore we have $c_{2}(1)=-1 / 6$ and $c_{3}(1)=1$. While the LL evolution of $c_{3}$ is zero, the

LL evolution of $c_{2}$ reads [22]

$$
\begin{equation*}
c_{2}(\nu)=-\frac{1}{6}-\frac{8 C_{F}}{3 \beta_{0}} \ln \left(\frac{z}{2-z}\right), \quad z \equiv \frac{\alpha_{s}\left(m_{t} \nu\right)}{\alpha_{s}\left(m_{t}\right)} . \tag{1.18}
\end{equation*}
$$

The terms $C_{V / A, 1}^{\mathrm{ew}}$ and $C_{V / A, 1}^{\mathrm{int}}$ involve NNLL order corrections originating from electroweak one-loop diagrams. While the terms $C_{V / A, 1}^{\mathrm{ew}}$ contain the real parts, $C_{V / A, 1}^{\mathrm{int}}$ originate from certain cuts through those diagrams. The computation of these coefficients will be done in Chaps. 2 and 3. The terms $\delta \tilde{c}_{1}(\Lambda)$ and $\delta \tilde{c}_{1}^{\text {int }}(\Lambda)$ depend on the external parameter $\Lambda$ that defines which physical processes are described by the absorptive effective theory action. The parameter $\Lambda$ is associated with experimental cuts on the dynamical variables of the $t \bar{t}$ pair or rather of its decay products. The determination of these $\Lambda$-dependent contributions will be done in Chap. 4. Note that in the second, third and fourth equation of (1.17) we omitted terms present in the first equation that do not lead to an effect up to NNLL order. Yet we included the $\mathrm{N}^{3} \mathrm{LO} \Lambda$-dependent terms in the first equation, because they arise in the formal treatment of instability effects by means of finite imaginary two-loop renormalization that will be carried out in Sec. 4.5. In the following we will use the abbreviation $C_{V / A, 1}(\nu) \equiv C_{V / A, 1}(\nu ; \Lambda)$ in the cases where the $\Lambda$ dependence is not relevant up to $\mathrm{N}^{3} \mathrm{LO}$.

For the computation of the $e^{+} e^{-} \rightarrow t \bar{t}$ cross section we will use the optical theorem. Therefore forward scattering operators $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$will become important. Because phase space matching described in Chap. 4 involves energydependent terms in the cross section, we need to define these operators such that they are capable to reproduce the energy-dependence in the formal effective theory treatment. They are defined as

$$
\begin{align*}
& \tilde{\mathcal{O}}_{V}^{(n)}=-\left[\bar{e}_{-} \gamma^{\mu} e_{+}\right]\left[\bar{e}_{+} \gamma_{\mu}\left(\hat{E} / m_{t}\right)^{n} e_{-}\right] \\
& \tilde{\mathcal{O}}_{A}^{(n)}=-\left[\bar{e}_{-} \gamma^{\mu} \gamma_{5} e_{+}\right]\left[\bar{e}_{+} \gamma_{\mu} \gamma_{5}\left(\hat{E} / m_{t}\right)^{n} e_{-}\right] \tag{1.19}
\end{align*}
$$

and thus pick up $n$ powers of the $t \bar{t}$ kinetic energy from the initial $e^{+} e^{-}$state. We also write $\tilde{\mathcal{O}}_{V / A} \equiv \tilde{\mathcal{O}}_{V / A}^{(0)}$ for the energy-independent operators. The normalization of the electron and positron fields ensures that we have

$$
\frac{1}{4} \sum_{\tau, \tau^{\prime}}\langle 0| a_{\tau}(\mathbf{k}) a_{\tau^{\prime}}^{c}\left(\mathbf{k}^{\prime}\right) \tilde{\mathcal{O}}_{V / A}^{(n)} a_{\tau^{\prime}}^{c \dagger}\left(\mathbf{k}^{\prime}\right) a_{\tau}^{\dagger}(\mathbf{k})|0\rangle=s\left(\frac{E}{m_{t}}\right)^{n}
$$

for the spin-averaged forward scattering amplitude.
The contribution of the forward scattering operators to the Lagrangian reads

$$
\begin{equation*}
\mathcal{L}_{\mathrm{fsc}}=\sum_{n} \tilde{C}_{V}^{(n)} \tilde{\mathcal{O}}_{V}^{(n)}+\tilde{C}_{A}^{(n)} \tilde{\mathcal{O}}_{A}^{(n)}, \tag{1.20}
\end{equation*}
$$

$\tilde{C}_{V / A}^{(n)}(\nu ; \Lambda)$ being the Wilson coefficients, which depend on the renormalization scale and on the cut $\Lambda$. We will use the abbreviation $\tilde{C}_{V / A} \equiv \tilde{C}_{V / A}^{(0)}$ for the
coefficients of the operators that do not contain the energy operator. The NLL running of $\tilde{C}_{V / A}$ is generated by NNLL finite lifetime and interference effects and is determined in Chap. 3. At the hard scale they are fixed by the $\Lambda$-dependent terms that are determined by phase space matching as described in Chap. 4.

### 1.5 Optical Theorem and Threshold Cross Section

For the computation of the total $e^{+} e^{-} \rightarrow t \bar{t}$ cross section the optical theorem is used. It implies that the total cross section is proportional to the imaginary part of the forward scattering amplitude $e^{+} e^{-} \rightarrow e^{+} e^{-}$where the initial and final $e^{+} e^{-}$states are identical and the interaction taking place between these states corresponds to the non-relativistic $t \bar{t}$ dynamics. The derivation of the optical theorem relies on the unitarity of the theory. As we will see in Chaps. 3 and 4 , the operators of our effective theory for unstable particles have complex Wilson coefficients that are not conjugated when the respective operators are conjugated. Although they render the Lagrangian formally non-Hermitian, the effective theory inherits unitarity from the underlying theory, and therefore it is possible to use the optical theorem [43]. Concerning phase space cuts, the $\Lambda$-dependent Lagrangian is constructed such that it correctly reproduces the full theory phase space upon the application of the optical theorem in the effective theory.

For $t \bar{t}$ production close to threshold up to NNLL we obtain the cross section

$$
\begin{align*}
\sigma_{\text {thr }}=\frac{1}{s} L^{l k} \operatorname{Im} & {\left[\left(C_{V, 1}(\nu ; \Lambda)^{2}+C_{A, 1}(\nu ; \Lambda)^{2}\right) \mathcal{A}_{1}^{l k}\right.} \\
& +\left(2 C_{V, 1}(\nu) C_{V, 1}^{(1)}(\nu)+2 C_{A, 1}(\nu) C_{A, 1}^{(1)}(\nu)\right)\left(E / m_{t}\right) \mathcal{A}_{1}^{l k} \\
& +\left(2 C_{V, 1}(\nu) C_{V, 2}(\nu)+2 C_{A, 1}(\nu) C_{A, 2}(\nu)\right) \mathcal{A}_{2}^{l k} \\
& \left.+\left(C_{V, 3}(\nu)^{2}+C_{A, 3}(\nu)^{2}\right) \mathcal{A}_{3}^{l k}\right] \\
+ & \sum_{n}\left(E / m_{t}\right)^{n} \operatorname{Im}\left[\tilde{C}_{V}^{(n)}(\nu ; \Lambda)+\tilde{C}_{A}^{(n)}(\nu ; \Lambda)\right] \\
& +\operatorname{Im}\left[\delta \tilde{C}_{V}(\nu)+\delta \tilde{C}_{A}(\nu)\right] \tag{1.21}
\end{align*}
$$

The spin-averaged lepton tensor reads

$$
\begin{align*}
L^{l k} & =\frac{1}{4} \sum_{\tau, \tau^{\prime}}\left[\bar{v}_{\tau^{\prime}}\left(\mathbf{k}^{\prime}\right) \gamma^{l}\left(\gamma_{5}\right) u_{\tau}(\mathbf{k})\right]\left[\bar{u}_{\tau}(\mathbf{k}) \gamma^{k}\left(\gamma_{5}\right) v_{\tau^{\prime}}\left(\mathbf{k}^{\prime}\right)\right] \\
& =\frac{1}{2}\left(k+k^{\prime}\right)^{2}\left(\delta^{l k}-\hat{e}_{z}^{l} \hat{e}_{z}^{k}\right) \tag{1.22}
\end{align*}
$$

with the definitions of electron/positron momenta given in Eqs. (1.12). The quantities $\mathcal{A}_{i}^{l k}$ are time-ordered products of the $t \bar{t}$ production and annihilation operators defined in Eq. (1.10), which describe the non-relativistic dynamical effects. Note that the electron and positron field operators, from which the operators in Eqs. $(1.11,1.14,1.19)$ are composed, only pick up the initial and final $e^{+} e^{-}$states and do not appear in the correlators $\mathcal{A}_{i}^{l k}$, i. e. at quantum loop order. They are treated as classic fields since pure QED effects are neglected in this work. Therefore the explicit expression for $\mathcal{A}_{i}^{l k}$ read

$$
\begin{align*}
& \mathcal{A}_{1}^{l k}=i \sum_{\mathbf{p}, \mathbf{p}^{\prime}} \int d^{4} x e^{-i \hat{q} \cdot x}\langle 0| T \mathcal{O}_{\mathbf{p}, 1}^{l}(0) \mathcal{O}_{\mathbf{p}^{\prime}, 1}^{k}(x)|0\rangle \\
& \mathcal{A}_{2}^{l k}=\frac{i}{2} \sum_{\mathbf{p}, \mathbf{p}^{\prime}} \int d^{4} x e^{-i \hat{q} \cdot x}\langle 0| T\left[\mathcal{O}_{\mathbf{p}, 1}^{l}(0) \mathcal{O}_{\mathbf{p}^{\prime}, 2}^{k}(x)+\mathcal{O}_{\mathbf{p}, 2}^{l \dagger}(0) \mathcal{O}_{\mathbf{p}^{\prime}, 1}^{k}(x)\right]|0\rangle, \\
& \mathcal{A}_{3}^{l k}=i \sum_{\mathbf{p}, \mathbf{p}^{\prime}} \int d^{4} x e^{-i \hat{q} \cdot x}\langle 0| T \mathcal{O}_{\mathbf{p}, 3}^{l \dagger}(0) \mathcal{O}_{\mathbf{p}^{\prime}, 3}^{k}(x)|0\rangle, \tag{1.23}
\end{align*}
$$

where $\hat{q} \equiv\left(\sqrt{s}-2 m_{t}, 0\right)$.
In Eq. (1.21) only the first line contains LL contributions. The terms in the second, third and fourth lines are $v^{2}$-suppressed and therefore contribute at NNLL order. This suppression originates from factors $E / m_{t}, \mathbf{p}^{2} / m_{t}^{2}$ appearing in $\mathcal{O}_{\mathbf{p}, 2}^{k}$ and $\boldsymbol{\sigma} \cdot \mathbf{p} / m_{t}$ appearing in $\mathcal{O}_{\mathbf{p}, 3}^{k}$, respectively. The fifth line contains phase space corrections starting at NLL order and in the last line we have written down explicitly the counterterms for the renormalized forward scattering operators, which will be determined in Sec. 3.3.

The correlators $\mathcal{A}_{i}^{l k}, i=1,2,3$, were determined in Refs. [22,43] up to NNLL order. They are related to the Green function $\tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)$ of the Schrödinger operator in momentum space containing effective potentials and corrections to bilinear operators originating from $\mathcal{L}_{u}, \mathcal{L}_{s}$ and $\mathcal{L}_{p}$. This relation is helpful for the determination of $\mathcal{A}_{i}^{l k}$. The Green function $\tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)$ fulfills

$$
\begin{align*}
-\left[E-\frac{\mathbf{p}^{2}}{m_{t}}\right. & \left.+\frac{\mathbf{p}^{4}}{4 m_{t}^{3}}+i \Gamma_{t}\left(1-\frac{\mathbf{p}^{2}}{2 m_{t}^{2}}\right)\right] \tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \\
& +\int D^{n} \mathbf{q} \mu_{S}^{2 \epsilon} \tilde{V}(\mathbf{p}, \mathbf{q}) \tilde{G}_{v, m_{t}, \nu}\left(\mathbf{q}, \mathbf{p}^{\prime}\right)=(2 \pi)^{3} \delta^{(3)}\left(\mathbf{p}-\mathbf{p}^{\prime}\right) \tag{1.24}
\end{align*}
$$

where $n=3-2 \epsilon$ and $D^{n} \mathbf{q}=e^{\epsilon \gamma_{E}}(4 \pi)^{-\epsilon} d^{n} \mathbf{q} /(2 \pi)^{n}$ and we suppressed the residual mass term $\delta m_{t}$ contained in $\mathcal{L}_{u}$. Since $t \bar{t}$ is produced via the intermediate photon or $Z$ boson in a spin triplet and color singlet state, we can simplify the spindependence of the subleading $1 / m_{t}^{2}$ potentials in $\mathcal{L}_{p}$ by projecting onto the spin triplet channel (i. e. we set $\mathbf{S}^{2}=2$ and neglect $\mathcal{V}_{\Lambda}$ and $\mathcal{V}_{t}$ ) and use Eq. (1.7) to obtain the effective potential in the color singlet representation. It reads

$$
\begin{equation*}
\tilde{V}(\mathbf{p}, \mathbf{q})=\tilde{V}_{c}(\mathbf{p}, \mathbf{q})+\frac{\mathcal{V}_{k}^{(s)} \pi^{2}}{m_{t}|\mathbf{k}|}+\frac{\mathcal{V}_{r}^{(s)}\left(\mathbf{p}^{2}+\mathbf{p}^{\prime 2}\right)}{2 m_{t}^{2} \mathbf{k}^{2}}+\frac{\mathcal{V}_{2}^{(s)}}{m_{t}^{2}}+\frac{2 \mathcal{V}_{s}^{(s)}}{m_{t}^{2}} \tag{1.25}
\end{equation*}
$$

with the NNLL order effective Coulomb potential

$$
\begin{align*}
& \tilde{V}_{c}(\mathbf{p}, \mathbf{q})=\frac{\mathcal{V}_{c}^{(s)}(\nu)}{\mathbf{k}^{2}}-\frac{4 \pi C_{F} \alpha_{s}\left(m_{t} \nu\right)}{\mathbf{k}^{2}}\left\{\frac{\alpha_{s}\left(m_{t} \nu\right)}{4 \pi}\left[-\beta_{0} \ln \left(\frac{\mathbf{k}^{2}}{m_{t}^{2} \nu^{2}}\right)+a_{1}\right]\right. \\
& \left.\quad+\left(\frac{\alpha_{s}\left(\mu_{S}\right)}{4 \pi}\right)^{2}\left[\beta_{0}^{2} \ln ^{2}\left(\frac{\mathbf{k}^{2}}{m_{t}^{2} \nu^{2}}\right)-\left(2 \beta_{0} a_{1}+\beta_{1}\right) \ln \left(\frac{\mathbf{k}^{2}}{m_{t}^{2} \nu^{2}}\right)+a_{2}\right]\right\}, \tag{1.26}
\end{align*}
$$

where $\mathbf{k}=\mathbf{p}-\mathbf{q}$ is the momentum transfer and the constants $a_{1}$ and $a_{2}$ are given in App. F. The second term in Eq. (1.26) contains the one- and two-loop corrections to the Coulomb potential $[70,71]$. In vNRQCD they arise in the time-ordered product by the interactions of the quarks with soft gluons [72]. The first term contains the scale-dependent Wilson coefficient of the "bare" Coulomb potential as given in $\mathcal{L}_{p}$, which is needed up to NNLL order. The evolution of the $v^{2}$ suppressed $1 / m_{t}$ and $1 / m_{t}^{2}$ potentials in Eq. (1.25), on the other hand, is needed only at LL order. These contributions were computed in Refs. [24, 26, 27, 72-74].

After solving Eq. (1.24) the correlators $\mathcal{A}_{i}^{l k}$ are obtained by Fourier transformations of $\tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)$ and an evaluation of the resulting expressions at zero distance $\mathbf{x}=\mathbf{x}^{\prime}$. By taking the traces of the sigma matrices of the currents in 3 dimensions they can be written as $3 \mathcal{A}_{i}^{l k}=\delta^{l k} \mathcal{A}_{i}$. Therefore one obtains

$$
\begin{aligned}
& \mathcal{A}_{1}\left(v, m_{t}, \nu\right)=6 N_{c} \mu_{S}^{4 \epsilon} \int D^{n} \mathbf{p} D^{n} \mathbf{p}^{\prime} \tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right) \\
& \mathcal{A}_{3}\left(v, m_{t}, \nu\right)=\frac{12 N_{c}}{m_{t}^{2} n} \mu_{S}^{4 \epsilon} \int D^{n} \mathbf{p} D^{n} \mathbf{p}^{\prime}\left(\mathbf{p} \cdot \mathbf{p}^{\prime}\right) \tilde{G}_{v, m_{t}, \nu}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)
\end{aligned}
$$

The factor $1 / n$ in the expression for $\mathcal{A}_{3}$ arises because after the trace over the sigma matrices in the $P$-wave current $\mathcal{O}_{\mathbf{p}, 3}^{j}$ has been taken in 3 dimensions, the dot product $\mathbf{p} \cdot \mathbf{p}^{\prime}$ must be projected back into $n$ dimensions. Due to the heavy quark equation of motion $\mathcal{A}_{1}$ and $\mathcal{A}_{2}$ are directly related,

$$
\begin{equation*}
\mathcal{A}_{2}\left(v, m_{t}, \nu\right)=v^{2} \mathcal{A}_{1}\left(v, m_{t}, \nu\right) \tag{1.27}
\end{equation*}
$$

Therefore only the LL terms in $\mathcal{A}_{1}$ are necessary to obtain the NNLL order contributions to $\mathcal{A}_{2}$.

The correlators $\mathcal{A}_{1}$, written as a sum of the various contributions in Eq. (1.24), and $\mathcal{A}_{3}$ read

$$
\begin{align*}
& \mathcal{A}_{1}\left(v, m_{t}, \nu\right)=6 N_{c}[ G^{c}\left(a, v, m_{t}, \nu\right)+\left(\mathcal{V}_{2}^{(s)}(\nu)+2 \mathcal{V}_{s}^{(s)}(\nu)\right) G^{\delta}\left(a, v, m_{t}, \nu\right) \\
&+\mathcal{V}_{r}^{(s)}(\nu) G^{r}\left(a, v, m_{t}, \nu\right)+\mathcal{V}_{k}^{(s)}(\nu) G^{k}\left(a, v, m_{t}, \nu\right) \\
&\left.+G^{\mathrm{kin}}\left(a, v, m_{t}, \nu\right)+G^{\mathrm{dil}}\left(a, v, m_{t}, \nu\right)\right] \\
& \mathcal{A}_{3}\left(v, m_{t}, \nu\right)=\frac{4 N_{c}}{m_{t}^{2}} G^{1}\left(a, v, m_{t}, \nu\right) . \tag{1.28}
\end{align*}
$$

where $a \equiv-\mathcal{V}_{c}^{(s)}(\nu) /(4 \pi)=C_{F} \alpha_{s}\left(m_{t} \nu\right)$. The contributions to $\mathcal{A}_{1}$ and $\mathcal{A}_{3}$ are called zero-distance $S$-wave and $P$-wave Green functions, respectively.

All LL order terms are contained in $G^{c}$ and are denoted as $G^{0}$. This function, as given in Eq. (D.1), was computed already in Refs. [75-77] except for the scheme-dependent regulator of the divergence. To obtain the $\overline{\mathrm{MS}}$ expression the $\mathcal{O}(a)$ term was matched to the two-loop graph with a single $\mathcal{V}_{c}$ insertion [78]. The resulting LL order cross section then reads

$$
\begin{equation*}
\sigma_{\mathrm{thr}}^{\mathrm{LL}}=2 N_{c}\left(\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\mathrm{born}}\right)^{2}\right) \operatorname{Im}\left[G^{0}\left(a, v, m_{t}, \nu\right)\right] . \tag{1.29}
\end{equation*}
$$

The NNLL order Coulomb Green function $G^{c}$ was computed in Ref. [22] by the exact solution of the corresponding Schrödinger equation using numerical techniques developed in Refs. [79, 80]. All other zero-distance Green functions are available in an analytic form. The terms $G^{\text {kin }}$ and $G^{\text {dil }}$ stem from the kinetic energy and lifetime dilatation corrections to the bilinear quark field operators in Eq. (1.3). $G^{\text {dil }}$ was determined in Ref. [43] and reads

$$
\begin{equation*}
G^{\text {dil }}=-i \frac{\Gamma_{t}}{2 m_{t}}\left[1+\frac{v}{2} \frac{\partial}{\partial v}+a \frac{\partial}{\partial a}\right] G^{0}\left(a, v, m_{t}, \nu\right) . \tag{1.30}
\end{equation*}
$$

The terms $G^{\delta}, G^{r}$ and $G^{k}$ correspond to corrections to the LL Coulomb Green function due to insertions of the $v^{2}$-suppressed potentials. The explicit expressions of these Green functions are given in Refs. [22, 43] and re-presented in App. D since our calculations in the following chapters will partly rely on them and partly reproduce them.

To understand instability and interference effects arising within the effective theory treatment at NNLL order, it is important to recognize the structure of the ultraviolet divergences contained in the Green functions. These divergences appear in two-loop graphs at $\mathcal{O}\left(\alpha_{s}\right)$ and cannot be subtracted by a proper current renormalization. In fact, for the computation of the $S$-wave Green functions the NLL renormalized current operators $\mathcal{O}_{\mathbf{p}, 1}$ are used and therefore subdivergences are automatically cancelled. The remaining divergences have the form $a v^{2} / \epsilon, a$
being scale-dependent. As long as the velocity $v$ is considered as a real number and the Wilson coefficients of the current operators appearing in Eq. (1.21) are real, the divergences appear only in the real parts of $\mathcal{A}_{i}$ and therefore do not enter the cross section through the optical theorem. However, neither of these two assumptions is true. The effective velocity $v$ is complex for an unstable top quark due to the imaginary electroweak matching condition $i \Gamma_{t}$ in Eq. (1.3). The current Wilson coefficients obtain imaginary parts through interference effects as we will see in Sec. 3.2. Through these mechanisms the divergences actually would enter the cross section at NNLL order, unless they are cancelled beforehand in a consistent field theoretical treatment. This will be discussed in Chap. 3.

### 1.6 Initial State Polarization

By polarizing the electron or positron beam, one can achieve a considerable rise of the threshold cross section. Therefore it is expected that also the numerical precision of the extracted physical parameters increases by polarizing, provided that the polarization itself is well understood experimentally.

To analyze the effects of polarization, we take a closer look at the derivation of Eq. (1.21) from the point of view of the full electroweak theory. We first concentrate on the LL order contribution and later discuss the remaining terms. For polarized electron and positron beams the LL contribution in the first line assumes the form

$$
\begin{align*}
& \sigma_{\rho \tau}=\frac{1}{s} \operatorname{Im}\left\{\left[\bar{v}_{\tau}\left(C_{V, 1}^{\text {born }} \gamma^{l}+C_{A, 1}^{\text {born }} \gamma^{l} \gamma_{5}\right) u_{\rho}\right]\right. \\
& \left.\times\left[\bar{u}_{\rho}\left(C_{V, 1}^{\mathrm{born}} \gamma^{k}+C_{A, 1}^{\mathrm{born}} \gamma^{k} \gamma_{5}\right) v_{\tau}\right] \mathcal{A}_{1}^{l k}\right\}, \tag{1.31}
\end{align*}
$$

where $u_{\rho}$ and $v_{\tau}$ denote Dirac spinors whose indices $\rho, \tau$ can take the values $\pm 1$ and whose 3 -momentum dependence is suppressed. They are eigenstates of the projection operators $\omega_{+}=\frac{1}{2}\left(1+\gamma_{5}\right)$ and $\omega_{-}=\frac{1}{2}\left(1-\gamma_{5}\right)$ and are related to physical states as (we assume massless leptons)

$$
\begin{array}{ll}
\omega_{+} u_{+}=u_{+} & \text {right-handed electron } \\
\omega_{+} v_{+}=v_{+} & \text {left-handed positron } \\
\omega_{-} u_{-}=u_{-} & \text {left-handed electron } \\
\omega_{-} v_{-}=v_{-} & \text {right-handed positron. }
\end{array}
$$

These eigenstates have the property

$$
\bar{u}_{\rho} \gamma^{\mu}\left(\gamma_{5}\right) v_{\tau}=0, \quad \bar{v}_{\tau} \gamma^{\mu}\left(\gamma_{5}\right) u_{\rho}=0 \quad \text { if } \rho \neq \tau
$$

which tells that electron and positron with the same physical handedness do not
give a contribution to the cross section. By means of simple algebra we obtain

$$
\begin{align*}
\sigma_{\rho \tau} & =\frac{1}{s} \operatorname{Im}\left\{\left(C_{V, 1}^{\text {born }}+\rho C_{A, 1}^{\text {born }}\right)^{2} \delta_{\rho \tau}\left[\bar{v}_{\rho} \gamma^{l} u_{\rho}\right]\left[\bar{u}_{\rho} \gamma^{k} v_{\rho}\right] \mathcal{A}_{1}^{l k}\right\}, \\
& =\frac{2}{3} \delta_{\rho \tau} \operatorname{Im}\left\{\left(C_{V, 1}^{\text {born }}+\rho C_{A, 1}^{\text {born }}\right)^{2} \mathcal{A}_{1}\right\}, \tag{1.32}
\end{align*}
$$

where the indices $\rho, \tau$ are not summed. To obtain the second equation we used $3 \mathcal{A}_{1}^{l k}=\delta^{l k} \mathcal{A}_{1}$ and $\left[\bar{v}_{\rho} \gamma^{k} u_{\rho}\right]\left[\bar{u}_{\rho} \gamma^{k} v_{\rho}\right]=2 s$ for $\rho= \pm 1$.

We define $P_{-}$and $P_{+}$as electron and positron polarization, respectively, such that $P_{-}>0$ if a fraction $\left|P_{-}\right|$of all electrons is (actively) polarized to be righthanded and $P_{-}<0$ if a fraction $\left|P_{-}\right|$of all electrons is polarized to be left-handed, accordingly $P_{+}>0$ corresponds to right-handed and $P_{+}<0$ to left-handed positron polarization. The cross section then reads

$$
\sigma\left(P_{-}, P_{+}\right)=\frac{1}{2}\left(1+P_{-}\right) \frac{1}{2}\left(1-P_{+}\right) \sigma_{++}+\frac{1}{2}\left(1-P_{-}\right) \frac{1}{2}\left(1+P_{+}\right) \sigma_{--} .
$$

With Eq. (1.32) this leads to ${ }^{2}$

$$
\begin{aligned}
\sigma\left(P_{-}, P_{+}\right)=\frac{1}{3} \operatorname{Im}\{[ & \left(1-P_{-} P_{+}\right)\left(\left(C_{V, 1}^{\text {born }}\right)^{2}+\left(C_{A, 1}^{\text {born }}\right)^{2}\right) \\
& \left.\left.+\left(P_{-}-P_{+}\right) 2 C_{V, 1}^{\text {born }} C_{A, 1}^{\text {born }}\right] \mathcal{A}_{1}\right\}
\end{aligned}
$$

Let us now discuss the remaining contributions to the cross section in the polarized case, corresponding to the terms $C_{V / A}^{\mathrm{ew}}, C_{V / A}^{\mathrm{int}}, E / m_{t}, \mathcal{A}_{2}, \mathcal{A}_{3}, \tilde{C}_{V / A}^{(n)}$ (and $\left.\delta \tilde{C}_{V / A}\right)$. As we have seen in the just given derivation of the LL order contributions for polarized beams, terms like $\left[\bar{v}_{\rho} \gamma^{l} \gamma_{5} u_{\rho}\right]\left[\bar{u}_{\rho} \gamma^{l} v_{\rho}\right]$ and $\left[\bar{v}_{\rho} \gamma^{l} u_{\rho}\right]\left[\bar{u}_{\rho} \gamma^{l} \gamma_{5} v_{\rho}\right]$ actually do not cancel by spin averaging as in the case of unpolarized beams. Therefore, in a formal treatment of polarization effects, one would have to take into account forward scattering operators of the form

$$
\left[\bar{e}_{-} \gamma^{\mu} \gamma_{5} e_{+}\right]\left[\bar{e}_{+} \gamma_{\mu}\left(\hat{E} / m_{t}\right)^{n} e_{-}\right], \quad\left[\bar{e}_{-} \gamma^{\mu} e_{+}\right]\left[\bar{e}_{+} \gamma_{\mu} \gamma_{5}\left(\hat{E} / m_{t}\right)^{n} e_{-}\right]
$$

in addition to the ones given in Eq. (1.19). This is necessary in particular for the cancellation of ultraviolet divergences related to e.g. $C_{V / A}^{\mathrm{int}}$. Here, we do not aim at a formal treatment and instead want to give a replacement rule that has to be

[^2]applied to the expression of the unpolarized cross section in order to account for polarization effects.

First, we write

$$
\begin{aligned}
C_{V, 12} & =C_{V, 1}^{\mathrm{born}}+C_{V, 1}^{\mathrm{ew}}+i C_{V, 1}^{\mathrm{int}}+\frac{E}{m_{t}} C_{V, 1}^{(1)}+v^{2} C_{V, 2}^{\mathrm{born}} \\
C_{A, 12} & =C_{A, 1}^{\mathrm{born}}+C_{A, 1}^{\mathrm{ew}}+i C_{A, 1}^{\mathrm{int}}+\frac{E}{m_{t}} C_{A, 1}^{(1)}+v^{2} C_{A, 2}^{\mathrm{born}}
\end{aligned}
$$

so that Eq. (1.21) reads

$$
\begin{aligned}
& \sigma_{\mathrm{thr}}=\frac{1}{s} L^{l k} \operatorname{Im}\left[\left(C_{V, 12}(\nu)^{2}+C_{A, 12}(\nu)^{2}\right) \mathcal{A}_{1}^{l k}+\left(C_{V, 3}(\nu)^{2}+C_{A, 3}(\nu)^{2}\right) \mathcal{A}_{3}^{l k}\right] \\
& +\sum_{n}\left(E / m_{t}\right)^{n} \operatorname{Im}\left[\tilde{C}_{V}^{(n)}(\nu ; \Lambda)+\tilde{C}_{A}^{(n)}(\nu ; \Lambda)\right] \\
& +\operatorname{Im}\left[\delta \tilde{C}_{V}(\nu)+\delta \tilde{C}_{A}(\nu)\right] .
\end{aligned}
$$

This equation is true up to NNLL order, if we do not consider (for simplicity) QCD corrections contained in $c_{\sigma}(\nu)$ and phase space corrections contained in $\delta \tilde{c}_{1}(\Lambda), \delta \tilde{c}_{1}^{\text {int }}(\Lambda)$ as given in Eq. (1.17). To obtain the cross section for polarized beams, one now has to substitute the terms in front of $\mathcal{A}_{1}$ and $\mathcal{A}_{3}$ in the first line and similar terms appearing in the combinations $\tilde{C}_{V}^{(n)}+\tilde{C}_{A}^{(n)}$ (see Eq. (3.9)) and $\delta \tilde{C}_{V}+\delta \tilde{C}_{A}$ as

$$
\begin{equation*}
C_{V}^{2}+C_{A}^{2} \xrightarrow{P_{-}, P_{+} \neq 0}\left(1-P_{-} P_{+}\right)\left(C_{V}^{2}+C_{A}^{2}\right)+\left(P_{-}-P_{+}\right) 2 C_{V} C_{A}, \tag{1.33}
\end{equation*}
$$

and the term $C_{V, 1}^{\text {born }} C_{V, 1}^{\mathrm{int}}+C_{A, 1}^{\text {born }} C_{A, 1}^{\mathrm{int}}$ appearing only in the second and third line as

$$
\begin{aligned}
C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}+C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{int}} \xrightarrow{P_{-}, P_{+} \neq 0} & \left(1-P_{-} P_{+}\right)\left(C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}+C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{int}}\right) \\
& +\left(P_{-}-P_{+}\right)\left(C_{V, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{int}}+C_{A, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}\right) .
\end{aligned}
$$

In particular, this substitution ensures that the ultraviolet divergences contained in $\mathcal{A}_{1}, \mathcal{A}_{3}$ and $\delta \tilde{C}_{V}+\delta \tilde{C}_{A}$ cancel.

Since the LL order terms $C_{V, 1}^{\text {born }}$ and $C_{A, 1}^{\text {born }}$ have opposite signs (see Tab. 2.1), we find that the cross section is large for negative electron and positive positron polarization, corresponding to left-handed electrons and right-handed positrons.

### 1.7 Overview of Electroweak Effects up to NNLL Order

In the previous sections we presented QCD effects that arise at LL, NLL and NNLL order of the cross section prediction. We already mentioned some electroweak effects. In this last section of the first chapter we give an overview of
electroweak effects, including QCD interferences, that have to be considered up to NNLL order. The following chapters will then be dedicated for the determination of several of these effects.

We start with the leading order effects mentioned already in the introduction. The QED beam effects, consisting of the machine-dependent beam energy spread, beamstrahlung and initial state radiation are implemented through a luminosity spectrum that is convolved into the theoretical predictions. In a complete treatment of pure QED effects beyond LL order, however, it is necessary to include also NNLL initial and final state effects as well as the QED Coulomb singularities. In this work we are not interested in these pure QED corrections and therefore will exclude them in the following analyses.

The second leading order electroweak effect mentioned in the introduction is the $e^{+} e^{-}$annihilation and $t \bar{t}$ production process mediated by the propagation of the virtual photon and $Z$ boson. It is described by the $\left(e^{+} e^{-}\right)(t \bar{t})$ effective NRQCD operators $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$. A NNLL order correction to this effect originates from the energy dependence of the photon and $Z$ boson propagator and is described by the $E / m_{t}$-suppressed $\left(e^{+} e^{-}\right)(t \bar{t})$ operators $\mathcal{O}_{V / A, \mathbf{p}, 1}^{(1)}$. Another NNLL electroweak correction affecting the production operators stems from real parts of one-loop electroweak diagrams describing the $e^{+} e^{-} \rightarrow t \bar{t}$ process. This usual NNLL electroweak correction ${ }^{3}$ is incorporated in the form of real matching conditions for the operators $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$ and will be determined in Chap. 2.

The third leading order effect stems from the finite lifetime of the top quark and is described by the imaginary matching condition $i \Gamma_{t} / 2$ for the bilinear quark operators. From QCD interferences this matching condition obtains $\mathcal{O}\left(\alpha_{s}\right)$ and $\mathcal{O}\left(\alpha_{s}^{2}\right)$ corrections leading to NLL and NNLL contributions to the cross section, respectively $[81,82]$. In the same way, electroweak $\mathcal{O}(\alpha)$ corrections lead to contributions at NNLL order. At this level, also the $\mathbf{p}^{2} / m_{t}^{2}$-suppressed effect of lifetime dilatation contributes, which was included in the ultrasoft Lagrangian in Eq. (1.3) and is eventually described by the correction $G^{\text {dil }}$ to the LL Green function $G^{0}$ for the cross section prediction.

Concerning instability effects at NNLL order it is not sufficient to take into account only corrections to the top quark width. Because the finite lifetime implies that only the top/antitop decay products are experimentally relevant, it is necessary to consider also those full theory $b W^{+} \bar{b} W^{-}$final state diagrams that lack one top/antitop line (i. e. either $b W^{+}$or $\bar{b} W^{-}$is not connected to an intermediate top/antitop quark). These diagrams give rise to the so-called interference effects, which are $\alpha \sim v^{2}$-suppressed and therefore contribute at NNLL order. They are incorporated into the effective theory as imaginary parts of the production operators $\mathcal{O}_{V / A, \mathbf{p}, 1}$ and will be discussed in Sec. 3.2.

[^3]Because NNLL order NRQCD forward scattering matrix elements contain ultraviolet phase space divergences they lead to a renormalization group running of the Wilson coefficients of the forward scattering operators $\tilde{\mathcal{O}}_{V / A}$ through a mixing effect. It will be analyzed in Sec. 3.3. The mismatch of the effective theory and the full theory phase space requires also a phase space matching procedure, which is carried out in Chap. 4. It is used to determine the hardscale Wilson coefficients of the forward scattering operators and corrections to those of the production currents. These phase space effects give contributions to the cross section at NLL order and beyond. They would not appear if the top quark were treated as a stable particle, and thus also constitute instability effects. Such phase space effects have not been treated in a systematic way in previous work and it has been stated frequently in the literature that at NLL order only $\mathcal{O}\left(\alpha_{s}\right)$ QCD corrections to the $i \Gamma_{t} / 2$ matching conditions for the bilinear quark operators have to be taken into account [83, 84]. However, it was also noted in Ref. [40] that the difference between the effective theory and the full theory phase space must be taken into account at NLL order. The NNLL evolution of the forward scattering operators is obtained from $\mathcal{O}\left(\alpha_{s}\right)$ corrections to the imaginary parts of the matching conditions for the production currents and the quark bilinear operators because these can give rise to ultraviolet divergences at the order $\mathrm{N}^{3} \mathrm{LL}$. It has not been determined yet.

Up to now we discussed the electroweak corrections to the NRQCD bilinear heavy quark operators, the top-antitop production currents as well as the forward scattering operators that can appear up to NNLL order. It remains to consider matching corrections to the potentials and heavy-quark-gluon interactions. It was shown that these do not give contributions to the total cross section at LL, NLL and NNLL order due to gauge cancellations [43, 83, 84]. We emphasize, however, that such cancellations indeed occur for the total cross section but are not expected in a differential consideration of the top-antitop production rate, in particular in our treatment using phase space cuts. Therefore a detailed analysis of contributions to the phase space matching originating from interactions of ultrasoft gluons with the heavy quarks or their decay products will have to be undertaken in another work. A simple estimate for the numerical size of these contributions is given in the outlook in Chap. 6.

## Chapter 2

## Usual Electroweak Effects at NNLL Order

For the determination of the real parts of the NNLL order electroweak matching conditions for the production currents we use the one-loop renormalized electroweak standard model as described in Ref. [85], which relies on the Glashow-Salam-Weinberg model [86-89]. It is based on a local $\operatorname{SU}(2) \times \mathrm{U}(1)$ gauge theory, whose gauge symmetry is spontaneously broken down to a $\mathrm{U}(1)_{\mathrm{em}}$ electromagnetic symmetry through a minimal Higgs mechanism [90-94]. The Higgs mechanism is employed to generate fermion and boson mass terms, since explicit mass terms in the Lagrangian would destroy gauge invariance and therefore renormalizability. The fermionic particle sector contains the spin- $\frac{1}{2}$ leptons $\left(\left(\nu_{e}, e\right),\left(\nu_{\mu}, \mu\right),\left(\nu_{\tau}, \tau\right)\right)$ and quarks $((u, d),(c, s),(t, b))$, which appear in three different generations. Lefthanded fermion field pairs transform as doublets under the $\mathrm{SU}(2)$ symmetry transformation, whereas right-handed fermions transform as singlets. It is assumed that there are no right-handed neutrinos. In the bosonic sector, for each group generator there is a spin-1 gauge boson field that transforms under the adjoint representation of the respective gauge group. After spontaneous symmetry breaking gauge bosons have obtained masses, but the original gauge boson fields are not the mass eigenstates of the theory. The eigenstates are obtained by diagonalizing the mass matrix and therefore by a rotation of the original fields, yielding the $W^{ \pm}$and $Z$ bosons with masses $M_{W}$ and $M_{Z}$, respectively, and a massless photon. Fermion masses, on the other hand, are obtained similarly from Yukawa interaction terms in the Lagrangian. By spontaneously breaking the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge invariance, unphysical Higgs fields arise, which are denoted as $\phi^{ \pm}$and $\chi$. They are no physical degrees of freedom as they can be eliminated by a suitable gauge transformation. They are absent in the so-called unitary gauge. Yet we will use the t' Hooft-Feynman gauge for our calculations since it leads to simpler expressions for the gauge boson propagators and no mixing of gauge fields and the Higgs field occurs. For the quantization of the classical Lagrangian composed of the gauge boson, Higgs and fermion part so-called gauge fixing terms (contain-
ing the gauge boson fields) are added. They lead to unphysical effects, which are compensated by adding Faddeev-Popov ghosts to the Lagrangian [95], denoted as $u_{\alpha}$ and $\bar{u}_{\alpha}(\alpha= \pm, \gamma, Z)$. The latter couple only to the gauge bosons and the physical and unphysical Higgs bosons. In our calculation they will appear only in photon and $Z$ boson self-energy diagrams.

This was a rough overview of the electroweak standard model that is used for the computations carried out in the following sections. For its complete description, including the explicit form of the Lagrangian, the renormalization conditions and the Feynman rules, we refer the reader to Ref. [85]. Still we will partly consider its renormalization conditions in the next section.

### 2.1 Renormalization Schemes

Renormalization of fields and couplings of a quantum field theory is necessary to absorb the ultraviolet divergences arising in loop integrals. Apart from this technical issue it actually completes the definition of a theory through renormalization conditions imposed on its parameters. If two different sets of renormalization conditions (i.e. different renormalization schemes) are used, then observables computed in one or the other scheme will in general differ from each other, except the same definition of the physical parameters is used in both schemes.

Our calculation of electroweak one-loop diagrams is done within the on-shell renormalization scheme used by Böhm, Hollik and Spiesberger (BHS scheme) in Ref. [85]. We carry out another, independent calculation using the automated packages FeynArts [96] and FormCalc [97], which are based on the slightly different on-shell scheme according to Denner [98]. If one sets the CKM matrix in the Denner scheme to the unit matrix, then both schemes use the same physical parameters

$$
\begin{equation*}
e, M_{W}, M_{Z}, M_{H}, m_{f, i} \tag{2.1}
\end{equation*}
$$

Here, $e$ is defined as the strength of the electromagnetic coupling in the Thomson limit and $M_{W}, M_{Z}, M_{H}, m_{f, i}$ are the gauge boson, Higgs and fermion mass parameters, respectively, defined as the real parts of the poles of the corresponding propagators. The photon mass is zero due to the residual $\mathrm{U}(1)_{\mathrm{em}}$ invariance.

Other parameters are treated differently in the two schemes. While in the Denner scheme all particle fields are renormalized such that the residues of the renormalized propagators are one, the BHS scheme uses only the minimal number of field renormalization constants for the benefit that renormalization conserves the gauge transformation properties of the fields and the Green functions. As a consequence, in the Denner scheme all photon and $Z$ boson field renormalization constants $\delta Z_{A A}, \delta Z_{A Z}, \delta Z_{Z A}, \delta Z_{Z Z}$, defined by

$$
\binom{Z_{0}}{A_{0}}=\left(\begin{array}{cc}
1+\frac{1}{2} \delta Z_{Z Z} & \frac{1}{2} \delta Z_{Z A} \\
\frac{1}{2} \delta Z_{A Z} & 1+\frac{1}{2} \delta Z_{A A}
\end{array}\right)\binom{Z}{A}
$$

(where $Z_{0}, A_{0}$ and $Z, A$ denote the unrenormalized and renormalized fields, respectively), have to be calculated independently from each other from self-energy diagrams according to

$$
\begin{array}{cl}
Z_{A A}=-\left.\frac{\partial \Sigma_{T}^{A A}\left(k^{2}\right)}{\partial k^{2}}\right|_{k^{2}=0}, & \delta Z_{Z A}=2 \frac{\Sigma_{T}^{A Z}(0)}{M_{Z}^{2}} \\
\delta Z_{Z Z}=-\left.\operatorname{Re} \frac{\partial \Sigma_{T}^{Z Z}\left(k^{2}\right)}{\partial k^{2}}\right|_{k^{2}=M_{Z}^{2}}, & \delta Z_{A Z}=-2 \operatorname{Re} \frac{\Sigma_{T}^{A Z}\left(M_{Z}^{2}\right)}{M_{Z}^{2}} \tag{2.2}
\end{array}
$$

(with $\Sigma_{T}^{i}\left(k^{2}\right), i=A A, A Z, Z Z, A Z$, being the transverse self-energies at momentum $k^{2}$ ). In the BHS scheme, on the other hand, two of them can be derived from the other two using the $Z$ and $W$ boson mass renormalization constants $\delta M_{Z}^{2}$ and $\delta M_{W}^{2}$, so that the two lower equations in (2.2) obtain a different form:

$$
\begin{align*}
& \delta Z_{A A}=-\left.\frac{\partial \Sigma_{T}^{A A}\left(k^{2}\right)}{\partial k^{2}}\right|_{k^{2}=0}, \quad \delta Z_{Z A}=2 \frac{\Sigma_{T}^{A Z}(0)}{M_{Z}^{2}}, \\
& \delta Z_{Z Z}=\delta Z_{A A}-\frac{c_{w}^{2}-s_{w}^{2}}{c_{w} s_{w}} \delta Z_{Z A}+\frac{c_{w}^{2}-s_{w}^{2}}{s_{w}^{2}}\left[\frac{\delta M_{Z}^{2}}{M_{Z}^{2}}-\frac{\delta M_{W}^{2}}{M_{W}^{2}}\right],  \tag{2.3}\\
& \delta Z_{A Z}=\delta Z_{Z A}-2 \frac{c_{w}}{s_{w}}\left[\frac{\delta M_{Z}^{2}}{M_{Z}^{2}}-\frac{\delta M_{W}^{2}}{M_{W}^{2}}\right]
\end{align*}
$$

where

$$
\delta M_{Z}^{2}=\operatorname{Re} \Sigma_{T}^{Z Z}\left(M_{Z}^{2}\right), \quad \delta M_{W}^{2}=\operatorname{Re} \Sigma_{T}^{W}\left(M_{W}^{2}\right)
$$

The fact that $\delta Z_{Z Z}$ and $\delta Z_{A Z}$ are different in the two schemes will lead to different results of parts of the calculation. For example the $\gamma t \bar{t}$ and $Z t \bar{t}$ vertex counterterms deviate. However, the sum of all electroweak one-loop $e^{+} e^{-} \rightarrow t \bar{t}$ amplitudes does not depend on the field renormalization of the intermediate photon or $Z$ boson. Therefore we finally find the same result for the sum of the amplitudes for the calculation carried out within the BHS scheme on the one hand and the Denner scheme on the other hand. The results derived within the BHS scheme are presented in detail in the next section.

### 2.2 Calculation of One-Loop Diagrams and Comparison to the Literature

In this section we calculate the NNLL order electroweak matching conditions $C_{V / A, 1}^{\mathrm{ew}}$ for the Wilson coefficients of the operators $\mathcal{O}_{V / A, \mathbf{p}, 1}$ appearing in Eqs. (1.15). They are obtained from the one-loop corrections to the $e^{+} e^{-} \rightarrow t \bar{t}$ amplitude for
on-shell external top quarks within the electroweak standard model. Due to the electroweak power counting $\alpha \sim v^{2}$ it is sufficient to consider only the limit $\sqrt{s} \rightarrow 2 m_{t}$ (i.e. $v \rightarrow 0$ ), where the top quarks are at rest. By neglecting pure QED corrections, contributions singular in $v$, which are associated with nontrivial NRQCD matrix elements, do not appear. These pure QED effects stem from full electroweak theory diagrams, where there is a photon but no other gauge boson in the loop (see Fig. C.1).

For the calculation of all the other electroweak one-loop diagrams we use standard techniques such as dimensional regularization [52] and Passarino-Veltman reduction of tensor to scalar integrals [99]. The scalar integrals are of two-point, three-point and four-point type, corresponding to self-energy, triangle and box diagrams. Since top and antitop are at rest and we use the c.m. frame for the calculation, the momenta appearing in the three- and four-point functions are linearly dependent and therefore a further reduction to two- and three-point functions, respectively, is possible, see App. A. For simplicity the electron mass is neglected (hence there is no coupling of the initial state leptons to the Higgs particle) except for the self-energy corrections. The CKM matrix is considered to be the unit matrix and the bottom quark mass is neglected in the $\gamma t \bar{t}$ and $Z t \bar{t}$ vertex corrections (including the top quark field renormalization).

Neglecting pure QED corrections the electroweak standard model amplitude can be written in the form

$$
\begin{equation*}
\mathcal{A}^{\mathrm{ew}}=i \frac{\alpha \pi}{m_{t}^{2}}\left[\bar{v}_{e^{+}}\left(k^{\prime}\right) \gamma^{\mu}\left(h_{R}^{\mathrm{ew}} \omega_{+}+h_{L}^{\mathrm{ew}} \omega_{-}\right) u_{e^{-}}(k)\right]\left[\bar{u}_{t}(p) \gamma_{\mu} v_{\bar{t}}(p)\right], \tag{2.4}
\end{equation*}
$$

where $\omega_{ \pm} \equiv \frac{1}{2}\left(1 \pm \gamma_{5}\right)$ and $k+k^{\prime}=2 p=\left(2 m_{t}, 0\right)$. By matching this to the amplitude obtained from the operators $\mathcal{O}_{V, \mathbf{p}, 1}$ and $\mathcal{O}_{A, \mathbf{p}, 1}$ in Eq. (1.15) we find

$$
\begin{align*}
& C_{V, 1}^{\mathrm{ew}}(\nu=1)=\frac{\alpha \pi}{2 m_{t}^{2}} \operatorname{Re}\left[h_{R}^{\mathrm{ew}}+h_{L}^{\mathrm{ew}}\right], \\
& C_{A, 1}^{\mathrm{ew}}(\nu=1)=\frac{\alpha \pi}{2 m_{t}^{2}} \operatorname{Re}\left[h_{R}^{\mathrm{ew}}-h_{L}^{\mathrm{ew}}\right] . \tag{2.5}
\end{align*}
$$

The real parts of the NNLL electroweak matching conditions $C_{V, 1}^{\mathrm{ew}}$ and $C_{A, 1}^{\mathrm{ew}}$ give rise to a shift of the normalization of the threshold cross section by a factor

$$
\begin{equation*}
\Delta^{\mathrm{ew}}=\frac{\delta \sigma_{\mathrm{thr}}^{\mathrm{ew}}}{\sigma_{\mathrm{thr}}}=\frac{2 C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{ew}}+2 C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{ew}}}{\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\mathrm{borrn}}\right)^{2}} . \tag{2.6}
\end{equation*}
$$

The coefficients $h_{L, R}^{\mathrm{ew}}$ were first computed in Ref. [38] in Feynman gauge using the BHS scheme. We have determined the real parts of the coefficients $h_{L, R}^{\mathrm{ew}}$ in the same gauge and renormalization scheme by hand and, independently, using the automated packages FeynArts [96] and FormCalc [97] (except for the $Z \gamma$ box diagrams). Apart from that we carried out another computation using FeynArts and FormCalc in the Denner scheme. While we find agreement of the results of
all our calculations, we find some discrepancies between our results and the ones given in Ref. [38]. For the presentation of our results we follow for the most part the placement used in Ref. [38].

The results for the coefficients $h_{L, R}^{\mathrm{ew}}$ can be cast into the form

$$
\begin{equation*}
h_{L, R}^{\mathrm{ew}}=h_{L, R}^{\mathrm{SE}}+h_{L, R}^{e+e^{-}}+h_{L, R}^{t \bar{t}}+h_{L, R}^{\mathrm{box}}, \tag{2.7}
\end{equation*}
$$

where

$$
\begin{align*}
h_{L, R}^{\mathrm{SE}}= & Q_{e}\left(-\frac{\Pi_{R}^{A A}}{4 m_{t}^{2}}\right) Q_{t}+\beta_{L, R}^{e} \frac{4 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}}\left(-\frac{\Pi_{R}^{Z Z}}{4 m_{t}^{2}-M_{Z}^{2}}\right) \frac{\beta_{R}^{t}+\beta_{L}^{t}}{2} \\
& -\left(Q_{e} \frac{\beta_{R}^{t}+\beta_{L}^{t}}{2}+\beta_{L, R}^{e} Q_{t}\right) \frac{\Pi_{R}^{Z A}}{4 m_{t}^{2}-M_{Z}^{2}},  \tag{2.8}\\
h_{L, R}^{e^{+}, e^{-}}= & F_{L, R}^{A} Q_{t}+F_{L, R}^{Z} \frac{4 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}} \frac{\beta_{R}^{t}+\beta_{L}^{t}}{2},  \tag{2.9}\\
h_{L, R}^{t \bar{t}}= & Q_{e} \frac{\alpha}{4 \pi} \sum a^{A}+\beta_{L, R}^{e} \frac{4 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}} \frac{\alpha}{4 \pi} \sum a^{Z},  \tag{2.10}\\
h_{L, R}^{\mathrm{box}}= & h_{L, R}^{W W}+h_{L, R}^{Z Z}+h_{L, R}^{Z \gamma} \tag{2.11}
\end{align*}
$$

and

$$
\begin{equation*}
\beta_{R}^{f}=-\frac{s_{w}^{2} Q_{f}}{s_{w} c_{w}}, \quad \beta_{L}^{f}=\frac{t_{3, f}-s_{w}^{2} Q_{f}}{s_{w} c_{w}} \tag{2.12}
\end{equation*}
$$

$t_{3, f}$ and $Q_{f}$ being the third component of the weak isospin and the electric charge of fermion $f$, respectively.

The terms $h_{L, R}^{\mathrm{SE}}$ describe the self-energy corrections corresponding to Figs. C. 2 and C. 3 in the Appendix, where $\Pi_{R}^{i}, i=A A, Z Z, Z A$, are the renormalized transverse photon and $Z$ boson self-energy corrections and the photon- $Z$ mixing correction, respectively. Compared to the expressions given in Ref. [38] our result does not sum the photon and $Z$ boson self-energy corrections into the denominator of the respective propagators since the resulting higher order corrections are beyond NNLL order and because at the $t \bar{t}$ threshold the intermediate photon and $Z$ boson are far off-shell. We obtain

$$
\begin{aligned}
& \Pi_{R}^{A A}=\Sigma_{T}^{A A}\left(4 m_{t}^{2}\right)+4 m_{t}^{2} \delta Z_{A A} \\
& \Pi_{R}^{Z Z}=\Sigma_{T}^{Z Z}\left(4 m_{t}^{2}\right)-\delta M_{Z}^{2}+\left(4 m_{t}^{2}-M_{Z}^{2}\right) \delta Z_{Z Z} \\
& \Pi_{R}^{Z A}=\Sigma_{T}^{A Z}\left(4 m_{t}^{2}\right)+\left(4 m_{t}^{2}-M_{Z}^{2}\right) \frac{1}{2} \delta Z_{Z A}+4 m_{t}^{2} \frac{1}{2} \delta Z_{A Z}
\end{aligned}
$$

The photon and $Z$ boson field renormalization constants are given in Eqs. (2.3) and $\Sigma_{T}^{i}\left(k^{2}\right)$ are the well-known transverse self-energy functions computed already
in Ref. [99, 100]. Since we are in agreement with Ref. [38] with respect to the $h_{L, R}^{\mathrm{SE}}$ part of the calculation, we abstain from giving the explicit expressions for $\Sigma_{T}^{i}\left(k^{2}\right)$ and refer the reader to Ref. [38].

The terms $h_{L, R}^{e^{+} e^{-}}$describe the corrections to the $e^{+} e^{-}$vertex corresponding to Fig. C.5, where $F_{L, R}^{i}, i=A, Z$, are the vertex corrections to the left- and righthanded $e^{+} e^{-} \gamma$ and $e^{+} e^{-} Z$ vertices. They include the respective counterterms for electron/positron and gauge boson wave function renormalization as well as for charge renormalization and are therefore ultraviolet finite. We find

$$
\begin{align*}
& F_{R}=\frac{\alpha}{4 \pi}\left[\binom{Q_{e}}{\beta_{R}^{e}}\left(\beta_{R}^{e}\right)^{2} \rho\right],  \tag{2.13}\\
& F_{L}=\frac{\alpha}{4 \pi}\left[\binom{Q_{e}}{\beta_{L}^{e}}\left(\beta_{L}^{e}\right)^{2} \rho+\binom{0}{\beta_{L}^{e}-2 t_{3}^{e} \frac{c_{w}}{s_{w}}} \frac{1}{2 s_{w}^{2}} \rho-\binom{1}{\frac{c_{w}}{s_{w}}} \frac{t_{3}^{e}}{s_{w}^{2}} \Lambda\right] \tag{2.14}
\end{align*}
$$

with

$$
\begin{align*}
& \rho=\frac{1}{2 m_{t}^{2}}\left\{-m_{t}^{2}+\left(6 m_{t}^{2}+M_{W}^{2}\right)\left(B_{0}\left(0, M_{W}^{2}, 0\right)-B_{0}\left(4 m_{t}^{2}, 0,0\right)\right)\right. \\
&\left.\quad-\left(4 m_{t}^{2}+M_{W}^{2}\right)^{2} C_{0}\left(4 m_{t}^{2}, 0,0,0,0, M_{W}^{2}\right)\right\}  \tag{2.15}\\
& \Lambda=\frac{1}{2 m_{t}^{2}}\left\{-3 m_{t}^{2}-\left(6 m_{t}^{2}+M_{W}^{2}\right) B_{0}\left(0, M_{W}^{2}, 0\right)+4 m_{t}^{2} B_{0}\left(0, M_{W}^{2}, M_{W}^{2}\right)\right. \\
&+\left(2 m_{t}^{2}+M_{W}^{2}\right) B_{0}\left(4 m_{t}^{2}, M_{W}^{2}, M_{W}^{2}\right) \\
&\left.\quad-M_{W}^{2}\left(8 m_{t}^{2}+M_{W}^{2}\right) C_{0}\left(0,4 m_{t}^{2}, 0,0, M_{W}^{2}, M_{W}^{2}\right)\right\} \tag{2.16}
\end{align*}
$$

The functions $B_{0}$ and $C_{0}$ are the scalar two- and three-point function, respectively. They are defined and expressed in terms of generalized logarithmic functions in App. A. Using the results of App. A, we obtain

$$
\Lambda=-\frac{5}{2}+\frac{2}{u}-\left(1+\frac{2}{u}\right) l \sqrt{1-\frac{4}{u}}-\left(1+\frac{1}{2 u}\right) \frac{4}{u} l^{2}
$$

where

$$
\begin{equation*}
l \equiv \ln \frac{\sqrt{1-4 / u}+1+i \epsilon}{\sqrt{1-4 / u}-1+i \epsilon}=\ln \frac{1+\sqrt{1-4 / u}}{1-\sqrt{1-4 / u}}-i \pi, \quad u \equiv \frac{4 m_{t}^{2}}{M_{W}^{2}} . \tag{2.17}
\end{equation*}
$$

This deviates from the expression given in Ref. [38]. For $\Delta^{\text {ew }}$, defined in Eq. (2.6), it leads to an absolute shift by +0.012 with respect to the results in Ref. [38].

The terms $h_{L, R}^{t \bar{t}}$ refer to the corrections to the $t \bar{t}$ vertices where

$$
\begin{align*}
\sum a^{A, Z} \equiv & a^{A, Z}(W)+a^{A, Z}(W, W)+a_{\mathrm{twr}}^{A, Z}(W)+a_{\mathrm{bwchr}} \\
& +a^{A, Z}(\phi)+a^{A, Z}(\phi, \phi)+a_{\mathrm{twr}}^{A, Z}(\phi)+a^{A, Z}(W, \phi) \\
& +a^{A, Z}(Z)+a_{\mathrm{twr}}^{A, Z}(Z) \\
& +a^{A, Z}(\chi)+a_{\mathrm{twr}}^{A, Z}(\chi) \\
& +a^{A, Z}(H)+a_{\mathrm{twr}}^{A, Z}(H) \\
& +a^{A, Z}(Z, H)+a^{A, Z}(\chi, H) . \tag{2.18}
\end{align*}
$$

The coefficients $a^{A, Z}$ correspond to the corrections to the $\gamma t \bar{t}$ and $Z t \bar{t}$ vertices, respectively. The arguments of the various terms in Eq. (2.18) indicate the virtual bosons that are being exchanged in the various triangle diagrams, e.g. $a^{A, Z}(W, W)$ refers to the vertex diagrams with exactly two internal $W$ lines and $a^{A, Z}(\chi)$ to those where the only internal boson is the neutral unphysical Higgs boson. We note that $a^{A, Z}(W, \phi), a^{A, Z}(\chi, H)$ and $a^{A}(Z, H)$ are identically zero in Feynman gauge. The subscript 'twr' corresponds to counterterms for the top/antitop wave function renormalization, the subscript 'bwchr' to the counterterm for gauge boson wave function plus charge renormalization. We note that the sum of the terms appearing in each line on the RHS of Eq. (2.18) is ultraviolet finite. Because the explicit expressions we find are rather lengthy, we do not give them in this section but refer the reader to App. B. Our result for $a(H)$ is consistent with Refs. [39, 101], but differs from the one given in Ref. [38]. Our result for $a_{\text {twr }}(Z)$ is consistent with Ref. [39], but differs from Ref. [38]. For $\Delta^{\text {ew }}$ these changes lead to an absolute shift by +0.076 with respect to the results in Ref. [38] for $M_{H}=130 \mathrm{GeV}$.

Finally, the terms $h_{L, R}^{\text {box }}$ describe the contributions from the $W W, Z Z$ and $Z \gamma$ box diagrams drawn in Fig. C.6. We note that the $Z \gamma$ box diagrams are infrared-finite for the $t \bar{t}$ pair being at rest. We find

$$
\begin{align*}
h_{R}^{W W} & =0 \\
h_{L}^{W W} & =\frac{\alpha}{4 \pi}\left(-\frac{1}{2 s_{w}^{4}}\right) F^{W W} \\
h_{R}^{Z Z} & =\frac{\alpha}{4 \pi}\left(-\left(\beta_{R}^{e}\right)^{2}\right)\left(\left(\beta_{R}^{t}\right)^{2}-\left(\beta_{L}^{t}\right)^{2}\right) F^{Z Z} \\
h_{L}^{Z Z} & =\frac{\alpha}{4 \pi}\left(\beta_{L}^{e}\right)^{2}\left(\left(\beta_{R}^{t}\right)^{2}-\left(\beta_{L}^{t}\right)^{2}\right) F^{Z Z}, \\
h_{R}^{Z \gamma} & =\frac{\alpha}{4 \pi}\left(-\beta_{R}^{e}\right)\left(\beta_{R}^{t}-\beta_{L}^{t}\right) Q_{e} Q_{t} 2 F^{Z \gamma}  \tag{2.19}\\
h_{L}^{Z \gamma} & =\frac{\alpha}{4 \pi}\left(\beta_{L}^{e}\right)\left(\beta_{R}^{t}-\beta_{L}^{t}\right) Q_{e} Q_{t} 2 F^{Z \gamma} \tag{2.20}
\end{align*}
$$

where

$$
\begin{align*}
F^{W W}= & \frac{2 m_{t}^{2}}{m_{t}^{2}}-M_{W}^{2}\left\{B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)-B_{0}\left(4 m_{t}^{2}, M_{W}^{2}, M_{W}^{2}\right)\right. \\
& \quad-M_{W}^{2} C_{0}\left(0,4 m_{t}^{2}, 0,0, M_{W}^{2}, M_{W}^{2}\right) \\
& \left.+m_{t}^{2} C_{0}\left(-m_{t}^{2}, 0, m_{t}^{2}, 0,0, M_{W}^{2}\right)\right\}  \tag{2.21}\\
F^{Z Z}= & \frac{1}{2 m_{t}^{2}-M_{Z}^{2}}\left\{4 m_{t}^{2} B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)-\left(2 m_{t}^{2}+M_{Z}^{2}\right) B_{0}\left(4 m_{t}^{2}, M_{Z}^{2}, M_{Z}^{2}\right)\right. \\
& \quad-\left(2 m_{t}^{2}-M_{Z}^{2}\right) B_{0}\left(-m_{t}^{2}, 0, m_{t}^{2}\right) \\
& \quad-M_{Z}^{2}\left(4 m_{t}^{2}-M_{Z}^{2}\right) C_{0}\left(0,4 m_{t}^{2}, 0,0, M_{Z}^{2}, M_{Z}^{2}\right) \\
& \left.+\left(8 m_{t}^{4}-4 m_{t}^{2} M_{Z}^{2}+M_{Z}^{4}\right) C_{0}\left(-m_{t}^{2}, m_{t}^{2}, 0,0, m_{t}^{2}, M_{Z}^{2}\right)\right\}  \tag{2.22}\\
F^{Z \gamma}= & \frac{1}{4} \begin{aligned}
m_{t}^{2}-M_{Z}^{2}
\end{aligned} 4 m_{t}^{2} B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)-\left(4 m_{t}^{2}+M_{Z}^{2}\right) B_{0}\left(4 m_{t}^{2}, M_{Z}^{2}, 0\right) \\
& +4 m_{t}^{2} B_{0}\left(m_{t}^{2}, m_{t}^{2}, 0\right)-\left(4 m_{t}^{2}-M_{Z}^{2}\right) B_{0}\left(-m_{t}^{2}, m_{t}^{2}, 0\right) \\
& \left.+\left(4 m_{t}^{2}-M_{Z}^{2}\right)^{2} C_{0}\left(-m_{t}^{2}, m_{t}^{2}, 0,0, m_{t}^{2}, M_{Z}^{2}\right)\right\} . \tag{2.23}
\end{align*}
$$

Our expression for $F^{Z \gamma}$ agrees with the one in Ref. [102], but differs from the one in Ref. [38]. We note that the same error is also contained in the analysis of Ref. [39]. For $\Delta^{\text {ew }}$ the changes lead to an absolute shift by +0.004 with respect to the result in Ref. [38].

Comparing our results to those of Ref. [39] we also find discrepancies. A detailed comparison can be found in Ref. [45].

## $2.3 \overline{\mathrm{MS}}$ Definition of Wilson Coefficients

By means of a one-loop renormalization group equation for the electromagnetic coupling within the $\overline{\mathrm{MS}}$ scheme we define the QED coupling at the scale $\mu=m_{t}$ as

$$
\begin{equation*}
\alpha^{n_{f}=8}(\mu)=\frac{\alpha}{1-\frac{\alpha}{3 \pi} \sum_{i=e, \mu, \tau} Q_{i}^{2} \ln \left(\frac{\mu^{2}}{m_{i}^{2}}\right)-\frac{\alpha}{3 \pi} \sum_{i=u, d, c, s, b} N_{c} Q_{i}^{2} \ln \left(\frac{\mu^{2}}{m_{i}^{2}}\right)}, \tag{2.24}
\end{equation*}
$$

where $\alpha=1 / 137.036$ is the fine structure constant and $N_{c}=3$ the number of colors. This definition absorbs the leading logarithmic vacuum polarization effects due to the three charged leptons and the quarks below the top quark scale. In this scheme the real NNLL electroweak matching conditions are modified according to

$$
\begin{equation*}
C_{V / A}^{\mathrm{ew}, \overline{\mathrm{MS}}}=C_{V / A}^{\mathrm{ew}}-C_{V / A}^{\mathrm{born}} \frac{\alpha^{n_{f}=8}\left(m_{t}\right)}{3 \pi} L\left(m_{t}\right) \tag{2.25}
\end{equation*}
$$

where

$$
L(\mu)=\sum_{i=e, \mu, \tau} Q_{i}^{2} \ln \left(\frac{\mu^{2}}{m_{i}^{2}}\right)+\sum_{i=u, d, c, s, b} N_{c} Q_{i}^{2} \ln \left(\frac{\mu^{2}}{m_{i}^{2}}\right) .
$$

We obtain for the correction to the threshold cross section in the unpolarized case

$$
\begin{align*}
\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}} & \equiv \frac{2 C_{V}^{\mathrm{born}} C_{V}^{\mathrm{ew}, \overline{\mathrm{MS}}}+2 C_{A}^{\mathrm{born}} C_{A}^{\mathrm{ew}, \overline{\mathrm{MS}}}}{\left(C_{V}^{\mathrm{born}}\right)^{2}+\left(C_{A}^{\mathrm{born}}\right)^{2}} \\
& =\Delta^{\mathrm{ew}}-2 \frac{\alpha^{n_{f}=8}\left(m_{t}\right)}{3 \pi} L\left(m_{t}\right) . \tag{2.26}
\end{align*}
$$

The use of the $\overline{\mathrm{MS}}$ scheme has the advantage that the dependence on the light fermion masses is substantially reduced for $\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}$ compared to $\Delta^{\mathrm{ew}}$. This is because most of the fermionic vacuum polarization effects are cancelled by the modification in Eq. (2.25). Moreover in this scheme $\Delta^{\mathrm{ew}}$ is by $17 \%$ smaller than in the one with the fine structure constant.

For the sum of the leading contribution $\sigma_{\text {thr }}$ and the electroweak correction $\Delta \sigma_{\text {thr }}^{\text {ew }}$ to the threshold cross section we have (note that $\sigma_{\text {thr }}$ contains a factor of $\alpha^{2}$ )

$$
\begin{align*}
\sigma_{\mathrm{thr}}+\Delta \sigma_{\mathrm{thr}}^{\mathrm{ew}} & =\sigma_{\mathrm{thr}}\left(1+\Delta^{\mathrm{ew}}\right) \\
& =\sigma_{\mathrm{thr}}\left(1+2 \frac{\alpha^{n_{f}=8}\left(m_{t}\right)}{3 \pi} L\left(m_{t}\right)\right)\left(1+\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}\right)+\mathcal{O}\left(\alpha^{4}\right) \\
& =\sigma_{\mathrm{thr}}\left(\frac{\alpha^{n_{f}=8}\left(m_{t}\right)}{\alpha}\right)^{2}\left(1+\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}\right)+\mathcal{O}\left(\alpha^{4}\right) \tag{2.27}
\end{align*}
$$

While the leading logarithmic vacuum polarization effects are contained in $\Delta^{\text {ew }}$ in the first line of Eqs. (2.27), they have been subtracted from $\Delta^{\text {ew, MS }}$ in the last line according to Eq. (2.26) and shifted to $\sigma_{\text {thr }}$ by the use of $\alpha^{n_{f}=8}\left(m_{t}\right)$ instead of the fine structure constant $\alpha$.

### 2.4 Numerical Analysis

In this section we give a brief numerical discussion of the real parts of the NNLL electroweak matching conditions obtained in this work. In Tab. 2.1 the numerical values for $C_{V / A}^{\mathrm{born}}$ and $C_{V / A}^{\mathrm{ew}}$ are displayed for various values for the top and the Higgs masses, $\alpha^{-1}=137.036$ and

$$
\begin{array}{lll}
M_{W}=80.425 \mathrm{GeV}, & M_{Z}=91.1876 \mathrm{GeV}, & c_{w}^{2}=M_{W}^{2} / M_{Z}^{2} \\
m_{e}=0.511 \mathrm{MeV}, & m_{\mu}=0.106 \mathrm{GeV}, & m_{\tau}=1.78 \mathrm{GeV}, \\
m_{u}=0.005 \mathrm{GeV}, & m_{d}=0.005 \mathrm{GeV}, & m_{s}=0.10 \mathrm{GeV},  \tag{2.28}\\
m_{c}=1.3 \mathrm{GeV}, & m_{b}=4.2 \mathrm{GeV} &
\end{array}
$$

| $m_{t}(\mathrm{GeV})$ | 170 |  |  |  |
| :--- | ---: | ---: | ---: | ---: |
| $M_{H}(\mathrm{GeV})$ | 115 | 150 | 200 | 1000 |
| $C_{V}^{\text {born }}\left(10^{-7} \mathrm{GeV}^{-2}\right)$ | -5.429 |  |  |  |
| $C_{A}^{\text {born }}\left(10^{-7} \mathrm{GeV}^{-2}\right)$ | 1.260 |  |  |  |
| $C_{V}^{\text {ew }}\left(10^{-8} \mathrm{GeV}^{-2}\right)$ | -4.562 | -4.073 | -3.702 | -3.060 |
| $C_{A}^{\text {ew }}\left(10^{-9} \mathrm{GeV}^{-2}\right)$ | -1.416 | -2.286 | -2.797 | -3.025 |
| $C_{V}^{C_{V}^{\text {ew }} / C_{V}^{\text {born }}}$ | 0.0840 | 0.0750 | 0.0682 | 0.0564 |
| $C_{A}^{\text {ew }} / C_{A}^{\text {born }}$ | -0.0112 | -0.0181 | -0.0222 | -0.0240 |
| $m_{t}(\mathrm{GeV})$ | 175 |  |  |  |
| $M_{H}(\mathrm{GeV})$ | 115 | 150 | 200 | 1000 |
| $C_{V}^{\text {born }}\left(10^{-7} \mathrm{GeV}^{-2}\right)$ | -5.123 |  |  |  |
| $C_{A}^{\text {born }}\left(10^{-7} \mathrm{GeV}^{-2}\right)$ | 1.184 |  |  |  |
| $C_{V}^{\text {ew }}\left(10^{-8} \mathrm{GeV}^{-2}\right)$ | -4.460 | -3.951 | -3.566 | -2.890 |
| $C_{A}^{\text {ew }}\left(10^{-9} \mathrm{GeV}^{-2}\right)$ | -1.408 | -2.335 | -2.904 | -3.260 |
| $C_{V}^{C_{V}^{\text {ev }} / C_{V}^{\text {born }}}$ | 0.0871 | 0.0771 | 0.0696 | 0.0564 |
| $C_{A}^{C^{\text {ew }} / C_{A}^{\text {born }}}$ | -0.0119 | -0.0197 | -0.0245 | -0.0275 |

Table 2.1: Numerical values for the tree level and real one-loop electroweak matching conditions $C_{V / A}^{\mathrm{born}}$ and $C_{V / A}^{\mathrm{ew}}$, respectively, and the $C_{V / A}^{\mathrm{ew}} / C_{V / A}^{\mathrm{born}}$ ratios for various values for the top and the Higgs masses, $\alpha^{-1}=137.036$ and the values given in Eq. (2.28). The coefficients $C_{V / A}^{\text {born }}$ do not depend on the Higgs mass.
for the gauge boson, the lepton and the quark masses. Note that the finite electron mass is applied in the calculation only for the self-energy corrections.

The vector coefficients dominate for the tree level as well as for the one-loop coefficients. The one-loop corrections show a significant Higgs mass dependence and vary in the vector (axial-vector) case between $8.6 \%(-1.2 \%)$ and $5.6 \%(-2.6 \%)$ for Higgs masses between 115 GeV and 1 TeV and $m_{t}=172.5 \mathrm{GeV}$.

A more transparent view on the impact of the real electroweak one-loop corrections on the predictions of the $t \bar{t}$ threshold cross section can be gained by considering the quantities $\Delta^{\mathrm{ew}}$ and $\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}$ defined in Eqs. (2.6) and (2.26), respectively, where for $\Delta^{\text {ew }}$ the fine structure constant $\alpha$ and for $\Delta^{\text {ew, MS }}$ the value $\alpha^{n_{f}=8}\left(m_{t}\right)=$ $1 / 125.926$ according to Eq. (2.24) is used. For $M_{H}=(115,150,200,1000) \mathrm{GeV}$ we have $\Delta^{\mathrm{ew}}=(0.161,0.142,0.128,0.104)$. As mentioned before, $\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}$ does not contain the LL polarization effects from light fermions and is therefore smaller by $17 \%$ in each case.

In Fig. 2.1 the dashed line represents $\Delta^{\mathrm{ew}}$ and the solid line $\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}$, given as functions of the Higgs mass for $m_{t}=172.5 \mathrm{GeV}$ and adopting the previous choices for the other parameters. The Higgs mass dependence is rather strong for small $M_{H}$ and drops quickly close to the decoupling limit for increasing $M_{H}$.


Figure 2.1: The correction $\Delta^{\text {ew }}$ in the scheme where the fine structure constant is used (dashed curve) and $\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}}$ in the scheme for the electromagnetic coupling defined in Eq. (2.24) (solid curve) as a function of $M_{H}$ for $m_{t}=172.5 \mathrm{GeV}$ and the values given in Eq. (2.28) for $\alpha=1 / 137.036$ and $\alpha^{n_{f}=8}\left(m_{t}\right)=1 / 125.926$, respectively.

The strong Higgs mass dependence for small Higgs masses is related to the fact that for a light Higgs boson the dominant effects of the virtual Higgs exchange between the $t \bar{t}$ pair could be described in NRQCD by a Yukawa potential that leads to a singularity $\propto m_{t} / M_{H}$ for $M_{H} \rightarrow 0$ (see e.g. Refs. [39, 79, 101, 103]). In an approach where the Higgs exchange is taken into account in NRQCD by a Yukawa potential the electroweak one-loop matching conditions would need to be modified. If the Higgs boson is indeed very close to the lower LEP bound this would be a viable alternative to our approach where all virtual electroweak effects are encoded in the NRQCD Wilson coefficients.

## Chapter 3

## Instability and Interference Effects

The idea of our effective theory treatment of unstable particles is to include the dynamical effects of the decay products present in the underlying theory in the coefficients (associated with the unstable particles) of the effective theory, in this way eliminating the dynamical degrees of freedom of the decay products. This is possible in those cases where the decay itself is an effect taking place at a scale well above the dynamical scales (i.e. scales of the dynamical degrees of freedom) of the effective theory. In the case of NRQCD this is true since top decay represents an effect acting at the scale $m_{t}$. A characteristic feature of our description is that Wilson coefficients and anomalous dimensions of the theory obtain imaginary parts that are associated with the instability effects. This is in analogy to the treatment of absorptive processes in the optical theory. If one is not interested in the details of the absorption mechanism, it is sufficient to write imaginary or "absorptive" parts into the coefficients appearing in the field equations (e.g. Maxwell's equations) to describe absorption effects. Here, the numerical size of the absorptive parts defines what effects of the underlying theory are described as absorption processes by the effective theory, thus defining the "effective absorption process." In our NRQCD treatment the Wilson coefficients will therefore obtain a dependence on an external (experimental) parameter.

The decay $t \rightarrow b W^{+}$and $\bar{t} \rightarrow \bar{b} W^{-}$in a restricted sense is not the only source of absorptive NRQCD matching conditions. Since it is necessary to consider the $b W^{+} \bar{b} W^{-}$final state ${ }^{1}$ in the underlying theory, apart from the literal decay channel $e^{+} e^{-} \rightarrow t \bar{t} \rightarrow b W^{+} \bar{b} W^{-}$shown in Fig. 3.1 (a) also those channels exist, where there is only a single intermediate top or antitop, $e^{+} e^{-} \rightarrow t+\bar{b} W^{-} \rightarrow$ $b W^{+} \bar{b} W^{-}$and $e^{+} e^{-} \rightarrow b W^{+} \bar{t} \rightarrow b W^{+} \bar{b} W^{-}$, shown in Figs. 3.1 (b) - (i). Diagrams

[^4]
(a)

(d)

(g)

(b)

(e)

(c)

(f)

(i)

Figure 3.1: Full theory Feynman diagrams describing the process $e^{+} e^{-} \rightarrow$ $b W^{+} \bar{b} W^{-}$with one or two intermediate top or antitop quark propagators. The circle in diagram (a) represents the QCD form factors for the $t \bar{t}$ vector and axialvector currents.
of that kind lead to $v^{2}$-suppressed contributions in the effective theory since the second resonant top quark propagator of the literal decay channel is missing. ${ }^{2}$ The interference of the double-resonant and single-resonant diagrams are also related to absorptive matching conditions referred to as interference effects, which contribute to the cross section as NNLL order corrections. They will be considered in Sec. 3.2.

Together with the subleading instability effects, these NNLL order corrections require a special treatment since they introduce ultraviolet phase space divergences not present at the LL level. In Sec. 3.3 it will be shown that due to a mixing effect they correspond to phase space logarithms which can be summed through renormalization group methods, yielding a NLL order contribution to the cross section. For the determination of the initial conditions of the renormalization group running we will develop the phase space matching procedure

[^5]

Figure 3.2: The one-loop contribution to the top self-energy induced by the $W$ boson and the bottom quark.
within NRQCD in Chap. 4.

### 3.1 Top Quark Decay at LL Order

We consider the total rate $\Gamma_{t}$ of the process $t \rightarrow W b$ where the top quark is at rest. Due to the optical theorem, $\Gamma_{t}$ is related to the imaginary part of the top quark self-energy contributions induced by the $W$ boson and the bottom quark. In Fig. 3.2 the electroweak one-loop contribution is shown. Since the energy scale of the loop integration is set by the top mass, the decay constitutes a hard effect for the effective theory. It is incorporated into NRQCD by matching the full theory diagram in Fig. 3.2 (and the according for the antitop) to the corresponding effective theory diagram. The latter is constructed from heavy quark bilinear operators $\psi_{\mathbf{p}}^{\dagger} \psi_{\mathbf{p}}$ (and $\chi_{\mathbf{p}}^{\dagger} \chi_{\mathbf{p}}$ ). This matching, obeying the velocity power counting, leads to the effective bilinear Lagrangian

$$
\begin{equation*}
\delta \mathcal{L}_{\text {bill,lead }}=\sum_{\mathbf{p}} \psi_{\mathbf{p}}^{\dagger}\left\{i D^{0}-\frac{(\mathbf{p}-i \mathbf{D})^{2}}{2 m_{t}}+\frac{i}{2} \Gamma_{t}\right\} \psi_{\mathbf{p}}+\left(\psi_{\mathbf{p}} \rightarrow \chi_{\mathbf{p}}\right), \tag{3.1}
\end{equation*}
$$

where the first two terms are obtained by tree-level matching and ultrasoft gauge invariance and we assumed that the real part of the self-energy is absorbed by a redefinition of the top mass using a pole mass description. The $\mathcal{O}(\alpha)$ top decay width $\Gamma_{t}$ reads

$$
\Gamma_{t}=\frac{\alpha\left|V_{t b}\right|^{2} m_{t}}{16 s_{w}^{2} x}(1-x)^{2}(1+2 x),
$$

where we set $V_{t b}=1$ and

$$
x \equiv M_{W}^{2} / m_{t}^{2} .
$$

According to Eqs. $(1.2,1.9)$ the counting $D^{0} \sim \mathbf{p}^{2} / m_{t} \sim \Gamma_{t} \sim m_{t} v^{2}$ implies that the full theory one-loop result $i \Gamma_{t}$ contributes already at leading order in the effective theory. Apart from the residual mass term $\delta m_{t}$ arising in a threshold mass scheme, the terms in Eq. (3.1) constitute the complete LL piece of the bilinear

Lagrangian, which is part of the ultrasoft Lagrangian, Eq. (1.3). Therefore we obtain as propagator of top or antitop (omitting $\delta m_{t}$ )

$$
\begin{equation*}
\frac{i}{p^{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} . \tag{3.2}
\end{equation*}
$$

The exchange of time-like $A^{0}$ gluons, which is possible due to $D^{0}=\partial^{0}+i g_{U} A^{0}$, contributes parametrically at LL order. However, their contribution at LL order can be removed from the quark-antiquark sector of the theory by a redefinition of the top and antitop fields $[104,105]$.

The presence of the $i \Gamma_{t}$ term effectively accounts for a shift of the kinetic energy into the complex plane according to the replacement $E \rightarrow E+i \Gamma_{t}[36]$. On this note one introduces the effective velocity

$$
\begin{equation*}
v=\sqrt{\frac{E+i \Gamma_{t}}{m_{t}}} \tag{3.3}
\end{equation*}
$$

The application of the $E \rightarrow E+i \Gamma_{t}$ rule to the zero-distance Green functions describing the non-relativistic $t \bar{t}$ dynamics is sufficient to account for all LL top quark decay effects. However, as we dill discuss in detail in Chap. 4, the $i \Gamma_{t}$ term renders the effective theory phase space infinite. We will show that the correct non-relativistic phase space is obtained by imposing additional matching conditions on the NRQCD Lagrangian.

### 3.2 Interference Effects at NNLL order

In Chap. 2 the NNLL order matching conditions $C_{V, 1}^{\mathrm{ew}}(\nu=1)$ and $C_{A, 1}^{\mathrm{ew}}(\nu=1)$ for the real parts of the Wilson coefficients of the dominant ${ }^{3} S_{1}$ currents have been determined from the real parts of one-loop electroweak diagrams. In this section we are interested in imaginary parts of those diagrams which are related according to the Cutkosky equations to cuts that put intermediate particles on their mass shells. Since we are interested in the $b W^{+} \bar{b} W^{-}$final state, we consider only cuts through $b$ and $W$ lines (and $\phi$ lines in the case of Feynman gauge), shown in Fig. 3.3. We note that by including graph (d) we correctly take into account the subleading $p^{2} / m_{t}^{2}$ contribution to the top decay rate $\Gamma_{t}$. This term is important to ensure electroweak gauge invariance at NNLL order. The associated imaginary matching conditions $C_{V, 1}^{\mathrm{int}}(\nu=1)$ and $C_{A, 1}^{\mathrm{int}}(\nu=1)$ will arise in the determination of the forward scattering amplitude within the effective theory and therefore in the cross section and reproduce the interference of full theory double- and singleresonant diagrams in the non-relativistic expansion.

The cuts shown in Fig. 3.3 correspond to the full theory amplitude

$$
\begin{equation*}
\mathcal{A}^{\mathrm{int}}=i\left[\bar{v}_{e^{+}}\left(k^{\prime}\right) \gamma^{\mu}\left(i C_{V, 1}^{\mathrm{int}}+i C_{A, 1}^{\mathrm{int}} \gamma_{5}\right) u_{e^{-}}(k)\right]\left[\bar{u}_{t}(p) \gamma_{\mu} v_{\bar{t}}(p)\right], \tag{3.4}
\end{equation*}
$$


(a)

(d)

(g)

(b)

(e)

(h)

(c)

(f)

(i)

Figure 3.3: Full theory diagrams in Feynman gauge that have to be considered to determine the electroweak absorptive parts in the Wilson coefficients $C_{V, 1}$ and $C_{A, 1}$ related to the physical $b W^{+}$and $\bar{b} W^{-}$intermediate states. Only the $b W^{+}$ cut is drawn explicitly. The symbol $\phi_{W} \equiv \phi$ refers to the charged unphysical Higgs boson.
where $k+k^{\prime}=2 p=\left(2 m_{t}, 0\right)$. As for the real parts discussed in Chap. 2 it is sufficient to consider on-shell tops at rest. Using the cutting rules, we find

$$
\begin{aligned}
i C_{V, 1}^{\mathrm{int}}= & -i \frac{\alpha^{2} \pi\left|V_{t b}\right|^{2}}{12 m_{t}^{2} s_{w}^{2} x\left(4 c_{w}^{2}-x\right)(1+x)}\left[\frac{3 x(1+x)}{(1-x)}\left(1+\frac{x-4}{4 s_{w}^{2}}\right) \ln \left(\frac{2-x}{x}\right)\right. \\
& +Q_{e} Q_{t}(1-x)(4-x)(1+2 x)\left(1+x+x^{2}\right) \\
& +Q_{e}(x-1)\left(1+4 x+2 x^{2}+2 x^{3}\right)+Q_{t}(1-x)(1+2 x)\left(1+x+x^{2}\right) \\
& \left.-\frac{1}{2}\left(1+12 x+9 x^{2}+2 x^{3}\right)+\frac{1}{8 s_{w}^{2}}\left(2+41 x+28 x^{2}-x^{3}+2 x^{4}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
i C_{A, 1}^{\mathrm{int}}= & i \frac{\alpha^{2} \pi\left|V_{t b}\right|^{2}}{12 m_{t}^{2} s_{w}^{2} x\left(4 c_{w}^{2}-x\right)(1+x)}\left[\frac{3 x(1+x)}{(1-x)}\left(1+\frac{x-4}{4 s_{w}^{2}}\right) \ln \left(\frac{2-x}{x}\right)\right. \\
& +Q_{t}(1-x)(1+2 x)\left(1+x+x^{2}\right) \\
& \left.-\frac{1}{2}\left(1+12 x+9 x^{2}+2 x^{3}\right)+\frac{1}{8 s_{w}^{2}}\left(2+41 x+28 x^{2}-x^{3}+2 x^{4}\right)\right] .
\end{aligned}
$$

These results are consistent with earlier work [38, 39].
The amplitude for the charge conjugated process describing $t \bar{t}$ annihilation reads

$$
\begin{equation*}
\overline{\mathcal{A}}^{\mathrm{int}}=i\left[\bar{u}_{e^{-}}(k) \gamma^{\mu}\left(i C_{V, 1}^{\mathrm{int}}+i C_{A, 1}^{\mathrm{int}} \gamma_{5}\right) v_{e^{+}}\left(k^{\prime}\right)\right]\left[\bar{v}_{\bar{t}}(p) \gamma_{\mu} u_{t}(p)\right] . \tag{3.5}
\end{equation*}
$$

By a calculation in both unitary and Feynman gauge it was checked that the diagrams in Fig. 3.3 form a gauge invariant set. In particular it was shown that the contributions arising from off-shell corrections in the top self-energy graphs are necessary for electroweak gauge invariance.

Eqs. $(3.4,3.5)$ are the matching conditions for the $t \bar{t}$ production and annihilation operators $\mathcal{O}_{V / A, \mathbf{p}, 1}$ and $\mathcal{O}_{V / A, \mathbf{p}, 1}^{\dagger}$ in Eq. (1.15). The fact that the sign of the imaginary part of the amplitude (3.4) does not change when going to the charge conjugated amplitude (3.5) is a consequence of unitarity of the full electroweak theory. It leads to the fact that the production and annihilation operators have the same imaginary matching conditions. The same mechanism happens already at LL level in the case of the bilinear quark field operators in Eq. (3.1).

In Ref. [43] we have computed also the NNLL zero-distance Green function describing the time dilatation effect, and therefore we include it in the following considerations. The shift of the cross section due to $C_{V / A, 1}^{\mathrm{int}}$ and $G^{\text {dil }}$ reads (see Sec. 1.5)

$$
\begin{align*}
\Delta \sigma_{\mathrm{thr}}^{\Gamma, 1}= & 2 N_{c} \operatorname{Im}\left\{2 i\left[C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}+C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{int}}\right] G^{0}\left(a, v, m_{t}, \nu\right)\right. \\
& \left.+\left[\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\mathrm{born}}\right)^{2}\right] G^{\mathrm{dil}}\left(a, v, m_{t}, \nu\right)\right\} . \tag{3.6}
\end{align*}
$$

Because the matching conditions are suppressed by $\alpha \sim v^{2}$ with respect to those of the Born level, it was sufficient to set $\mathcal{A}_{1}\left(v, m_{t}, \nu\right)=6 N_{c} G^{0}\left(a, v, m_{t}, \nu\right)$ in the first line. Apart from that, since the Wilson coefficients $C_{V / A, 1}$ do not run at LL order, only the matching conditions at $\nu=1$ appear in Eq. (3.6).

The terms in the first line in Eq. (3.6) reproduce the full theory matrix elements from the interference between the double-resonant amplitudes shown in Fig. 3.1 (a) and the single-resonant amplitudes shown in Figs. 3.1 (b) - (i) in the $t \bar{t}$ threshold limit for $m_{t} \rightarrow \infty$, where the $i \epsilon$ terms in the resonant full theory top propagators are replaced with the Breit-Wigner term $i m_{t} \Gamma_{t} / 2$. Diagram (a) also contains a subleading $v^{2}$-suppressed contribution that has to be
accounted for and corresponds to the top/antitop off-shell contributions shown in Figs. 3.3 (d) and (g). The circle shown in Fig. 3.1 (a) represents the QCD form factors for the $t \bar{t}$ vector and axial-vector currents. In the non-relativistic limit they reduce to the insertions of Coulomb potentials described by the higher order terms in $G^{0}$. Due to the cancellation of the QCD interference effects caused by gluons with ultrasoft momenta there are no further QCD corrections in the non-relativistic limit.

In contrast to the LL order cross section in Eq. (1.29), which is proportional to the imaginary part of $G^{0}$, the NNLL order correction $\Delta_{\text {thr }}^{\Gamma, 1}$ is sensitive to the real part. This has two remarkable consequences. First, since real and imaginary part of $G^{0}$ have a different line-shape, it gives rise to a shift of the peak position of the cross section in the numerical analysis and therefore to a change of the top mass prediction at the order of $30-50 \mathrm{MeV}$. Second, $\Delta_{\text {thr }}^{\Gamma, 1}$ contains $1 / \epsilon$-divergences. Their meaning and treatment will be discussed in the next section.

### 3.3 Phase Space Logarithms and NLL Renormalization Group Running

The $1 / \epsilon$-divergences encountered in the NNLL order correction given in Eq. (3.6) have conceptually interesting features. Since they stem from effective theory loop integrations involving top quark lines and enter the cross section through the optical theorem, they are related to the phase space available to the nonrelativistic quark pair in the effective theory and will therefore be referred to as phase space divergences. They have the form

$$
\begin{equation*}
\frac{1}{\epsilon} i C_{V / A, 1}^{\text {born }} C_{V / A, 1}^{\mathrm{int}} \mathcal{V}_{c}^{(s)}(\nu), \quad \frac{1}{\epsilon} i \Gamma_{t}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \mathcal{V}_{c}^{(s)}(\nu) \tag{3.7}
\end{equation*}
$$

Similar NNLL order divergences were observed in Refs. [22, 40, 106] and were mentioned in Sec. 1.5. They arise from the LL replacement $E \rightarrow E+i \Gamma_{t}$ in Eq. (3.3) inside the NNLL corrections $G^{r}, G^{\mathrm{kin}}, G^{\text {dil }}$ and $\mathcal{A}_{2}$ to the leading zerodistance $S$-wave Green function $G^{0}$ and the $v^{2}$-suppressed $P$-wave Green function $G^{1}$. The explicit forms of these subleading Green functions are given in App. D. The divergences we get from those expressions read

$$
\begin{align*}
& \frac{1}{\epsilon} i \Gamma_{t}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \mathcal{V}_{r}^{(s)}(\nu), \quad \frac{1}{\epsilon} i \Gamma_{t}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \mathcal{V}_{c}^{(s)}(\nu) \\
& \frac{1}{\epsilon} i \Gamma_{t}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \mathcal{V}_{c}^{(s)}(\nu) c_{2}(\nu), \quad \frac{1}{\epsilon} i \Gamma_{t}\left(C_{V / A, 3}^{\text {born }}\right)^{2} \mathcal{V}_{c}^{(s)}(\nu) \tag{3.8}
\end{align*}
$$

Here, only the LL renormalization group evolution of the Wilson coefficient of the subleading $S$-wave current $c_{2}$ enters since we have $c_{1}(\nu)=1$ and $c_{3}(\nu)=1$ at LL order for the Wilson coefficients of the leading $S$-wave and the $P$-wave currents as we saw in Sec. 1.4.

The physical origin of the divergences is a logarithmic high energy behavior of the top-antitop effective theory phase space integration for the respective matrix elements. All of them are proportional to to either $\Gamma_{t}$ or $C_{V / A, 1}^{\mathrm{int}}$, so they are immediately related to the finite lifetime of the top or to interference effects and would not arise if the top quark were treated as a stable particle. We postpone a detailed investigation of the phase space available to the decaying non-relativistic quark pair to Chap. 4 and turn our attention to the formal treatment of the divergences in this section.

Phase space logarithms are known in the literature and can be resummed with renormalization group techniques [107]. The divergences here, on the other hand, are special since they exist only if the top quark is considered as unstable. Yet they can be handled with the renormalization techniques known from effective theories for stable particles. The only difference is that the renormalization procedure will involve operators having non-Hermitian Wilson coefficients.

Because the divergences arise in the $e^{+} e^{-}$forward scattering amplitude, they renormalize $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$operators. In the case of unpolarized (spin-averaged) $e^{+} e^{-}$states it is sufficient to consider only $\tilde{\mathcal{O}}_{V / A}$ defined in Eqs. (1.19), whose Wilson coefficients are $\tilde{C}_{V / A}$. Since we neglect QED effects, the electron and the positron act as classic fields, hence $\tilde{C}_{V / A}$ run only through mixing due to the scale-dependence of the divergences in Eqs. (3.7, 3.8). Since only the imaginary parts of the coefficients $\tilde{C}_{V / A}$ can contribute to the cross section through the optical theorem we neglect the real contributions in the following. Using the $\overline{\mathrm{MS}}$ subtraction method we find for the counterterms of the renormalized $\tilde{\mathcal{O}}_{V / A}$ operators

$$
\begin{aligned}
\delta \tilde{C}_{V / A}= & i \frac{N_{c} m_{t}^{2}}{32 \pi^{2} \epsilon}\left[\left(C_{V / A, 1}^{\text {born }}\right)^{2} \frac{\Gamma_{t}}{m_{t}}+2 C_{V / A, 1}^{\text {born }} C_{V / A, 1}^{\mathrm{int}}\right] \mathcal{V}_{c}^{(s)}(\nu) \\
& +i \frac{N_{c} m_{t}^{2}}{32 \pi^{2} \epsilon}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{\Gamma_{t}}{m_{t}}\left[\left(2 c_{2}(\nu)-1\right) \mathcal{V}_{c}^{(s)}(\nu)+\mathcal{V}_{r}^{(s)}(\nu)\right] \\
& +i \frac{N_{c} m_{t}^{2}}{48 \pi^{2} \epsilon}\left(C_{V / A, 3}^{\mathrm{born}}\right)^{2} \frac{\Gamma_{t}}{m_{t}} \mathcal{V}_{c}^{(s)}(\nu),
\end{aligned}
$$

which subtract the $1 / \epsilon$ divergences in Eqs. (3.7, 3.8). The renormalization group equations for the coefficients $\tilde{C}_{V / A}$ read

$$
\begin{aligned}
\frac{d \tilde{C}_{V / A}(\nu)}{d \ln \nu}= & i \frac{N_{c} m_{t}^{2}}{8 \pi^{2}}\left\{\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{\Gamma_{t}}{m}\left(2 c_{2}(\nu) \mathcal{V}_{c}^{(s)}(\nu)+\mathcal{V}_{r}^{(s)}(\nu)\right)\right. \\
& \left.+2 C_{V / A, 1}^{\mathrm{born}} C_{V / A, 1}^{\mathrm{int}} \mathcal{V}_{c}^{(s)}(\nu)\right\}+i \frac{N_{c} m_{t}^{2}}{12 \pi^{2}}\left\{\left(C_{V / A, 3}^{\mathrm{born}}\right)^{2} \frac{\Gamma_{t}}{m_{t}} \mathcal{V}_{c}^{(s)}(\nu)\right\}
\end{aligned}
$$

where the LL running of $\mathcal{V}_{r}^{(s)}$ and $c_{2}(\nu)$ is given in Eqs. (1.8, 1.18). Solving the
renormalization group equations one obtains [43]

$$
\begin{align*}
\tilde{C}_{V / A}(\Lambda ; \nu)= & \tilde{C}_{V / A}(\Lambda)+i \frac{2 N_{c} m_{t}^{2} C_{F}}{3 \beta_{0}}\left\{\left[\left(\left(C_{V / A, 1}^{\text {born }}\right)^{2}+\left(C_{V / A, 3}^{\text {born }}\right)^{2}\right) \frac{\Gamma_{t}}{m_{t}}\right.\right. \\
& \left.+3 C_{V / A, 1}^{\text {born }} C_{V / A, 1}^{\mathrm{int}}\right] \ln (z)-\frac{4 C_{F}}{\beta_{0}} \frac{\Gamma_{t}}{m_{t}}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \ln ^{2}(z) \\
& \left.+\frac{4\left(C_{A}+2 C_{F}\right)}{\beta_{0}} \frac{\Gamma_{t}}{m_{t}}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \rho(z)\right\} \tag{3.9}
\end{align*}
$$

where

$$
\begin{align*}
\rho(z) & =\frac{\pi^{2}}{12}-\frac{1}{2} \ln ^{2} 2+\ln 2 \ln (z)-\operatorname{Li}_{2}\left(\frac{z}{2}\right), \\
z & \equiv \frac{\alpha_{s}\left(m_{t} \nu\right)}{\alpha_{s}\left(m_{t}\right)} . \tag{3.10}
\end{align*}
$$

Here, we have introduced the $\Lambda$-dependent hard scale ( $\nu=1$ ) matching conditions $\tilde{C}_{V / A}(\Lambda)$. They are determined by the phase space matching procedure which will be developed in Chap. 4. We note that it was already shown in Ref. [40] that the difference between the full theory phase space and the effective theory phase space contributes to these matching conditions.

The contribution of the operators $\tilde{\mathcal{O}}_{V / A}$ to the cross section according to Eq. (1.21) reads

$$
\begin{equation*}
\Delta \sigma_{\mathrm{thr}}^{\Gamma, 2}=\operatorname{Im}\left[\tilde{C}_{V}(\Lambda ; \nu)+\tilde{C}_{A}(\Lambda ; \nu)\right] . \tag{3.11}
\end{equation*}
$$

Parametrically $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ is of order $\alpha^{3} \sim v^{6}$. Compared to the LL cross section, which counts as $\alpha^{2} v \sim v^{5}$, it constitutes a NLL contribution. This is due to the fact that the running was generated by divergences appearing in effective theory matrix elements at the order NNLL. We checked by an explicit calculation that the scale-dependence of $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ compensates the logarithmic scale-dependence in the NNLL order matrix elements.

### 3.4 Numerical Analysis

In Fig. 3.4 we have plotted $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ and $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ in picobarn in the $1 S$ mass scheme [40, 63] for $M_{1 \mathrm{~S}}=175 \mathrm{GeV}, \alpha=1 / 125.7, s_{w}^{2}=0.23120, V_{t b}=1$ and $M_{W}=$ 80.425 GeV with the renormalization scaling parameter $\nu=0.1$ (solid curves), 0.2 (dashed curves) and 0.3 (dotted curves). The divergences in $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ are subtracted minimally. For the QCD coupling we used $\alpha_{s}\left(M_{Z}\right)=0.118$ as an input and employed four-loop renormalization group running. In Fig. 3.4 (a) the sum of $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ and $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ is shown while in Fig. 3.4 (b) both contributions are presented
separately. For the top quark width we adopted the value $\Gamma_{t}=1.43 \mathrm{GeV} .{ }^{3}$ We find that the sum of the corrections is negative and shows a moderate $\nu$ dependence. Compared to the NNLL QCD predictions for the total cross section given in Ref. [34] the corrections are around $-10 \%$ for energies below the peak, between $-2 \%$ and $-4 \%$ close to the peak and about $-2 \%$ above the peak. Their magnitude is comparable to the NNLL QCD corrections. The peculiar energydependence of the corrections, caused by the dependence on the real part of the Green function $G^{0}$, leads to a slight displacement of the peak position. Relative to the peak position of the LL cross section one obtains a shift of $(30,35,47) \mathrm{MeV}$ for $\nu=(0.1,0.2,0.3)$. This shift is comparable to the expected experimental uncertainties of the top mass measurements from the threshold scan [8].

[^6]

Figure 3.4: The corrections $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ and $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ in picobarn for $M_{1 \mathrm{~S}}=175 \mathrm{GeV}$, $\alpha=1 / 125.7, s_{w}^{2}=0.23120, V_{t b}=1, M_{W}=80.425 \mathrm{GeV}, \Gamma_{t}=1.43 \mathrm{GeV}$ and $\nu=0.1$ (solid curves), 0.2 (dashed curves) and 0.3 (dotted curves) in the energy range $346 \mathrm{GeV}<\sqrt{s}<354 \mathrm{GeV}$. Panel (a) shows the sum of both corrections and panel (b) the individual size of $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ (energy-dependent lines) and $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ (straight lines).

## Chapter 4

## Phase Space Matching

The aim of this chapter is to develop a systematic approach to the computation of the NRQCD phase space. Already the appearance of the initial condition for the NLL renormalization group running in Eq. (3.9) indicates the necessity of a matching prescription for the phase space of the effective theory, which contains unstable particles. One approach would be to match the matrix elements describing the effective theory phase space to the corresponding ones describing the full theory phase space of the $b W^{+} \bar{b} W^{-}$final state as described in Ref. [108] in the case of $W^{+} W^{-}$threshold production. We will go a different way in this work for basically two reasons. First, the computation of the full theory matrix elements is quite cumbersome and probably much more extensive than the calculations necessary in our approach. Second, our approach involves techniques of operator product expansion and finite imaginary renormalization describing phase space cuts, which are interesting from the formal point of view of an effective theory treatment of unstable particles.

The basic idea of our approach is as follows. Imaginary parts associated with the top quark instability affect the NRQCD phase space in such a way that it substantially differs from the corresponding one for stable particles. As a consequence, an additional parameter is introduced that specifies which physical setup is described by the effective theory. This parameter is associated with a cut on the invariant masses of the decaying top and antitop and therefore leads to a dependence of the cross section on the invariant masses of those experimentally observed $b W^{+}$and $\bar{b} W^{-}$pairs which are associated with a top pair threshold event. For the effective theory the parameter acts as a restriction of the phase space because it cuts off unphysical parts. Since the required computations can be carried out using NRQCD Feynman rules, our approach can be considered as a matching procedure within the effective theory.

Formally the unphysical part of the NRQCD phase space is subtracted from the full NRQCD phase space by an operator product expansion including $e^{+} e^{-}$ forward scattering operators with imaginary coefficients. As we will see, the cut introduces a breaking of the NRQCD power counting, which has to be dealt with.

A characteristic feature of all the phase space effects we will consider in this work is the fact that they are proportional to at least one power of $\Gamma_{t}$ or of $C_{V / A, 1}^{\mathrm{int}}$, hence they are directly related to the top quark instability. We will find that they are numerically important for the determination of the cross section because their contribution starts already at NLO.

### 4.1 Basic Idea

As the simplest example for the phase space integration in the effective theory we consider the correlator of the dominant $S$-wave annihilation and production currents, neglecting QCD interactions. It is obtained by a loop integration of the top and antitop propagator in Eq. (3.2), where half of the kinetic energy $E$ flows through each propagator. Using dimensional regularization, $d=4-2 \epsilon, \tilde{\mu}$ being the mass scale parameter, we obtain

$$
\begin{equation*}
G^{0,0}=i \tilde{\mu}^{2 \epsilon} \int \frac{d^{d} p}{(2 \pi)^{d}} \frac{i}{\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \frac{i}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}}=\frac{m_{t}^{2}}{4 \pi} i v, \tag{4.1}
\end{equation*}
$$

where we denote $G^{0,0}$ as the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ piece of the zero-distance $S$-wave Green function $G^{0}$ and $v=\sqrt{\left(E+i \Gamma_{t}\right) / m_{t}}$ and we could set $\epsilon=0$. The corresponding contribution to the cross section is proportional to the imaginary part of $G^{0,0}$ through the optical theorem and reads

$$
\begin{equation*}
\sigma_{\mathrm{thr}}^{0,0}=N_{c}\left(\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\text {born }}\right)^{2}\right) 2 \operatorname{Im} G^{0,0} . \tag{4.2}
\end{equation*}
$$

We obtain the same result by a 4 -particle phase space integration starting from the full theory squared diagrams in Fig. 3.1 (a) neglecting the QCD form factors. Within a non-relativistic expansion, which assumes top quarks close to their mass shells, we obtain top quark propagators like in Eq. (3.2) (here we explicitly have to replace the $i \epsilon$ terms with the Breit-Wigner terms $i m_{t} \Gamma_{t} / 2$ ) and contract the two appearing $b W$ subdiagrams to expressions proportional to $\Gamma_{t}$. This yields

$$
\begin{equation*}
\operatorname{Im} G^{0,0}=\frac{1}{(2 \pi)^{3}} \int_{-\infty}^{+\infty} d p_{0} \int_{0}^{+\infty} d|\mathbf{p}| \mathbf{p}^{2} \frac{\Gamma_{t}^{2}}{\left|\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right|^{2}\left|\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right|^{2}} \tag{4.3}
\end{equation*}
$$

Here, the 4 -dimensional $p^{\mu}$ integration is a phase space integration. It is the only one of the initially four integrations of the 4 -particle phase space that is left. (The others have gone by the contraction of the $b W$ subdiagrams and the overall 4 -momentum conserving delta function.) The integrand is the differential cross section depending on $p_{0}$ and $\mathbf{p}$.

This formula exactly represents the effective theory phase space calculation in Eqs. (4.1, 4.2) and therefore allows for a physical interpretation of that approach. We define the variables

$$
\begin{equation*}
t_{1}=2 m_{t}\left[\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}\right], \quad t_{2}=2 m_{t}\left[\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}\right] \tag{4.4}
\end{equation*}
$$

which characterize the offshellness of top and antitop, respectively, in the phase space integration. Using $t_{1,2}$ we divide the full phase space into different regions, namely the one where $t_{1,2}$ are small (both top and antitop are resonant), the one where either $t_{1}$ or $t_{2}$ is small (only one quark is resonant) and the one where $t_{1,2}$ are not small (non-resonant),

$$
\begin{aligned}
\left|t_{1}\right| & <\Lambda^{2} \text { and }\left|t_{2}\right| \\
\left(\left|t_{1}\right|\right. & \left.<\Lambda^{2} \text { and }\left|t_{2}\right|>\Lambda^{2}\right) \quad \text { or } \quad\left(\left|t_{1}\right|>\Lambda^{2} \text { and }\left|t_{2}\right|<\Lambda^{2}\right) \quad \text { (single-resonant) }, \\
\left|t_{1}\right| & >\Lambda^{2} \text { and }\left|t_{2}\right|>\Lambda^{2} \quad \text { (non-resonant) } .
\end{aligned}
$$

Here, we introduced a cut $\Lambda>0$ that defines the boundaries of the regions. The regions are schematically shown in Fig. 4.1.

We consider the black lines in Fig. 4.1, which correspond to the conditions $t_{1}=0$ and $t_{2}=0$, respectively, where top and antitop are on their mass shells. In a full theory tree level computation of stable quark pair production close to threshold, the phase space consists only of the intersection point of the two lines in the right panel (where $E>0$ ), whereas there is no phase space available below threshold as the curves have no intersection in the left panel $(E<0)$. This is related to the cutting rule

$$
\frac{i\left(\not q-m_{t}\right)}{q^{2}-m_{t}^{2}+i \epsilon} \rightarrow\left[i\left(\not q-m_{t}\right)\right](-2 \pi i) \delta\left(q^{2}-m_{t}^{2}\right) \theta\left(q_{0}\right)
$$

for the full theory tree-level quark propagator. In the non-relativistic expansion of $\left(q^{2}-m_{t}^{2}\right)$ where $q^{\mu}=\left(\frac{E}{2} \pm p_{0}+m_{t}, \mathbf{p}\right)$ using $p_{0} \sim m_{t} v^{2}, \mathbf{p} \sim m_{t} v$ the term inside the delta function reads

$$
\begin{equation*}
q^{2}-m_{t}^{2}=2 m_{t}\left[\frac{E}{2} \pm p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}\right]=t_{1,2}, \tag{4.5}
\end{equation*}
$$

confirming the conditions $t_{1,2}=0$. Physically speaking, the full theory phase space for stable tops is bounded by these conditions.

This boundary is macerated by the $i \Gamma_{t}$ terms in the Wigner form (4.3), and hence the effective theory phase space is infinite, $\infty<t_{1,2}<\infty$. This led to the ultraviolet phase space divergences that appeared for specific matrix elements encountered in Sec. 1.5. The high energy behavior of the phase space has been accounted for by means of renormalization group methods in Sec. 3.3. The hard


Figure 4.1: 3-momentum-energy planes showing the various regions of the phase space integration. The kinetic Energy $E$ is negative in the left panel and positive in the right one. The solid lines correspond to $t_{1}=0$ and $t_{2}=0$, respectively. Yellow bands indicate single-resonant regions, the red sector corresponds to the double-resonant region. Numerical values have been chosen in this example as $E= \pm 5 \mathrm{GeV}, \Lambda^{2} /\left(2 m_{t}\right)=20 \mathrm{GeV}$.
matching conditions are now obtained from the physical phase space available to the $t \bar{t}$ system, that means the double-resonant region,

$$
-\Lambda^{2}<t_{1,2}<\Lambda^{2}
$$

where the parameter characterizing the phase space is at the order of the hard scale. From this we obtain the formal power counting

$$
\begin{equation*}
\Lambda \sim m_{t} \tag{4.6}
\end{equation*}
$$

It will be useful for the classification of the various phase space corrections. At this point we note that the introduction of the explicit cut $\Lambda$ for loop integrations involves power counting breaking effects, which will be investigated in Sec. 4.6. In order to achieve a suppression of these power counting effects we will choose numerical values for the cut that deviate from the formal relation (4.6). Therefore the values will fulfill at least $\Lambda \lesssim m_{t}$. We postpone the issue of the allowed numerical range to Sec. 4.6, where explicit analytic formulae from the matching computation will be available.

Let us now ask what role the choice of the numerical value for the cut plays from the physical point of view. Because the cut implies a limitation of the phase space integration, it is clear that the cross section $\sigma_{\text {thr }}$ will pick up a dependence on $\Lambda$. Due to Eq. (4.5) one obtains

$$
\begin{equation*}
-\Lambda^{2}<q^{2}-m_{t}^{2}<\Lambda^{2} \tag{4.7}
\end{equation*}
$$

for the physical region. Therefore $\sigma(\Lambda)$ corresponds to those measured $t \bar{t}$ events with the property (4.7) for the squared 4 -momentum $q^{2}$ of top/antitop. In the experiment, where only the $b W^{+} \bar{b} W^{-}$final state can be seen, this corresponds to cuts on the invariant masses of the $b W^{+}$and $\bar{b} W^{-}$systems. Therefore the numerical choice of $\Lambda$ defines what kind of $t \bar{t}$ events are described by the effective theory treatment using absorptive matching conditions. The interpretation of the cut will be discussed in more detail in Sec. 4.7.

### 4.2 Phase Space Corrections to Green Functions vs. Operator Product Expansion

Let us consider the imaginary part of the LL order zero-distance $S$-wave (Coulomb) Green function, which enters the cross section in Eq. (1.29). It can be written in the form

$$
\begin{equation*}
\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)=\frac{\Gamma_{t}^{2}}{2} \int \frac{d^{4} p}{(2 \pi)^{4}}\left|\tilde{G}_{v, m_{t}, \nu}^{0}\left(p_{0}, \mathbf{p}\right)\right|^{2} \tag{4.8}
\end{equation*}
$$

where the 4 -momentum integration corresponds to the phase space integration and $\tilde{G}_{v, m_{t}, \nu}^{0}\left(p_{0}, \mathbf{p}\right)$ is the effective theory correlator which corresponds to the full theory diagram in Fig. 3.1 (a) containing the QCD form factors in the nonrelativistic limit. It describes energy and momentum distribution of the top quark pair and reads

$$
\begin{align*}
\tilde{G}_{v, m_{t}, \nu}^{0}\left(p_{0}, \mathbf{p}\right)= & \frac{1}{\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \frac{1}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \\
& \times\left[\frac{\mathbf{p}^{2}}{m_{t}}-\left(E+i \Gamma_{t}\right)\right] \tilde{G}_{v, m_{t}, \nu}^{0}(0, \mathbf{p}), \tag{4.9}
\end{align*}
$$

where

$$
\tilde{G}_{v, m_{t}, \nu}^{0}(0, \mathbf{q})=\int D^{n} \mathbf{p} \tilde{G}_{v, m_{t}, \nu}^{0}(\mathbf{p}, \mathbf{q})
$$

is the partially Fourier transformed momentum space Green function, evaluated at $\mathbf{x}=0$, obtained from the LL version of Eq. (1.24). For example in the $\mathcal{O}\left(\alpha_{s}^{0}\right)$
case in Eq. (4.3) the form factor in the second line of Eq. (4.9) is one and we simply have

$$
\begin{equation*}
\tilde{G}_{v, m_{t}, \nu}^{0,0}\left(p_{0}, \mathbf{p}\right)=\frac{1}{\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \frac{1}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} . \tag{4.10}
\end{equation*}
$$

As we showed in the last section, the physical phase space region corresponding to a hard cut $\Lambda$ reads

$$
\begin{equation*}
\Delta(\Lambda)=\left\{\left(p_{0}, \mathbf{p}\right) \in \mathbb{R}^{4}:\left|t_{1,2}\right|<\Lambda^{2}\right\} \tag{4.11}
\end{equation*}
$$

where $p_{0}$ and $\mathbf{p}^{2}$ are understood as functions of $t_{1,2}$ according to Eqs. (4.4). Therefore we define

$$
\begin{equation*}
\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)=\frac{\Gamma_{t}^{2}}{2} \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}}\left|\tilde{G}_{v, m_{t}, \nu=1}^{0}\left(p_{0}, \mathbf{p}\right)\right|^{2} \tag{4.12}
\end{equation*}
$$

as the imaginary part of the leading $S$-wave Coulomb Green function describing the physical phase space at the hard scale $\Lambda \sim m_{t}$ and $\nu=1$. For a practical implementation of the phase space corrections it is sufficient to add the correction term of the form

$$
\begin{equation*}
\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right](\Lambda)=\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)-\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1} \tag{4.13}
\end{equation*}
$$

to the imaginary part of $G^{0}$ in the LL order expression for the cross section, Eq. (1.29). Here, $\left.G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}$ denotes the zero-distance Green function in the $\overline{\mathrm{MS}}$ scheme, evaluated at the hard scale $\nu=1$.

The other phase space contributions to the cross section in Eq. (1.21), which involve $v^{2}$-suppressed operators originating from the kinetic energy correction, time dilatation, subleading currents, suppressed potentials or interference effects, in general do not have the simple form of Eq. (4.12). They can be described by interferences of different form factors, so that Eq. (4.12) would look like

$$
\begin{equation*}
\sim \sum_{i j} \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \tilde{G}_{v, m_{t}, \nu=1}^{i}\left(p_{0}, \mathbf{p}\right)\left(\tilde{G}_{v, m_{t}, \nu=1}^{j}\left(p_{0}, \mathbf{p}\right)\right)^{*} \tag{4.14}
\end{equation*}
$$

where $i$ and $j$ denote the sort of correction. For example the phase space correction to the real part of $G^{0}$ in Eq. (3.6) arising from interference effects reads

$$
\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right](\Lambda)=\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)-\left.\operatorname{Re} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1},
$$

where

$$
\begin{align*}
{\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)=\frac{\Gamma_{t}}{2} \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} } & {\left[\tilde{G}_{v, m_{t}, \nu=1}^{0}\left(p_{0}, \mathbf{p}\right)\left(\tilde{G}_{v, m_{t}}^{\mathrm{int}}\left(p_{0}, \mathbf{p}\right)\right)^{*}\right.} \\
+ & \left.\tilde{G}_{v, m_{t}}^{\mathrm{int}}\left(p_{0}, \mathbf{p}\right)\left(\tilde{G}_{v, m_{t}, \nu=1}^{0}\left(p_{0}, \mathbf{p}\right)\right)^{*}\right] \tag{4.15}
\end{align*}
$$

Here, the $1 / \epsilon$ divergence in $\left.G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}$ is subtracted minimally for consistency with the $\overline{\mathrm{MS}}$ treatment in Sec. 3.3. The form factor for a vertex cut, which is associated with the interference effects, can be derived from Eq. (4.23) and leads to

$$
G_{v, m_{t}}^{\mathrm{int}}\left(p_{0}, \mathbf{p}\right)=-\frac{1}{2}\left[\frac{1}{\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}}+\frac{1}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}}\right]
$$

In the case of $\operatorname{Im} G^{0}$ and $\operatorname{Re} G^{0}$ this approach to the implementation of phase space effects is possible immediately since an analytic expression for $\tilde{G}_{v, m_{t}, \nu}^{0}(0, \mathbf{p})$ is well known [109]. For the NLL and NNLL versions of $\tilde{G}_{v, m_{t, \nu}}^{0}(0, \mathbf{p})$ numerical routines are available [80]. The phase space integrations in Eqs. (4.12) and (4.15) can be done numerically. We will use this approach for the numerical analysis of Sec. 4.6. In principle, phase space corrections associated with other Green functions can be implemented in analogy, once the according form factors appearing in Eq. (4.14) are known analytically or numerically.

The just described approach is a practical implementation of the phase space effects by a manipulation of the effective theory correlators. In our opinion it is more satisfactory from the formal point of view to write the corrections directly into the Wilson coefficients of effective theory operators. This allows us to stick to the usual use of the optical theorem and determine the cross section from the imaginary part of the $e^{+} e^{-}$forward scattering amplitude derived from the effective theory action. This approach corresponds to an operator product expansion of the above defined quantities $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$ and $\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)$ (and according ones for the other corrections) such that phase space effects are encoded in the coefficients of the appearing operators. In our computations at the formal NLO and NNLO levels these operators are the currents $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$ and $\mathcal{O}_{V / A, \mathbf{p}, \sigma}^{\dagger}$ in Eq. (1.15) and $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$forward scattering operators $\tilde{\mathcal{O}}_{V / A}^{(n)}$ in Eq. (1.20). We will find that the contributions to the coefficients are again imaginary and associated with top instability. In the following we will refer to this mechanism also as finite imaginary renormalization. We will explain in detail how this renormalization procedure works in Secs. 4.4 and 4.5.

### 4.3 Integration Techniques

The aim of this section is to introduce the basic techniques of calculation that allow for an analytic integration over the physical phase space region. Because most of the integrands depend only on $p_{0}$ and $|\mathbf{p}|$ (this is not true in the case of $P$ wave current renormalization) we use spherical coordinates for the 3-momentum integration. Due to the structure of $\Delta(\Lambda)$, it is natural to change integration
variables using the linear transformation in Eq. (4.4), which leads to

$$
\begin{aligned}
& \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} f\left(m_{t} E+2 m_{t} p_{0}-\mathbf{p}^{2}, m_{t} E-2 m_{t} p_{0}-\mathbf{p}^{2}, \cos \vartheta, \varphi\right)= \\
& =\frac{1}{(2 \pi)^{4}} \frac{1}{8 m_{t}} \iint_{\mathbb{R}^{2}} d t_{1} d t_{2} \theta\left(\Lambda^{2}-\left|t_{1}\right|\right) \theta\left(\Lambda^{2}-\left|t_{2}\right|\right) \theta\left(m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)\right) \\
& \quad \times \sqrt{m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)} \int_{-1}^{1} d(\cos \vartheta) \int_{0}^{2 \pi} d \varphi f\left(t_{1}, t_{2}, \cos \vartheta, \varphi\right),
\end{aligned}
$$

where $\vartheta$ and $\varphi$ denote the angles of the spherical coordinates. The step function $\theta\left(m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)\right)$ guarantees that its argument, which corresponds to $|\mathbf{p}|$, is always positive. On the RHS the cut enters explicitly in the form of the step functions $\theta\left(\Lambda^{2}-\left|t_{1,2}\right|\right)$. If the integrand $f\left(t_{1}, t_{2}, \cos \vartheta, \varphi\right)$ is independent of $\vartheta$ and $\varphi$, the angular integration gives the trivial factor $4 \pi$.

In all cases we will have to consider the $t_{1}-t_{2}$-part of the integration can be written as a combination of the basic integrals defined as

$$
\begin{align*}
I_{k}^{p_{1} q_{1} p_{2} q_{2}, x}= & \iint_{\mathbb{R}^{2}} d t_{1} d t_{2} \theta\left(\Lambda^{2}-\left|t_{1}\right|\right) \theta\left(\Lambda^{2}-\left|t_{2}\right|\right) \theta\left(m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)\right) \\
& \times \frac{p P_{k}^{x}\left(t_{1}, t_{2}\right)}{\left(t_{1}+i m_{t} \Gamma_{t}\right)^{p_{1}}\left(t_{1}-i m_{t} \Gamma_{t}\right)^{q_{1}}\left(t_{2}+i m_{t} \Gamma_{t}\right)^{p_{2}}\left(t_{2}-i m_{t} \Gamma_{t}\right)^{q_{2}}} \tag{4.16}
\end{align*}
$$

where

$$
\begin{aligned}
P_{0}^{x}\left(t_{1}, t_{2}\right) & =1 \\
P_{1}^{x}\left(t_{1}, t_{2}\right) & =\frac{1}{p} i \ln \frac{V_{x}+p}{V_{x}-p} \\
P_{2}^{x}\left(t_{1}, t_{2}\right) & =\frac{1}{t_{1}+t_{2}+2 i m_{t} \Gamma_{t}}
\end{aligned}
$$

if $p_{1}, p_{2}, q_{1}, q_{2} \in \mathbb{N}_{0}$ and $x \in[0,1]$. We use the abbreviations

$$
p \equiv \sqrt{m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)}, \quad V_{x} \equiv \sqrt{m_{t} E+x i m_{t} \Gamma_{t}}
$$

and $I_{0 / 2}^{p_{1} q_{1} p_{2} q_{2}} \equiv I_{0 / 2}^{p_{1} q_{1} p_{2} q_{2}, x}$. We note that there are a number of symmetry relations concerning the exchange $t_{1} \leftrightarrow t_{2}$ and complex conjugation that connect several of these integrals. The integration region in the $t_{1}-t_{2}$-plane is shown in Fig. 4.2.

The integrals with $x=1$ will appear in the computation of vacuum polarization diagrams that lead to a renormalization of forward scattering operators, whereas the integrals with $x=0$ will be needed for the computation of vertex corrections that correspond to a renormalization of the production currents. The



Figure 4.2: Integration region in the $t_{1}-t_{2}$-plane of the integrals defined in Eq. (4.16). The left and right panel shows the cases $E<0$ and $E>0$, respectively. The integration region corresponds to the double-resonant region in the $p_{0}-\mathbf{p}$-plane shown in Fig. 4.1.

Integrals $I_{0}^{p_{1} q_{1} p_{2} q_{2}}$ correspond to $\mathcal{O}\left(\alpha_{s}^{0}\right)$ diagrams, whereas $I_{1}^{p_{1} q_{1} p_{2} q_{2}, x}$ and $I_{2}^{1111}$ are needed for $\mathcal{O}\left(\alpha_{s}^{1}\right)$ computations.

The integral $I_{0}^{1111}$, for instance, corresponds to $G^{0,0}$,

$$
\operatorname{Im} G^{0,0}(\Lambda)=\frac{\Gamma_{t}^{2}}{2} \frac{\left(2 m_{t}\right)^{4}}{(2 \pi)^{4}} \frac{4 \pi}{8 m_{t}} I_{0}^{1111}
$$

according to Eqs. (4.10, 4.12).
Our method of calculation of the required double integrals $I_{k}$ is the following. One of the integration, say $t_{2}$, can be carried out directly. For the remaining $t_{1}$-integration it is helpful to understand that the integral contains two different scales,

$$
m_{t} E \sim m_{t} \Gamma_{t} \quad \text { "soft", } \quad \Lambda^{2} \quad \text { "hard" }
$$

We define a cut $L^{2}$ such that it separates the scales,

$$
m_{t} E \sim m_{t} \Gamma_{t} \ll L^{2} \ll \Lambda^{2} .
$$

The integration is then divided into two regions,

$$
\left|t_{1}\right|<L^{2} \quad \text { region one, } \quad L^{2}<\left|t_{1}\right| \quad \text { region two. }
$$

Before we carry out the integration, we perform a power expansion of the integrand based on the wide separation of the hard and the soft scale. The expansion
is different in the two regions: Since

$$
\frac{m_{t} E}{\Lambda^{2}}, \frac{m_{t} \Gamma}{\Lambda^{2}}, \frac{t_{1}}{\Lambda^{2}} \ll 1 \quad \text { (region one) }, \quad \frac{m_{t} E}{\Lambda^{2}}, \frac{m_{t} \Gamma}{\Lambda^{2}} \ll 1 \quad \text { (region two) }
$$

we expand simultaneously in three parameters (including the integration variable) in region one and in two parameters in region two. The resulting terms are complex logarithms and trivial functions, which can be integrated in an analytic way. After the integration has been carried out we expand in powers of $1 / L^{2}$. We find that the artificially introduced cut $L^{2}$ cancels in the sum of the integrals over the two different regions. This cancellation works up to a power corresponding to the order at which the initial expansions were stopped.

In the course of the calculation complex logarithms, dilogarithms and trivial functions show up. We note that in the case of the holomorphic functions that are defined only on a subset of the complex number plane the expansion is indeed straightforward, but nevertheless quite subtle. In particular, since we use the computer algebra program Mathematica, it is important to make sure explicitly that the expansion is done in the right way. For the complex logarithmic integrations we use techniques similar to those described in Ref. [110]. The results of our computations can be found in App. E.

### 4.4 One-Loop Renormalization

In this section we determine cuts through the effective theory one-loop diagrams shown in Fig. 4.3, which describe the physical phase space available to the topantitop pair if a cut $\Lambda$ is applied. The task of our operator product expansion is to write every diagram as a sum of the well-known zero-distance Green function and the correction originating from the cut. This correction contributes to the finite imaginary renormalization of the coefficients $\tilde{C}_{V / A}^{(n)}$ of the $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$forward scattering operators $\tilde{\mathcal{O}}_{V / A}^{(n)}$ in Eq. (1.20). At one-loop level they are the only operators that appear. The coefficients have the form

$$
\tilde{C}_{V / A}^{(n)}(\Lambda)=\sum_{j} \tilde{C}_{V / A}^{(n), j}(\Lambda),
$$

where $j$ denotes the type of correction.
Because such a relation between the physical phase space and the theory components fixes a theory parameter, it can be considered as a renormalization condition for this theory parameter. This is in analogy e.g. to the condition that physical masses are equivalent to the real parts of poles of propagators in a certain pole-mass renormalization scheme. The one-loop renormalization condition for $\tilde{C}_{V / A}^{(n)}$ is illustrated in Fig. 4.5.

(a)

(d)

(b)

(e)

(c)

(f)

Figure 4.3: Cuts through effective theory $\mathcal{O}\left(\alpha_{s}^{0}\right)$ diagrams associated with a $t_{1}$ and $t_{2}$ integration up to $\Lambda^{2}$. Crosses indicate top-antitop production or annihilation. Graphs (a)-(c) involve only the leading $S$-wave current $\mathcal{O}_{\mathbf{p}, 1}^{j}$ including no insertion of a bilinear operator (a), a kinetic energy insertion (b) and a time dilatation insertion (c). Graphs (d) and (e) indicate the correlator of leading and subleading $\left(\mathcal{O}_{\mathbf{p}, 2}^{j}\right) S$-wave vector current and the $P$-wave axial-vector current $\left(\mathcal{O}_{\mathbf{p}, 3}^{j}\right)$ correlator, respectively. Graph (f) involves a cut through a vertex of the plain $S$-wave correlator and corresponds to an interference effect. Diagrams (b), (c) and (f) include also other placements of the insertion or the cut, respectively.

The computation of the various cut diagrams is done by the application of cutting rules to the according propagators using the integration techniques described in Secs. 4.2 and 4.3. In the case of the interference effects the cutting rules for propagators are not sufficient and therefore we will use a full theory computation to obtain the necessary amplitudes.

### 4.4.1 Plain $S$-Wave

The plain dominant $S$-wave contribution is obtained from the leading ${ }^{3} S_{1}$ production and annihilation graph without any insertions of additional operators shown in Fig. 4.3 (a). We apply the cutting rules

$$
\frac{i}{\frac{E}{2} \pm p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \rightarrow-2 \operatorname{Im}\left[\frac{1}{\frac{E}{2} \pm p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}}\right]
$$



Figure 4.4: Effective theory $\mathcal{O}\left(\alpha_{s}^{0}\right)$ top quark loop diagrams. The appearing symbols are explained in the capture of Fig. 4.3. Graphs (b) and (c) include also insertions in the antitop propagators.
to the propagators in Eq. (4.1) and obtain the expression

$$
\begin{align*}
C_{4.3(\mathrm{a})}(\Lambda)= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\Gamma_{t}}{\left(\frac{E}{2}+p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}+p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \\
& \times \frac{\Gamma_{t}}{\left(\frac{E}{2}-p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \\
= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)} . \tag{4.17}
\end{align*}
$$

The integrand in the first equation is the one of the non-relativistic limit of the full theory calculation in Eq. (4.3). In the second equation $t_{1}$ and $t_{2}$ are understood as functions of $p_{0}$ and $\mathbf{p}^{2}$ according to Eqs. (4.4). A change of variables immediately yields

$$
\begin{aligned}
& C_{4.3(a)}(\Lambda)= \frac{\left(2 m_{t}\right)^{4}}{(2 \pi)^{4}} \\
& \frac{4 \pi}{8 m_{t}} \Gamma_{t}^{2} I_{0}^{1111} \\
&= 2 \frac{m_{t}^{2}}{4 \pi}\left(\operatorname{Re}[v]-\frac{2 \sqrt{2}}{\pi} \frac{\Gamma_{t}}{\Lambda}+\frac{4+2 \sqrt{2} \operatorname{arsinh}(1)}{3 \pi^{2}} \frac{m_{t} \Gamma_{t}^{2}}{\Lambda^{3}}\right. \\
&\left.-\frac{2 \sqrt{2}}{3 \pi} \frac{m_{t} E \Gamma_{t}}{\Lambda^{3}}+\mathcal{O}\left(v^{6} \frac{m_{t}^{5}}{\Lambda^{5}}\right)\right),
\end{aligned}
$$

where we inserted in the second line analytic expression from App. E for the function $I_{0}^{1111}$.

The term $\operatorname{Re}[v]$ is the only one that cannot be written as a contribution from the local operators $\tilde{\mathcal{O}}_{V / A}^{(n)}$ since $v=\sqrt{\left(E+i \Gamma_{t}\right) / m_{t}}$. On the other hand, this is
not necessary because this term is delivered by the correlator of the production and annihilation currents,

$$
2 \operatorname{Im} G^{0,0}=C_{4.3(\mathrm{a})}(\infty)=2 \operatorname{Im}\left[\frac{m_{t}^{2}}{4 \pi} i v\right],
$$

where $G^{0,0}$ is known from Eq. (4.1) and displayed in Fig. 4.4 (a). Therefore it is possible to write the contribution to the cross section in Eq. (1.21) in the form

$$
\begin{align*}
N_{c}\left(\left(C_{V, 1}^{\text {born }}\right)^{2}+\left(C_{V, 1}^{\text {born }}\right)^{2}\right) & C_{4.3(\mathrm{a})}(\Lambda)=2 N_{c} \operatorname{Im}\left[\left(\left(C_{V, 1}^{\text {born }}\right)^{2}+\left(C_{V, 1}^{\text {born }}\right)^{2}\right) G^{0,0}\right. \\
& \left.+\sum_{n}\left(E / m_{t}\right)^{n}\left(\tilde{C}_{V}^{(n), 0,0}(\Lambda)+\tilde{C}_{A}^{(n), 0,0}(\Lambda)\right)\right] \tag{4.18}
\end{align*}
$$

From the point of view of renormalization this equation is the renormalization condition for $C_{V / A}^{(n), 0,0}$. Fig. 4.5 gives a graphical illustration.

We obtain

$$
\begin{gather*}
\sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), 0,0}(\Lambda)=N_{c}\left(C_{V / A, 1}^{\text {born }}\right)^{2} i\left[C_{4.3(\mathrm{a})}(\Lambda)-2 \operatorname{Im} G^{0,0}\right] \\
=2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left(-\frac{2 \sqrt{2}}{\pi} \frac{\Gamma_{t}}{\Lambda}+\frac{4+2 \sqrt{2} \operatorname{arsinh}(1)}{3 \pi^{2}} \frac{m_{t} \Gamma_{t}^{2}}{\Lambda^{3}}\right. \\
\left.\quad-\frac{2 \sqrt{2}}{3 \pi} \frac{m_{t} E \Gamma_{t}}{\Lambda^{3}}+\mathcal{O}\left(v^{6} \frac{m_{t}^{5}}{\Lambda^{5}}\right)\right) \tag{4.19}
\end{gather*}
$$

By counting powers of $v$ and using the formal counting $\Lambda \sim m_{t}$ we find that the coefficients $\tilde{C}_{V / A}^{(0), 0,0}$ contribute to the cross section already at next-to-leading order (NLO).

### 4.4.2 Kinetic Energy Corrections to $S$-Wave

The cut graphs in Fig. 4.3 (b) comprising the kinetic energy correction $\mathbf{p}^{4} /\left(8 m_{t}^{3}\right)$ from Eq. (1.3) can be written in the form

$$
\begin{aligned}
C_{4.3(\mathrm{a})}(\Lambda)+ & C_{4.3(\mathrm{~b})}(\Lambda)= \\
= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\Gamma_{t}}{\left(\frac{E}{2}+p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+\frac{\mathrm{p}^{4}}{8 m_{t}^{3}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}+p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+\frac{\mathrm{p}^{4}}{8 m_{t}^{3}}-i \frac{\Gamma_{t}}{2}\right)} \\
& \quad \times \frac{\Gamma_{t}}{\left(\frac{E}{2}-p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+\frac{\mathrm{p}^{4}}{8 m_{t}^{3}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}-p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}+\frac{\mathrm{p}^{4}}{8 m_{t}^{3}}-i \frac{\Gamma_{t}}{2}\right)}, \\
= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)}\left[1+\Delta^{\mathrm{kin}]}\right.
\end{aligned}
$$



Figure 4.5: One-loop renormalization condition for $\tilde{C}_{V / A}^{(n)}(\Lambda) \simeq \tilde{C}(\Lambda)$. The phase space integration up to $\Lambda^{2}$ on the LHS contributing to the cross section is reproduced in the effective theory by the RHS. Since the effective theory uses the optical theorem, the RHS is the imaginary part of the forward scattering amplitude. The latter is written as the sum of the correlator of production and annihilation currents (Fig. $4.4(\mathrm{a}))$ and the amplitude arising from $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$ operators (displayed as the crossed circle). Phase space corrections on the RHS are encoded in the coefficients $\tilde{C}(\Lambda)$. For this illustration and others we use a slightly different normalization of these coefficients and suppress the energydependence arising from the $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$operators.
with

$$
\begin{equation*}
\Delta^{\mathrm{kin}}=-\frac{\mathbf{p}^{4}}{2 m_{t}^{2}}\left(\frac{t_{1}}{t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}}+\frac{t_{2}}{t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}}\right) . \tag{4.20}
\end{equation*}
$$

The correction $\Delta^{\text {kin }}$ contains only the leading terms in the $\mathbf{p}^{2} / m_{t}^{2}$ expansion, "one $\mathbf{p}^{4} /\left(8 m_{t}^{3}\right)$ insertion." By means of partial fraction the integral can be decomposed to the basic integrals.

Similar to the case without kinetic energy insertion, our result contains the "non-local" term $2 \operatorname{Im} G^{\text {kin,0 }}$ where $G^{\text {kin,0 }}$ is the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ term (shown in Fig. 4.4 (b)) of the Green function given in Eq. (D.3). It also contains a linear divergence $\Lambda / m_{t}$, which cannot appear in $G^{\mathrm{kin}, 0}$ since $G^{\mathrm{kin}, 0}$ was computed using dimensional regularization. The difference $C_{4.3(\mathrm{~b})}(\Lambda)-2 \operatorname{Im} G^{\mathrm{kin}, 0}$ renormalizes the Wilson coefficients of the forward scattering operators. Our explicit results read

$$
\begin{aligned}
\operatorname{Im} G^{\mathrm{kin}, 0}= & \operatorname{Im}\left[\frac{5 m_{t}^{2}}{32 \pi} i v^{3}\right] \quad \text { (reproduced) } \\
\sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \text { kin }, 0}(\Lambda)= & 2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{5 m_{t}^{2}}{32 \pi} i\left[\frac{9}{5 \sqrt{2} \pi} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}\right. \\
& +\left(\frac{-86+105 \sqrt{2} \operatorname{arsinh}(1)}{30 \pi^{2}} \frac{\Gamma_{t}^{2}}{m_{t} \Lambda}-\frac{7}{\sqrt{2} \pi} \frac{E \Gamma_{t}}{m_{t} \Lambda}\right) \\
& \left.+\mathcal{O}\left(v^{6} \frac{m_{t}^{3}}{\Lambda^{3}}\right)\right]
\end{aligned}
$$

where the coefficients $\tilde{C}_{V / A}^{(n), \text { kin, } 0}$ contribute at the relative parametric order $\Lambda^{2} / m_{t}^{2}$ compared to the plain $S$-wave result.

Here, we also give results originating from the subleading terms in the $\mathbf{p}^{2} / m_{t}^{2}$ expansion "two $\mathbf{p}^{4} /\left(8 m_{t}^{3}\right)$ insertions," which are not explicitly written down in the formula for $\Delta^{\text {kin }}$ above, and the result originating from the higher order kinetic energy correction $\mathbf{p}^{6} /\left(16 m_{t}^{3}\right)$ contained in Eq. (1.3), but not explicitly written down. The result for two $\mathbf{p}^{4} /\left(8 m_{t}^{3}\right)$ insertions reads

$$
\begin{aligned}
& \operatorname{Im} G^{\mathrm{kin}, 0,\left[2 \times \mathbf{p}^{4} /\left(8 \mathrm{~m}_{t}^{3}\right)\right]}= \operatorname{Im}\left[\frac{63 m_{t}^{2}}{512 \pi} i v^{5}\right] \\
& \begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \mathrm{kin}, 0,\left[2 \times \mathbf{p}^{4} /\left(8 \mathrm{~m}_{t}^{3}\right)\right]}(\Lambda)= \\
&= 2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{63 m_{t}^{2}}{512 \pi} i\left[\frac{53}{252 \sqrt{2} \pi} \frac{\Gamma_{t} \Lambda^{3}}{m_{t}^{4}}\right. \\
&+\left(\frac{5(754-483 \sqrt{2} \operatorname{arsinh}(1))}{1176 \pi^{2}} \frac{\Gamma_{t}^{2} \Lambda}{m_{t}^{3}}+\frac{115}{28 \sqrt{2} \pi} \frac{E \Gamma_{t} \Lambda}{m_{t}^{3}}\right) \\
&\left.+\mathcal{O}\left(v^{6} \frac{m_{t}}{\Lambda}\right)\right]
\end{aligned}
\end{aligned}
$$

One $\mathbf{p}^{6} /\left(16 m_{t}^{3}\right)$ insertion leads to

$$
\begin{aligned}
& \operatorname{Im} G^{\mathrm{kin}, 0,\left[1 \times \mathbf{p}^{6} /\left(16 \mathrm{~m}_{t}^{5}\right)\right]}= \operatorname{Im}\left[-\frac{7 m_{t}^{2}}{64 \pi} i v^{5}\right] \\
& \begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \operatorname{kin}, 0,\left[1 \times \mathbf{p}^{6} /\left(16 \mathrm{~m}_{t}^{5}\right)\right]}(\Lambda)= \\
&= 2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{-7 m_{t}^{2}}{64 \pi} i\left[\frac{11}{42 \sqrt{2} \pi} \frac{\Gamma_{t} \Lambda^{3}}{m_{t}^{4}}\right. \\
&+\left(\frac{514-315 \sqrt{2} \operatorname{arsinh}(1)}{140 \pi^{2}} \frac{\Gamma_{t}^{2} \Lambda}{m_{t}^{3}}+\frac{9}{2 \sqrt{2} \pi} \frac{E \Gamma_{t} \Lambda}{m_{t}^{3}}\right) \\
&\left.+\mathcal{O}\left(v^{6} \frac{m_{t}}{\Lambda}\right)\right] .
\end{aligned}
\end{aligned}
$$

These corrections are of relative parametric order $\Lambda^{4} / m_{t}^{4}$ compared to the plain $S$-wave result.

### 4.4.3 Time Dilatation Corrections to $S$-Wave

The time dilatation corrections of the form $-i \Gamma_{t} \mathbf{p}^{2} /\left(4 m_{t}^{2}\right)$ in Eq. (1.3) contained in the cut graphs in Fig. 4.3 (c) are obtained by the application of the cutting rule

$$
\frac{i}{\frac{E}{2} \pm p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\left(1-\frac{\mathbf{p}^{2}}{2 m_{t}^{2}}\right)} \rightarrow-2 \operatorname{Im}\left[\frac{1}{\frac{E}{2} \pm p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\left(1-\frac{\mathbf{p}^{2}}{2 m_{t}^{2}}\right)}\right]
$$

to the propagators in the uncut graph. We get

$$
\begin{align*}
C_{4.3(\mathrm{a})}(\Lambda)+C_{4.3(\mathrm{c})}(\Lambda) & =\int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)}\left[1+\Delta^{\mathrm{dil}}\right] \\
\Delta^{\mathrm{dil}} & =\frac{\mathbf{p}^{2}}{m_{t}^{2}} \frac{m_{t}^{4} \Gamma_{t}^{4}-t_{1}^{2} t_{2}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)} \tag{4.21}
\end{align*}
$$

The correction $\Delta^{\text {dil }}$ contains only the leading terms in the $\mathbf{p}^{2} / m_{t}^{2}$ expansion. Using again partial fraction the integral is split into the basic integrals. We give the explicit expression,

$$
\begin{aligned}
C_{4.3(\mathrm{c})}(\Lambda)=\frac{i \Gamma_{t}}{16 \pi^{3}} & {\left[-\left(m_{t} E+i m_{t} \Gamma_{t}\right)\left(I_{0}^{1020}+I_{0}^{2010}\right)\right.} \\
& +\left(m_{t} E-i m_{t} \Gamma_{t}\right)\left(I_{0}^{0102}+I_{0}^{0201}\right) \\
& \left.+m_{t} E\left(I_{0}^{0120}+I_{0}^{2001}-I_{0}^{0210}-I_{0}^{1002}\right)+I_{0}^{1010}-I_{0}^{0101}\right] .
\end{aligned}
$$

Due to symmetry relations between several of the basic integrals, we need to compute only $I_{0}^{1010}, I_{0}^{2010}$ and $I_{0}^{2001}$.

Finally, our result reproduces the non-local $\mathcal{O}\left(\alpha_{s}^{0}\right)$ term $2 \operatorname{Im} G^{\text {dil,0 }}$ in Eq. (D.4). $G^{\text {dil,0 }}$ is pictorially shown in Fig. 4.4 (b). We obtain

$$
\begin{aligned}
\operatorname{Im} G^{\text {dil }, 0}= & \operatorname{Im}\left[\frac{3 m_{t} \Gamma_{t}}{16 \pi} v\right] \quad \text { (reproduced) } \\
\sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n) \text { dil }, 0}(\Lambda)= & 2 N_{c}\left(C_{V / A}^{\text {born }}\right)^{2} \frac{3 m_{t}^{2}}{16 \pi} i\left[-\frac{2 \sqrt{2}}{3 \pi} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}\right. \\
& +\left(\frac{4(2-3 \sqrt{2} \operatorname{arsinh}(1))}{3 \pi^{2}} \frac{\Gamma_{t}^{2}}{m_{t} \Lambda}+\frac{2 \sqrt{2}}{\pi} \frac{E \Gamma_{t}}{m_{t} \Lambda}\right) \\
& \left.+\mathcal{O}\left(v^{6} \frac{m_{t}^{3}}{\Lambda^{3}}\right)\right] .
\end{aligned}
$$

Therefore the coefficients $\tilde{C}_{V / A}^{(n) \text { dil,0 }}$ contribute at the relative parametric order $\Lambda^{2} / m_{t}^{2}$ compared to the plain $S$-wave case.

### 4.4.4 $P$-Wave and Subleading $S$-Wave

The correlator of leading $\left(\mathcal{O}_{\mathbf{p}, 1}^{j}\right)$ and subleading $\left(\mathcal{O}_{\mathbf{p}, 2}^{j}\right)$ vector currents, " $v^{2}$ suppressed $S$-wave," shown in Fig. 4.3 (d) and the axial-vector correlator in Fig. 4.3 (e) lead to essentially the same analytic expressions, $C_{4.3(\mathrm{e})}=m_{t}^{2} C_{4.3 \text { (d) }}$. We obtain

$$
\begin{align*}
C_{4.3(\mathrm{a})}(\Lambda)+C_{4.3(\mathrm{~d})}(\Lambda) & =\int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)}\left[1+\Delta^{v^{2}}\right], \\
\Delta^{v^{2}} & =\frac{\mathbf{p}^{2}}{m_{t}^{2}}, \quad \Delta^{P-\text { wave }}=\mathbf{p}^{2} . \tag{4.22}
\end{align*}
$$

Our result reproduces the terms $2 \operatorname{Im}\left[v^{2} G^{0,0}\right]=2 \operatorname{Im}\left[G^{1,0} / m_{t}^{2}\right]$, where $v^{2} G^{0,0}$ and $G^{1,0}$ are the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ terms (shown in Figs. $\left.4.4(\mathrm{~d}, \mathrm{e})\right)$ that can be obtained using Eqs. (D.1, D.2, 1.27). It reads

$$
\begin{aligned}
& \operatorname{Im}\left[v^{2} G^{0,0}\right]=\operatorname{Im}\left[G^{1,0} / m_{t}^{2}\right]=\operatorname{Im}\left[\frac{m_{t}^{2}}{4 \pi} i v^{3}\right] \quad \text { (reproduced) } \\
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), v^{2}, 0}(\Lambda)= 2 N_{c} 2 C_{V / A, 1}^{\text {born }} C_{V / A, 2}^{\text {born }} \frac{m_{t}^{2}}{4 \pi} i\left[\frac{\sqrt{2}}{\pi} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}\right. \\
&+\left(\frac{-2+3 \sqrt{2} \operatorname{arsinh}(1)}{\pi^{2}} \frac{\Gamma_{t}^{2}}{m_{t} \Lambda}-\frac{3 \sqrt{2}}{\pi} \frac{E \Gamma_{t}}{m_{t} \Lambda}\right) \\
&\left.+\mathcal{O}\left(v^{6} \frac{m_{t}^{5}}{\Lambda^{5}}\right)\right], \\
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), P-\text { wave }, 0}(\Lambda)= \frac{4}{3} N_{c}\left(C_{V / A, 3}^{\text {born }}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[\frac{\sqrt{2}}{\pi} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}\right. \\
&+\left(\frac{-2+3 \sqrt{2} \operatorname{arsinh}(1)}{\pi^{2}} \frac{\Gamma_{t}^{2}}{m_{t} \Lambda}-\frac{3 \sqrt{2}}{\pi} \frac{E \Gamma_{t}}{m_{t} \Lambda}\right) \\
&\left.+\mathcal{O}\left(v^{6} \frac{m_{t}^{5}}{\Lambda^{5}}\right)\right],
\end{aligned}
$$

where $C_{V / A, 2}^{\mathrm{born}}=-1 / 6 C_{V / A, 1}^{\mathrm{born}}$. Therefore the coefficients $\tilde{C}_{V / A}^{(n), v^{2}, 0}$ and $\tilde{C}_{V / A}^{(n), P \text {-wave } 0}$ contribute at the relative parametric order $\Lambda^{2} / m_{t}^{2}$ compared to the plain $S$-wave result.

### 4.4.5 Interferences

Last we consider the diagrams in Fig. 4.3 (f) containing one propagator cut and one vertex cut. They correspond to the interferences of full theory doubleresonant and single-resonant diagrams shown in Fig. 3.1. We derive the according amplitudes from the non-relativistic limit of these full theory diagrams.

We obtain four contributions that correspond to the four possible ways to cut one propagator and one vertex in the associated effective theory diagram. Their sum reads

$$
C_{4.3(\mathrm{f})}=C_{4.3 \text { (f) }}^{\mathrm{left}, \mathrm{pp}}+C_{4.3(\mathrm{f})}^{\mathrm{left}, \mathrm{dn}}+C_{4.3(\mathrm{f})}^{\mathrm{right}, \mathrm{up}}+C_{4.3(\mathrm{f})}^{\mathrm{right}, \mathrm{dn}},
$$

where "left" and "right" indicate which vertex is cut and "up" and "dn" indicate whether the upper or the lower propagator is cut, respectively. They have the explicit forms

$$
\begin{align*}
C_{4.3(\mathrm{f})}^{\text {left,up }}(\Lambda)= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\Gamma_{t}}{\left(\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}+p_{0}-\frac{\mathrm{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \\
& \times(-i) \cdot \frac{-i}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}}, \\
C_{4.3(\mathrm{f})}^{\text {right,up }}(\Lambda)= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\Gamma_{t}}{\left(\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \\
& \times i \cdot \frac{i}{\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} . \tag{4.23}
\end{align*}
$$

The two other cuts are obtained from these by the replacement $p_{0} \rightarrow-p_{0}$ and have the same value, i. e. $C_{4.3(\mathrm{f})}^{\mathrm{left/} / \mathrm{fight} \text {,up }}=C_{4.3(\mathrm{f})}^{\mathrm{left} / \text { right,dn }}$, because the integration region is symmetric under this transformation. Apart from that we note that $C_{4.3(\mathrm{f})}^{\text {left, } \mathrm{mp} / \mathrm{dn}}=\left(C_{4.3(\mathrm{f})}^{\text {right } \mathrm{mp} / \mathrm{dn}}\right)^{*}$.

After change of variables this yields

$$
\begin{align*}
C_{4.3(\mathrm{f})}(\Lambda) & =\int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)} \Delta^{\mathrm{int}} \\
\Delta^{\mathrm{int}} & =-\frac{t_{1}+t_{2}}{m_{t} \Gamma_{t}} \tag{4.24}
\end{align*}
$$

Our result for $C_{4.3(\mathrm{f})}(\Lambda)$ contains the term $4 \operatorname{Re} G^{0,0}$ and therefore reproduces
the contribution given in Eq. (3.6). We find

$$
\begin{aligned}
\sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \text { int }, 0}(\Lambda)=2 & N_{c} 2 i C_{V / A, 1}^{\mathrm{born}} C_{V / A, 1}^{\mathrm{int}}\left[\frac{m_{t} \Lambda}{2 \sqrt{2} \pi^{2}}\right. \\
& +\left(-\frac{m_{t}^{2} E}{2 \sqrt{2} \pi^{2} \Lambda}+\frac{(-2+3 \sqrt{2} \operatorname{arsinh}(1)) m_{t}^{2} \Gamma_{t}}{4 \pi^{3} \Lambda}\right) \\
& \left.+\mathcal{O}\left(v^{4} \frac{m_{t}^{5}}{\Lambda^{3}}\right)\right]
\end{aligned}
$$

We note that the first term in the $\Lambda / m_{t}$ expansion is not proportional to $\Gamma_{t}$ in the case of the interference phase space corrections. The role of $\Gamma_{t}$ to characterize a correction as an instability effect is here played by the coefficients $C_{V / A, 1}^{\text {int }}$. These also bring the additional power of $\alpha$ that was delivered by $\Gamma_{t}$ in the corrections considered above, and we have the counting

$$
\begin{equation*}
C_{V / A, 1}^{\mathrm{int}} \sim C_{V / A, 1}^{\mathrm{born}} \frac{\Gamma_{t}}{m_{t}} \tag{4.25}
\end{equation*}
$$

Thus, this result is again of the relative parametric order $\Lambda^{2} / m_{t}^{2}$ compared to the plain $S$-wave.

### 4.5 Two-Loop Renormalization

Before we carry out the two-loop renormalization of the forward scattering operators we will determine the renormalization of the production currents (vertex renormalization) originating from one-loop graphs. This is in analogy to the usual renormalization procedure associated with two-loop vacuum diagrams, where subdivergences are removed by vertex counterterms that are obtained from one-loop diagrams.

### 4.5.1 One-Loop Current Renormalization

The finite imaginary renormalization of the currents is reflected in the correction terms $\delta \tilde{c}_{i}(\Lambda)$ and $\delta \tilde{c}_{1}^{\text {int }}(\Lambda)$ in Eqs. (1.17). They have the form

$$
\begin{equation*}
i \delta \tilde{c}_{i}(\Lambda)=\sum_{j} i \delta \tilde{c}_{i}^{j}(\Lambda), \quad i \delta \tilde{c}_{1}^{\mathrm{int}}(\Lambda)=\sum_{j} i \delta \tilde{c}_{1}^{\mathrm{int}, j}(\Lambda) \tag{4.26}
\end{equation*}
$$

where the index $i=1,2,3$ indicates the type of current and $j$ the source of the correction. Because they enter the cross section as factors multiplying the Green functions and they are associated with at least one power of $\Gamma_{t}$ or $C_{V / A, 1}^{\mathrm{int}}$ and with one power of $\alpha_{s}$, their contribution to the cross section starts at $\mathrm{N}^{3} \mathrm{LO}$. We will

(a)

(b)

(c)

Figure 4.6: Cuts through effective theory $\mathcal{O}\left(\alpha_{s}^{1}\right)$ diagrams associated with a $t_{1}$ and $t_{2}$ integration up to $\Lambda^{2}$. In this illustration the crossed vertex indicates topantitop production in an $S$-wave or $P$-wave state. The dot indicates a potential interaction. The cross in the quark propagator in graph (b) indicates one of the corrections shown in Figs. 4.3 (b) - (e). Other possible placements of the cross are included.


Figure 4.7: Cut full theory $e^{+} e^{-} \rightarrow t \bar{t}$ diagram associated with $b W^{+} \bar{b} W^{-}$final states involving one gluon exchange.
restrict the following calculations to this order. In particular, we do not consider additional factors of $m_{t} E / \Lambda^{2}$ that lead to corrections starting at $N^{5} \mathrm{LO}$. These would imply a renormalization of the operators $\mathcal{O}_{V / A, \mathbf{p}, 1}^{(1)}$ in Eq. (1.14) and the according operators for more powers of $m_{t} E / \Lambda^{2}$.

As a simple example we consider the one-loop diagram describing $t \bar{t}$ production via the dominant $S$-wave current, shown in Fig 4.6 (a), where the interaction is due to the Coulomb potential in Eq. (1.4),

$$
\tilde{V}_{c}^{(s)}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)=\frac{\mathcal{V}_{c}^{(s)}}{\left(\mathbf{p}-\mathbf{p}^{\prime}\right)^{2}}
$$

This vertex correction corresponds to the leading contribution of the full theory diagram shown in Fig. 4.7 in the non-relativistic limit. By an explicit calculation of the full theory amplitude in this limit we find the expression

$$
\begin{align*}
C_{4.6(\mathrm{a})}^{c}(\Lambda)= & \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\Gamma_{t}}{\left(\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}+p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \\
& \times \frac{\Gamma_{t}}{\left(\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}\right)\left(\frac{E}{2}-p_{0}-\frac{\mathbf{p}^{2}}{2 m_{t}}-i \frac{\Gamma_{t}}{2}\right)} \frac{-i \mathcal{V}_{c}^{(s)}}{\left(\mathbf{p}-\mathbf{p}^{\prime}\right)^{2}}, \tag{4.27}
\end{align*}
$$

where $\pm \mathbf{p}^{\prime}$ is the 3-momentum of the top and the antitop quark in the $\mathrm{c} . \mathrm{m}$. frame, respectively, if the quarks are on-shell. For the determination of the current renormalization it is sufficient to consider on-shell quarks and therefore we can set $\mathbf{p}^{\prime 2}=m_{t} E, E>0$. In Eq. (4.27) we suppressed the spinors for top-antitop production.

By carrying out the angular integration and changing variables we obtain

$$
C_{4.6(\mathrm{a})}^{c}(\Lambda)=\frac{1}{(2 \pi)^{4}} \frac{1}{8 m_{t}} \iint_{\Delta(\Lambda)} d t_{1} d t_{2} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2} p}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)} \Delta^{c},
$$

where

$$
\tilde{\Delta}(\Lambda)=\left\{\left(t_{1}, t_{2}\right) \in \mathbb{R}^{2}:\left|t_{1,2}\right|<\Lambda^{2}\right\}
$$

and

$$
\Delta^{c}=-i \mathcal{V}_{c} \frac{2 \pi}{p \sqrt{m_{t} E}} \ln \left|\frac{\sqrt{m_{t} E}+p}{\sqrt{m_{t} E}-p}\right|
$$

and we used the abbreviation $p=\sqrt{m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)}$. Written in the form of a basic integral and applying $\mathcal{V}_{c}^{(s)}=-4 \pi C_{F} \alpha_{s} \equiv-4 \pi a$, we obtain the result

$$
\begin{aligned}
C_{4.6(\mathrm{a})}^{c}(\Lambda)= & 4 \pi i a \frac{\left(2 m_{t}\right)^{4}}{(2 \pi)^{4}} \frac{\Gamma_{t}^{2}}{8 m_{t}} \frac{2 \pi}{\sqrt{m_{t} E}} \operatorname{Im}\left[I_{1}^{1111,0}\right] \\
= & i a\left[\sqrt{\frac{m_{t}}{E}} \operatorname{Re}\left[\ln \frac{m_{t} v+\sqrt{m_{t} E}}{m_{t} v-\sqrt{m_{t} E}}\right]-\frac{8 \sqrt{2}}{3 \pi} \frac{m_{t}^{2} \Gamma_{t}}{\Lambda^{3}}\right. \\
& \left.+\mathcal{O}\left(v^{4} \frac{m_{t}^{5}}{\Lambda^{5}}\right)\right] .
\end{aligned}
$$

In the limit $\Lambda \rightarrow \infty$ only the logarithmic term remains. We obtain exactly this term from the uncut one-loop diagram calculated in dimensional regularization if we use the optical theorem: For the uncut one-loop diagram in Fig. 4.8 we have the expression

$$
\begin{align*}
D^{c}\left(\left|\mathbf{p}^{\prime}\right|\right) & =\tilde{\mu}^{2 \epsilon} \int \frac{d^{d} p}{(2 \pi)^{d}} \frac{i}{\frac{E}{2}+p_{o}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \frac{i}{\frac{E}{2}-p_{o}-\frac{\mathbf{p}^{2}}{2 m_{t}}+i \frac{\Gamma_{t}}{2}} \frac{-i \mathcal{V}_{c}^{(s)}}{\left(\mathbf{p}-\mathbf{p}^{\prime}\right)^{2}} \\
& =a \frac{m_{t}}{2\left|\mathbf{p}^{\prime}\right|} i \ln \frac{m_{t} v+\left|\mathbf{p}^{\prime}\right|}{m_{t} v-\left|\mathbf{p}^{\prime}\right|} \tag{4.28}
\end{align*}
$$



Figure 4.8: One-loop vertex correction diagram.


Figure 4.9: One-loop renormalization condition for the Wilson coefficient of a $t \bar{t}$ production operator.
and therefore we obtain the optical theorem

$$
C_{4.6(\mathrm{a})}^{c}(\infty)=2 i \operatorname{Im}\left[D^{c}\left(\sqrt{m_{t} E}\right)\right]
$$

The difference $C_{4.6(\mathrm{a})}^{c}(\Lambda)-2 i \operatorname{Im}\left[D^{c}\left(\sqrt{m_{t} E}\right)\right]$ contains the corrections coming from the phase space cut and gives a finite renormalization to the current operator. The result reads

$$
\begin{align*}
i \delta \tilde{c}_{1}^{c, 1}(\Lambda) & =\frac{1}{2}\left(C_{4.6(\mathrm{a})}^{c}(\Lambda)-2 i \operatorname{Im}\left[D^{c}\left(\sqrt{m_{t} E}\right)\right]\right) \\
& =-i a \frac{4 \sqrt{2}}{3 \pi} \frac{m_{t}^{2} \Gamma_{t}}{\Lambda^{3}}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{5}}{\Lambda^{5}}\right) \tag{4.29}
\end{align*}
$$

The factor $1 / 2$ is required to remove the factor 2 that appears in the optical theorem. In Fig. 4.9 the renormalization condition for the Wilson coefficient of a current operator is illustrated. We note that the coefficient $a=C_{F} \alpha_{s}$ appearing in the matching conditions is evaluated at the hard scale $\nu=1$.

The according contributions to the leading vector current renormalization originating from the diagram in Fig. 4.6 (a) with insertions of the other relevant potentials in Eq. (1.4),

$$
\frac{\mathcal{V}_{r}^{(s)}\left(\mathbf{p}^{2}+\mathbf{p}^{\prime 2}\right)}{2 m_{t}^{2} \mathbf{k}^{2}}, \frac{\mathcal{V}_{s}^{(s)}}{m_{t}^{2}},
$$

is given in the following. We note that due to Eq. (1.5) the potential $\mathcal{V}_{k}^{(s)}$ is $\alpha_{s}$-suppressed compared to the other potentials and therefore we do not take it into account here. Due to Eqs. (1.6) the potential $\mathcal{V}_{2}^{(s)}$ is zero at the hard scale,
and thus does not contribute, too. We obtain

$$
\begin{aligned}
& i \delta \delta_{1}^{r, 1}(\Lambda)=-i a \frac{\sqrt{2}}{\pi} \frac{\Gamma_{t}}{\Lambda}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right) \\
& i \delta \delta_{1}^{s, 1}(\Lambda)=i a \frac{4 \sqrt{2}}{3 \pi} \frac{\Gamma_{t}}{\Lambda}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right)
\end{aligned}
$$

where we used $\mathcal{V}_{r}^{(s)}(\nu=1)=-4 \pi a$ and $\mathcal{V}_{s}^{(s)}(\nu=1)=(4 \pi / 3) a$ according to Eq. (1.6). We note that because the potential $\mathcal{V}_{s}^{(s)}$ is momentum-independent, the calculation of $\delta \delta_{1}^{s, 1}$ can be reduced to the calculation of the cut one-loop plain $S$-wave diagram in Fig. 4.3 (a).

The computation of the diagrams in Fig. 4.6 (b) where the potential is the Coulomb potential and the cross in the propagator indicates one of the corrections shown in Figs. $4.3(\mathrm{~b})-(\mathrm{e})$ is done in a similar way. In the case of the $P$-wave current $\mathcal{O}_{\mathrm{p}, 3}^{i}$ only one of the angular integrations is trivial because the $\mathbf{p}$ term breaks down spherical symmetry to cylindrical symmetry. We obtain

$$
\begin{aligned}
i \delta \tilde{c}_{1}^{\mathrm{kin}, 1}(\Lambda) & =-i a \frac{7}{4 \sqrt{2} \pi} \frac{\Gamma_{t}}{\Lambda}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right), \\
i \delta \tilde{c}_{1}^{\mathrm{dil}, 1}(\Lambda) & =i a \frac{\sqrt{2}}{\pi} \frac{\Gamma_{t}}{\Lambda}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right), \\
i \delta \delta_{1}^{v^{2}, 1}(\Lambda) & =-i a \frac{2 \sqrt{2}}{\pi} \frac{\Gamma_{t}}{\Lambda}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right), \\
i \delta \tilde{c}_{3}^{P-\text { wave }, 1}(\Lambda) & =0+\mathcal{O}\left(a v^{4} \frac{m_{t}^{3}}{\Lambda^{3}}\right) .
\end{aligned}
$$

Note that the cut diagram in Fig. 4.6 (b) where the insertion is $\mathbf{p}^{2} / m_{t}^{2}$ originating from the current operators $\mathcal{O}_{V / A, \mathbf{p}, 2}$ gives rise to a renormalization of the operators $\mathcal{O}_{V / A, \mathbf{p}, 1}$. The interference diagrams in Fig. 4.6 (c) with a Coulomb potential inserted yields

$$
i \delta \delta^{\mathrm{c}^{\mathrm{int}, 1}}(\Lambda)=-i a \frac{2 \sqrt{2}}{\pi} \frac{m_{t}}{\Lambda}+\mathcal{O}\left(a v^{2} \frac{m_{t}^{3}}{\Lambda^{3}}\right) .
$$

We note that all of these corrections are of relative order $\Lambda^{2} / m_{t}^{2}$ compared to the result for the dominant $S$-wave current renormalization in Eq. (4.29). In the case of the interference correction this is true because again we can use Eq. (4.25).

All of the terms $i \delta \tilde{c}_{i}^{j}$ contribute to the imaginary parts of the Wilson coefficients of the production currents according to Eq. (1.15). A crucial point to note here is that we obtain the same coefficients for the according annihilation currents. This was already the case in Eqs. $(3.4,3.5)$ where we had absorptive parts coming from $b W$ cuts through full theory diagrams where no phase space cut was applied.

### 4.5.2 Two-Loop Renormalization of Forward Scattering Operators


(a)

(d)

(g)

(b)

(e)

(h)

(c)

(f)

(i)

(j)

Figure 4.10: Cuts through effective theory $\mathcal{O}\left(\alpha_{s}^{1}\right)$ two-loop diagrams associated with a $t_{1}$ and $t_{2}$ integration up to $\Lambda^{2}$ for the cut loop and an integration in dimensional regularization for the uncut loop. The crossed vertex indicates top-antitop production via the dominant $S$-wave current. The dot indicates a potential. The crosses in the quark propagators in graphs (c)-(h) indicate kinetic energy insertions. The graphs (c)-(j) include those where the insertions are put into the antitop lines or the antitop is cut, respectively. Similar diagrams corresponding to the other corrections shown in Fig. 4.3 (c) - (e) must be taken into account for two-loop renormalization, but are not displayed here.

For the two-loop renormalization of the forward scattering operators we have to compute graphs as shown in Figs. 4.10. We start with diagrams (a) and (b). The subdiagram in graph (a) not containing the cut was already computed in Eq. (4.28). In analogy to Eq. (4.17) we obtain the integral

$$
\begin{equation*}
C_{4.10(\mathrm{a})}(\Lambda)=\int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4} \Gamma_{t}^{2}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)} D^{c}(p) \tag{4.30}
\end{equation*}
$$

for graph (a). Because in graph (b) the potential is on the RHS of the cut, we have $C_{4.10(\mathrm{~b})}=\left(C_{4.10(\mathrm{a})}\right)^{*}$ and obtain

$$
\begin{aligned}
C_{4.10(\mathrm{a})}(\Lambda)+C_{4.10(\mathrm{~b})}(\Lambda)=2 \frac{m_{t}^{2}}{4 \pi} & {\left[-a \operatorname{Im}[\ln (-i v)]-2 a \frac{m_{t} \Gamma_{t}}{\Lambda^{2}}\right.} \\
& \left.-a \frac{8 \sqrt{2}}{3 \pi} \frac{m_{t}^{2} \Gamma_{t}}{\Lambda^{3}} \operatorname{Re}[i v]+\mathcal{O}\left(a v^{4} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right]
\end{aligned}
$$

for the sum of both graphs. As we did in the one-loop renormalization procedure we analyze this expression with respect to non-local terms. The $\Lambda$-independent non-local term is accounted for by the imaginary part

$$
\operatorname{Im}\left[G^{0,1}\right]=\operatorname{Im}\left[-\frac{m_{t}^{2}}{4 \pi} a \ln (-i v)\right]
$$

of the $\mathcal{O}\left(\alpha_{s}^{1}\right)$ term of the dominant $S$-wave Green function $G^{0}$ in Eq. (D.1). The second term renormalizes the coefficients $\tilde{C}_{V / A, 1}$ of the forward scattering operators.

We are left with the second non-local term $\sim \operatorname{Re}[i v]$. We find that it is equal to the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ contribution

$$
2 \operatorname{Im}\left[2 i \delta \delta_{1}^{c, 1} G^{0,0}\right]
$$

and therefore is accounted for by the one-loop diagram in Fig. 4.4 (a), where one of the currents is multiplied by the imaginary renormalization correction obtained in Eq. (4.29). In Fig. 4.11 we illustrate the mechanism of the operator product expansion at two-loop level.

Hence, we obtain for the coefficients $\tilde{C}_{V / A}^{0,1}$ of the operators $\tilde{\mathcal{O}}_{V / A}$

$$
\tilde{C}_{V / A}^{0,1}(\Lambda)=2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[-2 a \frac{m_{t} \Gamma_{t}}{\Lambda^{2}}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right] .
$$

They contribute to the cross section at relative order $\mathcal{O}\left(\pi a m_{t} / \Lambda\right)$ compared to the plain $S$-wave result. Therefore they constitute a parametric NNLO contribution.

The computation of the other diagrams in Fig. 4.10 and the remaining diagrams associated with time dilatation, the $v^{2}$-suppressed $S$-wave current and the $P$-wave current is done in analogy. ${ }^{1}$ For every cut two-loop diagram that we cal-

[^7]

Figure 4.11: Two-loop renormalization condition for $\tilde{C}(\Lambda)$. The phase space integration on the LHS is reproduced in the effective theory by the RHS. As in the one-loop case shown in Fig. 4.5 the forward scattering operators contribute with imaginary Wilson coefficients. In the two-loop case also the current operators contribute with imaginary coefficients. As in Fig. 4.5 we slightly changed the normalization of $\tilde{C}(\Lambda)$ for this and the next illustration.
culated we reproduced the $\mathcal{O}\left(\alpha_{s}^{1}\right)$ and the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ non-local term of the associated Green function.

At this point we take a closer look at the sum of the four graphs (i) and (j) in Fig. 4.10. We find

$$
\begin{aligned}
C_{4.10(\mathrm{i})}^{\mathrm{up}}(\Lambda)+C_{4.10(\mathrm{i})}^{\mathrm{dn}}(\Lambda) & +C_{4.10(\mathrm{j})}^{\mathrm{up}}(\Lambda)+C_{4.10(\mathrm{j})}^{\mathrm{dn}}(\Lambda)= \\
=4 & \frac{m_{t}^{2}}{4 \pi}\left[-a\left(\operatorname{Re}\left[\ln \frac{-i v m_{t}}{\Lambda}\right]+\frac{3}{2} \ln 2\right)\right. \\
& \left.-a \frac{2 \sqrt{2}}{\pi} \frac{m_{t}}{\Lambda} \operatorname{Re}[i v]+\mathcal{O}\left(a v^{2} \frac{m_{t}^{2}}{\Lambda^{2}}\right)\right],
\end{aligned}
$$

and therefore it contains a cut-dependent logarithm, $\ln \left(m_{t} / \Lambda\right)$. Because we have the scaling $\Lambda \sim m_{t}$ for the matching procedure, the logarithm is small. From the technical point of view this logarithm corresponds to the $1 / \epsilon$ and $\ln (\nu)$ terms that we find for the corresponding uncut two-loop graph computed in dimensional regularization,

$$
\begin{equation*}
G^{0,1}=a \frac{m_{t}^{2}}{4 \pi}\left[\frac{1}{4 \epsilon}-\ln \left(\frac{-i v}{\nu}\right)+\frac{1}{2}-\ln 2\right] . \tag{4.31}
\end{equation*}
$$

We emphasize that the two logarithms play different roles from the strategic point of view of our treatment. While the logarithm in Eq. (4.31) is associated with the ultraviolet phase space divergences that led to the running of the coefficients



$+i \delta \tilde{c}_{1}^{\text {kin }, 1}(\Lambda)$

$+i \delta \check{c}_{1}^{c, 1}(\Lambda)$


Figure 4.12: Illustration of the operator product expansion involving the Coulomb potential and the kinetic energy insertion. The coefficient $i \delta \delta_{1}^{\text {kin, } 1}(\Lambda)$ is obtained from cut one-loop vertex diagrams with kinetic energy insertions. The coefficient $i \delta c_{1}^{c, 1}(\Lambda)$ is obtained from the cut one-loop vertex diagram without insertion. The coefficients $\tilde{C}^{[c, k i n], 1}(\Lambda)$ are obtained from the equation shown here.
$\tilde{C}_{V / A}(\nu)$, the logarithm $\ln \left(m_{t} / \Lambda\right)$ contributes to the hard matching condition $\tilde{C}_{V / A}(1)$. The same is true for the other divergences investigated in Sec. 3.3.

Finally, our results for the two-loop renormalization of $\tilde{C}_{V / A}^{(n)}$ arising from the remaining diagrams read

$$
\begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), r, 1}(\Lambda)= \\
&= 2 N_{c}\left(C_{V / A, 1}^{\mathrm{born}}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[-a\left(\ln \frac{m_{t}}{\Lambda}+\frac{1}{2}+\frac{1}{2} \ln 2\right) \frac{\Gamma_{t}}{m_{t}}\right. \\
&\left.+a\left(\frac{1}{\pi} \frac{\Gamma_{t}^{2}}{\Lambda^{2}}-\frac{2 E \Gamma_{t}}{\Lambda^{2}}\right)+\mathcal{O}\left(a v^{6} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right]
\end{aligned}
$$

$$
\sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n) s, 1}(\Lambda)=0
$$

$$
\begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \text { kin }, 1}(\Lambda)= \\
& = \\
& =2 N_{c}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[-a\left(\ln \frac{m_{t}}{\Lambda}+\frac{3}{8}+\frac{1}{2} \ln 2\right) \frac{\Gamma_{t}}{m_{t}}\right. \\
& \left.\quad+a\left(\frac{1}{\pi} \frac{\Gamma_{t}^{2}}{\Lambda^{2}}-\frac{5}{2} \frac{E \Gamma_{t}}{\Lambda^{2}}\right)+\mathcal{O}\left(a v^{6} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right] \\
& \begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \text { dil, } 1}(\Lambda)= \\
&= 2 N_{c}\left(C_{V / A, 1}^{\text {born }}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[a\left(\ln \frac{m_{t}}{\Lambda}+\frac{1}{2}+\frac{1}{2} \ln 2\right) \frac{\Gamma_{t}}{m_{t}}\right. \\
&\left.\quad a\left(\frac{2}{\pi} \frac{\Gamma_{t}^{2}}{\Lambda^{2}}-\frac{E \Gamma_{t}}{\Lambda^{2}}\right)+\mathcal{O}\left(a v^{6} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right] \\
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), v^{2}, 1}(\Lambda)= \\
&= 2 N_{c} 2 C_{V / A, 1}^{\text {born }} C_{V / A, 2}^{\text {born }} \frac{m_{t}^{2}}{4 \pi} i\left[-a\left(\ln \frac{m_{t}}{\Lambda}+\frac{1}{2}+\frac{1}{2} \ln 2\right) \frac{\Gamma_{t}}{m_{t}}\right. \\
&\left.+a\left(\frac{1}{\pi} \frac{\Gamma_{t}^{2}}{\Lambda^{2}}-\frac{2 E \Gamma_{t}}{\Lambda^{2}}\right)+\mathcal{O}\left(a v^{6} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right]
\end{aligned}
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), P-\text { wave }, 1}(\Lambda)= \\
&= \frac{4}{3} N_{c}\left(C_{V / A, 3}^{\mathrm{born}}\right)^{2} \frac{m_{t}^{2}}{4 \pi} i\left[-a\left(\ln \frac{m_{t}}{\Lambda}+\frac{2}{3}+\frac{1}{2} \ln 2\right) \frac{\Gamma_{t}}{m_{t}}\right. \\
&\left.+a\left(\frac{1}{\pi} \frac{\Gamma_{t}^{2}}{\Lambda^{2}}-\frac{2 E \Gamma_{t}}{\Lambda^{2}}\right)+\mathcal{O}\left(a v^{6} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \sum_{n}\left(E / m_{t}\right)^{n} \tilde{C}_{V / A}^{(n), \text { int }, 1}(\Lambda)= \\
&= 2 N_{c} 2 i C_{V / A, 1}^{\mathrm{born}} C_{V / A, 1}^{\mathrm{int}} \frac{m_{t}^{2}}{4 \pi}\left[-a\left(\ln \frac{m_{t}}{\Lambda}+\frac{1}{2}+\frac{1}{2} \ln 2\right)\right. \\
&\left.+a \frac{1}{\pi} \frac{m_{t} \Gamma_{t}}{\Lambda^{2}}+\mathcal{O}\left(a v^{4} \frac{m_{t}^{4}}{\Lambda^{4}}\right)\right]
\end{aligned}
$$

where $C_{V / A, 2}^{\mathrm{born}}=-1 / 6 C_{V / A, 1}^{\mathrm{born}}$. The spin-dependent, momentum-independent po-
tential $\mathcal{V}_{s}^{(s)}$ does not give a contribution. ${ }^{2}$ Except the latter one these potential corrections lead to contributions to the cross section at relative order $\mathcal{O}\left(\pi a \Lambda / m_{t}\right)$ compared to the plain $S$-wave result.

In Fig. 4.12 we illustrate the mechanism of the operator product expansion for the two-loop $S$-wave diagrams with the Coulomb potential exchange and one kinetic energy insertion. The illustration corresponds to the renormalization condition for the coefficients $\tilde{C}^{(n),[c, k i n], 1}$. Since the terms $i \delta \tilde{c}_{1}^{c, 1} G^{\text {kin,0 }}$ contribute only at $\mathcal{O}\left(\pi a v^{5} m_{t}^{3} / \Lambda^{3}\right)$, i. e. at formal $\mathrm{N}^{5} \mathrm{LO}$, it was not necessary to include such terms in our treatment. This is the reason why we implicitly used abbreviations such as $\tilde{C}^{(n), k i n, 1} \equiv \tilde{C}^{(n),[c, k i n], 1}$ throughout this section.

### 4.6 Mild Power Counting Breaking and Numerical Analysis

In this section we analyze the phase space effects derived in the last section in respect of their parametric and numerical behaviour. First of all, we observe that they are free of large logarithms that involve the decay width $\Gamma_{t}$ or the interference coefficients $C_{V / A, 1}^{\mathrm{int}}$. The only logarithm that appears is of the form $\ln \left(\Lambda / m_{t}\right)$, which is small due to the scaling $\Lambda \sim m_{t}$. We find that the limit $\Gamma_{t} \rightarrow 0$ and $C_{V / A, 1}^{\mathrm{int}} \rightarrow 0$ reproduces the (standard) NRQCD prediction not containing phase space effects.

A parametric classification is possible by counting powers of $\pi a$ and $\Lambda / m_{t}$, as shown in Tabs. 4.1. The tables contain the contributions to the cross section coming from the various phase space effects. The normalization is chosen such that the cross section originating from the leading $S$-wave Green function reads $\sigma^{\mathrm{LL}} \simeq v$. The left table contains contributions due to the forward scattering operator coefficients $\tilde{C}_{V / A}^{(0)}$, which do not contribute to the energy-dependence of the cross section. For simplicity we neglect higher orders of the $m_{t} E / \Lambda^{2}$ and the $m_{t} \Gamma_{t} / \Lambda^{2}$ expansion everywhere in this table. The right table contains corrections due to the leading terms of the current coefficients $\delta \tilde{c}_{i}$. At lowest order they contribute to the cross section through a multiplication with $G^{0,0}$ and therefore they are suppressed by $v$. Going one column to the right in the left table one obtains a factor $\pi a m_{t} / \Lambda$. This corresponds to a potential insertion. If the row is unchanged, this potential is a Coulomb potential. Due to the scaling $\Lambda \sim m_{t}$ every additional Coulomb potential brings a formal suppression by $\pi a$. This behavior at the hard scale is in contrast to the low-energy $t \bar{t}$ dynamics where every additional Coulomb exchange yields a factor $\pi a / v \sim 1\left(\right.$ where $\left.a=C_{F} \alpha_{s}\left(\mu_{S}\right)\right)$

[^8]| $\frac{1}{\pi} \frac{\Gamma_{t}}{\Lambda} \times$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{0}$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{2}$ |
| :---: | :---: | :---: | :---: |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{0}$ | $\frac{1}{\pi} \frac{\Gamma_{t}}{\Lambda}$ | $a \frac{m_{t} \Gamma_{t}}{\Lambda^{2}}$ | $\pi a^{2} \frac{m_{t}^{2} \Gamma_{t}}{\Lambda^{3}}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\frac{1}{\pi} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}$ | $a \frac{\Gamma_{t}}{m_{t}}$ | $\pi a^{2} \frac{\Gamma_{t}}{\Lambda}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{2}$ | $\frac{1}{\pi} \frac{\Gamma_{t} \Lambda^{3}}{m_{t}^{4}}$ | $a \frac{\Gamma_{t} \Lambda^{2}}{m_{t}^{3}}$ | $\pi a^{2} \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}$ |


| $\frac{1}{\pi} \frac{\Gamma_{t}}{\Lambda} \times$ | $v \frac{1}{\pi} \frac{m_{t}}{\Lambda}\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ |
| :--- | :---: |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{0}$ | $v \frac{1}{\pi} a \frac{m_{t}^{2} \Gamma_{t}}{\Lambda^{3}}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $v \frac{1}{\pi} a \frac{\Gamma_{t}}{\Lambda}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{2}$ | $v \frac{1}{\pi} a \frac{\Gamma_{t} \Lambda}{m_{t}^{2}}$ |

Table 4.1: Classification of the phase space effects with respect to their parametric contributions to the cross section assuming the scaling $\Lambda \sim m_{t}$ (in particular for the $\ln \left(m_{t} / \Lambda\right)$ terms $)$ and $m_{t} C_{V / A, 1}^{\text {int }} \sim C_{V / A, 1}^{\text {born }} \Gamma_{t}$ for the interference correction. We show only the leading terms of the $m_{t} E / \Lambda^{2}$ and $m_{t} \Gamma_{t} / \Lambda^{2}$ expansion. The normalization is chosen such that $\sigma^{\mathrm{LL}} \simeq v+\mathcal{O}\left(\alpha_{s}\right)$. The left and right table refers to the contributions due to $\tilde{C}_{V / A}^{(0)}$ and $\delta \tilde{c}_{i}$, respectively. Going one column to the right in the same row accounts for an additional Coulomb potential exchange. Going one row down accounts for a $\mathbf{p}^{2} / m_{t}^{2}$ insertion (associated with a bilinear insertion, a suppressed current or a suppressed potential). In this illustration we included also contributions that have not been computed.
and a Coulomb resummation is indispensable. ${ }^{3}$ Going down one row in any of the tables brings a factor $\Lambda^{2} / m_{t}^{2}$. This corresponds to a $\mathbf{p}^{2} / m_{t}^{2}$ suppression. It originates from a bilinear insertion or a subleading current or a potential other than Coulomb (except for $\mathcal{V}_{k}^{(s)}$ ). ${ }^{4}$ Due to power counting breaking of the form $\Lambda \sim m_{t}$ there is actually no formal suppression of contributions coming from all the insertions of the higher-dimensional operators. To summarize, the entries of the first column of the left table contributes at formal NLO, the second at NNLO and so on. The entries of the right table contribute at $\mathrm{N}^{3} \mathrm{LO}$.

The problem of higher-dimensional operators can be solved pragmatically by a proper choice of the numerical value of the cut $\Lambda$. Lowering the value leads to a suppression of the $\left(\Lambda / m_{t}\right)^{n}$ terms and therefore softens the power counting breaking. We will refer to this effect as "mild power counting breaking." The price one has to pay for this suppression, however, is an enhancement of the terms that contain powers of $m_{t} / \Lambda$. These powers proliferate by adding Coulomb

[^9]$$
\frac{\mathcal{V}_{k}^{(s)} \pi^{2}}{m_{t}\left|\mathbf{p}-\mathbf{p}^{\prime}\right|}
$$
in Eq. (1.4) is at relative order $\mathcal{O}\left(a^{2}\right)$ compared to the plain $S$-wave contribution.
potentials. (Adding any other potentials, on the other hand, takes away powers of $m_{t} / \Lambda$.) Our aim is to find the right balance between the desired suppression of $\Lambda / m_{t}$ terms and the required non-enhancement of ( $\pi a m_{t} / \Lambda$ ) terms.

In Tabs. 4.2, 4.3, 4.4, 4.5 the numerical values of the various phase space effects to the threshold cross section that were computed in the last section are given for different values of the cut, $\Lambda=60 \mathrm{GeV}, \Lambda=100 \mathrm{GeV}, \Lambda=140 \mathrm{GeV}$ and $\Lambda=160 \mathrm{GeV}$ and the kinetic energy, $E=0 \mathrm{GeV}, E=-5 \mathrm{GeV}$, with the parameter choices $m_{t}=172 \mathrm{GeV}, \Gamma_{t}=1.36 \mathrm{GeV}, M_{W}=80.425 \mathrm{GeV}, M_{Z}=$ $91.1876 \mathrm{GeV}, c_{w}=M_{W} / M_{Z}, \alpha=1 / 125.7$ and $a=C_{F} \alpha_{s}\left(m_{t}\right)=0.1436$. They include higher terms in the $m_{t} E / \Lambda^{2}$ and the $m_{t} \Gamma_{t} / \Lambda^{2}$ expansion up to $\mathrm{N}^{3} \mathrm{LO}$. However, these values do not include the energy-dependence originating from the intermediate photon and $Z$ boson in the $e^{+} e^{-} \rightarrow t \bar{t}$ process. For the following numerical discussion we devise the value of 5 fb as our desired accuracy of phase space effects. This corresponds to a relative uncertainty of about $0.5 \%$ compared to the current NNLL prediction of the cross section at the peak position.

As expected from the parametric counting we find that the contribution from $\tilde{C}_{V / A}^{0,0}$ is dominant as long as the cut $\Lambda$ is not too large. However, we also observe that the interference terms give rise to contributions almost as large as the dominant ones for $\Lambda=(60,100) \mathrm{GeV}$ and even larger for $\Lambda=(140,160) \mathrm{GeV}$ although they are $\mathbf{p}^{2} / m_{t}^{2}$-suppressed (i. e. $\left.\mathcal{O}\left(\Lambda^{2} / m_{t}^{2}\right)\right)$. The reason for this enhancement is the numerical relation

$$
2 C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}+2 C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{int}} \approx-4.7\left(\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\mathrm{born}}\right)^{2}\right) \frac{\Gamma_{t}}{m_{t}}
$$

which tracks down a factor larger than 4 in the scaling relation Eq. (4.25). Therefore the interference contributions also will be considered as dominant phase space effects.

Next we analyze the tables with respect to the $\Lambda^{2} / m_{t}^{2}$ and the $\pi a m_{t} / \Lambda$ expansion. As expected from the mild power counting breaking, we observe a good quality of the $\Lambda^{2} / m_{t}^{2}$ expansion for small values of $\Lambda$. Even for large values the convergence is acceptable. In particular, the terms $\tilde{C}_{V / A}^{\mathrm{kin}, 0,\left[2 \times \mathrm{pp}^{4} /\left(8 \mathrm{~m}_{\mathrm{t}}^{3}\right)\right]}$ and $\tilde{C}_{V / A}^{\mathrm{kin}, 0,\left[1 \times \mathrm{p}^{4} /\left(16 \mathrm{~m}_{t}^{3}\right)\right]}$ at $\mathcal{O}\left(\Lambda^{4} / m_{t}^{4}\right)$ give contributions already below 0.5 fb , which is one tenth of our desired precision. Since we expect the other terms at $\mathcal{O}\left(\Lambda^{4} / m_{t}^{4}\right)$ to be at a similar numerical level, we have not included them in our calculations above. This reasoning excludes $\mathcal{O}\left(\Lambda^{4} / m_{t}^{4}\right)$ terms originating from interference effects. We expect these contributions to be as important as other $\mathcal{O}\left(\Lambda^{2} / m_{t}^{2}\right)$ terms (e.g. from one kinetic energy insertion), thus at the level of few fb. Therefore it should be possible to neglect them, too, for all of our four numerical choices of the cut. However, to be conservative we set a limit of $\Lambda<120 \mathrm{GeV}$ in order to have a reliable general suppression of higher dimensional operators. As we will see in Sec. 4.8, this also guarantees a good suppression of background events.

Concerning the $\pi a m_{t} / \Lambda$ expansion, we find that its convergence is sufficient at least for $\Lambda=(140,160) \mathrm{GeV}$ for the non-dominant contributions. In these



Table 4.2: Contributions to the cross section at threshold in femtobarn originating from phase space effects.

| $\Lambda=140 \mathrm{GeV}, E=0 \mathrm{GeV}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $\frac{1}{\pi} \frac{\Gamma_{t}}{\Lambda} \times$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{0}$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ | $v \frac{1}{\pi} \frac{m_{t}}{\Lambda}\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{0}$ | $\tilde{C}_{V / A}^{0,0} \quad-16.89$ | $\tilde{C}_{V / A}^{0,1} \quad-6.64$ | $\begin{array}{lll}\delta \tilde{c}_{1}^{c} & 0.31\end{array}$ |
|  | Summed Coul.: -26.35 , Diff.: -3.13 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {int,0 }}$ | $\tilde{C}_{V / A}^{\text {int,1 }}$ 10.85 <br> din: 5.18  | $\delta \tilde{c}_{1}^{\text {int,1 }} \quad-0.72$ |
|  | Summed Coul.: -11.11 , Diff.: 5.18 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {kin,0 }}$ 3.17 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ -2.83 <br> $\tilde{C}_{V / A}^{v 2}$ -1.88 <br> $\tilde{C}_{V / A}^{P-\text { wave }, 0}$ 1.17 | $\tilde{C}_{V / A}^{\text {kin,1 }}$ -2.04 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ 2.31 <br> $\tilde{C}_{V, A}^{2,1}$ 0.77 <br> $\tilde{C}_{V / A}^{P \text { wave, }}$ -0.55 <br> $\tilde{C}_{V / A}^{r, 1}$ -2.31 <br> $\tilde{C}_{V / A}^{S, 1}$ 0 | $\delta \tilde{c}_{1}^{\text {kin }, 1}$ 0.13 <br> $\delta \tilde{c}_{1}^{\text {dil, }}$ -0.15 <br> $\delta \tilde{c}_{2}^{v^{2}, 1}$ -0.05 <br> $\delta \tilde{c}_{3}^{P \text { wave }, 1}$ 0 <br> $\delta \tilde{c}_{1}^{r, 1}$ 0.15 <br> $\delta \tilde{c}_{1}^{s, 1}$ -0.10 |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{2}$ | $\begin{array}{lr}\tilde{C}_{V / A}^{\text {kin, } 0\left[2 \times \mathbf{p}^{4} /\left(8 \mathrm{~m}_{\mathrm{t}}^{3}\right)\right]} & 0.20 \\ \tilde{C}_{V / A}^{\text {kin } 0,\left[1 \times \mathbf{p}^{6} /\left(16 \mathrm{~m}_{\mathrm{t}}^{5}\right)\right]} & -0.22\end{array}$ |  |  |



Table 4.3: Contributions to the cross section at threshold in femtobarn originating from phase space effects.

| $\Lambda=60 \mathrm{GeV}, E=-5 \mathrm{GeV}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $\frac{1}{\pi} \frac{\Gamma_{t}}{\Lambda} \times$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{0}$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{\text {I }}$ | $v \frac{1}{\pi} \frac{m_{t}}{\Lambda}\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{0}$ | $\tilde{C}_{V / A}^{0,0} \quad-35.74$ | $\tilde{C}_{V / A}^{0,1} \quad-36.14$ | $\delta \tilde{c}_{1}^{c} \quad 10.70$ |
|  | Summed Coul.: -87.72, Diff.: -26.53 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {int,0 }}$ | $\begin{array}{ll}\tilde{C}_{V / A}^{\text {int,1 }} & 19.58\end{array}$ | $\delta \mathrm{c}_{1}^{\text {int, } 1} \quad-4.58$ |
|  | Summed Coul.: 9.34, Diff.: 8.59 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {kin }}$, 2.64 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ -2.12 <br> $\tilde{C}_{V / A}^{v 2}$ -1.40 <br> $\tilde{C}_{V / A}^{P \text {-wave }, 0}$ 0.87 | $\tilde{C}_{V / A}^{\text {kin, }}$ -3.90 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ 4.18 <br> $\tilde{C}_{V / A}^{v, 1}$ 1.40 <br> $\tilde{C}_{V / A}^{P \text { wave }, 1}$ -0.94 <br> $\tilde{C}_{V / A}^{r, 1}$ -4.18 <br> $\tilde{C}_{V / A}^{s, 1}$ 0 |   <br> $\delta \tilde{c}_{1}^{\text {kin, }} 1$ 0.85 <br> $\delta \tilde{c}_{1,1}^{\text {dil }}$ -0.98 <br> $\delta \tilde{c}_{2}^{v^{2}, 1}$ -0.33 <br> $\delta \tilde{c}_{3}^{P \text { wave }, 1}$ 0 <br> $\delta \tilde{c}_{1}^{r, 1}$ 0.98 <br> $\delta \tilde{c}_{1}^{s, 1}$ -0.65 |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{2}$ |   <br> $\tilde{C}_{V / A}^{\text {kin } 0,\left[2 \times \mathbf{p}^{4} /\left(8 \mathrm{~m}_{\mathrm{t}}^{3}\right)\right]}$ -0.05 <br> $\tilde{C}_{V / A}^{\mathrm{kin} 0,\left[1 \times \mathbf{p}^{6} /\left(16 \mathrm{~m}_{\mathrm{t}}^{5}\right)\right]}$ 0.05 |  |  |


| $\Lambda=100 \mathrm{GeV}, E=-5 \mathrm{GeV}$ |  |  |  |
| :---: | :---: | :---: | :---: |
| $\frac{1}{\pi} \frac{\Gamma}{\Lambda} \times$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{0}$ | $\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ | $v \frac{1}{\pi} \frac{m_{t}}{\Lambda}\left(\pi a \frac{m_{t}}{\Lambda}\right)^{1}$ |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{0}$ | $\tilde{C}_{V / A}^{0,0} \quad-22.90$ | $\tilde{C}_{V / A}^{0,1} \quad-13.01$ | $\delta \widetilde{c}_{1}^{c} \quad 2.31$ |
|  | Summed Coul.: -41.17, Diff.: -7.57 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {int,0 }} \quad-20.57$ | $\tilde{C}_{V / A}^{\text {int,1 }} \quad 14.32$ | $\delta \tilde{c}_{1}^{\text {int,1 }} \quad-2.75$ |
|  | Summed Coul.: -2.25 , Diff.: 6.75 |  |  |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{1}$ | $\tilde{C}_{V / A}^{\text {kin,0 }}$ 3.03 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ -2.56 <br> $\tilde{C}_{V / A}^{v 2}$ -1.69 <br> $\tilde{C}_{V / A}^{P-\text { wave }, 0}$ 1.05 | $\tilde{C}_{V / A}^{\text {kin, }}$ -2.78 <br> $\tilde{C}_{V / A}^{\text {dil, }}$ 3.05 <br> $\tilde{C}_{V / A}^{v, 1}$ 1.02 <br> $\tilde{C}_{V / A}^{P \text { wave }, 1}$ -0.71 <br> $\tilde{C}_{V / A}^{r, 1}$ -3.05 <br> $\tilde{C}_{V / A}^{s, 1}$ 0 |   <br> $\delta \tilde{c}_{1}^{\text {kin, }}$ 0.51 <br> $\delta \tilde{c}_{1}^{\text {dil }}, 1$ -0.59 <br> $\delta \tilde{c}_{2}^{v^{2}, 1}$ -0.20 <br> $\delta \tilde{c}_{3}^{P \text {-wave }, 1}$ 0 <br> $\delta \tilde{c}_{1}^{r, 1}$ 0.59 <br> $\delta \tilde{c}_{1}^{s, 1}$ -0.39 |
| $\left(\frac{\Lambda^{2}}{m_{t}^{2}}\right)^{2}$ | $\tilde{C}_{V / A}^{\text {kin } 0,\left[2 \times \mathbf{p}^{4} /\left(8 \mathrm{~m}_{\mathrm{t}}^{3}\right)\right]}$ -0.05 <br> $\tilde{C}_{V / A}^{\text {kin } 0,\left[1 \times \mathbf{p}^{6} /\left(16 \mathrm{~m}_{\mathrm{t}}^{5}\right)\right]}$ 0.03 |  |  |

Table 4.4: Contributions to the cross section below threshold, $E=-5 \mathrm{GeV}$, in femtobarn originating from phase space effects.



Table 4.5: Contributions to the cross section below threshold, $E=-5 \mathrm{GeV}$, in femtobarn originating from phase space effects.
cases the terms $\tilde{C}_{V / A}^{\text {kin }, 1}, \tilde{C}_{V / A}^{\text {dil, }}, \tilde{C}_{V / A}^{v^{2}, 1}, \tilde{C}_{V / A}^{P \text {-wave, } 1}, \tilde{C}_{V / A}^{r, 1}, \tilde{C}_{V / A}^{s, 1}$ and the according current coefficients give rise to corrections already below the 5 fb level and therefore it was not necessary to calculate $\mathcal{O}\left(\alpha_{s}^{2}\right)$ contributions. Since we have excluded the values $\Lambda=(140,160) \mathrm{GeV}$ before, we will now concentrate on the $\Lambda<120 \mathrm{GeV}$ region. Here it is not guaranteed that the $\pi a$ suppression is strong enough to reduce the $m_{t} / \Lambda$ enhancement sufficiently. In fact, we observe that the $\mathcal{O}\left(\alpha_{s}^{1}\right)$ interference term gives a larger contribution than the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ interference term, and thus we cannot anticipate in which way the series evolves beyond $\mathcal{O}\left(\alpha_{s}\right)$. For this reason we will carry out an exact (numerical) phase space integration including all Coulomb rungs and compare it to our analytic prediction that contains only the zeroth and the first Coulomb rung $\left(\mathcal{O}\left(\alpha_{s}^{0}\right)\right.$ and $\left.\mathcal{O}\left(\alpha_{s}^{1}\right)\right)$. This comparison will be done for the dominant terms $\tilde{C}_{V / A}^{0,0}, \tilde{C}_{V / A}^{0,1}$ and $\tilde{C}_{V / A}^{\text {int,0 }}, \tilde{C}_{V / A}^{\text {int, }}$.

The summation of the Coulomb rungs is contained in the solution $\tilde{G}_{v, m_{t}, \nu}^{0}\left(\mathbf{p}, \mathbf{p}^{\prime}\right)$ of the leading order version of the Schrödinger equation (1.24). As we already showed in Sec. 4.2, the physical phase space up to the scale $\Lambda$ is contained in $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$ given in Eq. (4.12) and $\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)$ given in Eq. (4.15) (this refers to $\tilde{C}_{V / A}^{0}$ and $\tilde{C}_{V / A}^{\text {int }}$, respectively). They read

$$
\begin{align*}
& {\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)=\frac{\Gamma_{t}^{2}}{2} \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)}\left|f_{v, m_{t}}(|\mathbf{p}|)\right|^{2}} \\
& {\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)=\frac{\Gamma_{t}}{2} \int_{\Delta(\Lambda)} \frac{d^{4} p}{(2 \pi)^{4}} \frac{\left(2 m_{t}\right)^{4}}{\left(t_{1}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)\left(t_{2}^{2}+m_{t}^{2} \Gamma_{t}^{2}\right)}} \\
&  \tag{4.32}\\
& \quad \times\left[-\frac{t_{1}+t_{2}}{2 m_{t}} \operatorname{Re}\left[f_{v, m_{t}}(|\mathbf{p}|)\right]-\Gamma_{t} \operatorname{Im}\left[f_{v, m_{t}}(|\mathbf{p}|)\right]\right]
\end{align*}
$$

$f_{v, m_{t}}(|\mathbf{p}|)$ being the form factor associated with the LL Coulomb Green function,

$$
f_{v, m_{t}}(|\mathbf{p}|)=\left[\frac{|\mathbf{p}|^{2}}{m_{t}}-\left(E+i \Gamma_{t}\right)\right] \tilde{G}_{v, m_{t}, \nu=1}^{0}(0,|\mathbf{p}|)
$$

which is one at $\mathcal{O}\left(\alpha_{s}^{0}\right)$. For the partially Fourier transformed Green function that does not depend on the direction of $\mathbf{p}$ we use [109]

$$
\begin{aligned}
G_{v, m_{t}, \nu}^{0}(0,|\mathbf{p}|)=-\frac{i m_{t}}{4 k|\mathbf{p}|} \frac{1}{1-\lambda}[ & { }_{2} F_{1}\left(2,1 ; 2-\lambda ; \frac{1}{2}\left(1+\frac{i|\mathbf{p}|}{k}\right)\right) \\
& \left.-{ }_{2} F_{1}\left(2,1 ; 2-\lambda ; \frac{1}{2}\left(1-\frac{i|\mathbf{p}|}{k}\right)\right)\right]
\end{aligned}
$$

with the hypergeometric function [111]

$$
{ }_{2} F_{1}(a, b ; c ; z)=\frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_{0}^{1} t^{b-1}(1-t)^{c-b-1}(1-t z)^{-a} d t
$$



Figure 4.13: Summation of the Coulomb rungs: $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$ (black curve), $\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}+\tilde{C}^{0,0}(\Lambda)+\tilde{C}^{0,1}(\Lambda)$ (blue curve) and the difference of both (red curve) as functions of the kinetic energy $E$ for various values of the cut $\Lambda$. Here, $\tilde{C}^{0,0 / 1}(\Lambda)$ denote the corrections to $\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)$ due to $\tilde{C}_{V / A}^{0,0 / 1}(\Lambda)$. The relation between $\Lambda$ and $\Lambda_{c}$ is given in Eq. (4.37).

In Fig. 4.13 we have plotted the exact summation $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$ and in comparison the estimate obtained from $\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}$ and our analytic NLO and NNLO results $\tilde{C}_{V / A}^{0,0}(\Lambda), \tilde{C}_{V / A}^{0,1}(\Lambda)$ as functions of the kinetic energy $E$ for various values of the cut $\Lambda$. We have also plotted the difference of both curves. Fig. 4.14 shows the analogous curves for $\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)$. As can be seen already from a qualitative analysis, the difference, which corresponds to corrections beyond NNLO, is reduced by the choice of higher values for $\Lambda$ as expected from our parametric counting. Apart from that we find that the difference is roughly energy-independent, indicating that higher order terms can be described by the energy-independent forward scattering operators.

From a quantitative analysis we obtain that the contributions beyond NNLO contained in $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$ at threshold, $E=0$, are at the level of

$$
(4.3 \%, 1.7 \%, 0.9 \%, 0.5 \%) \quad \text { for } \quad \Lambda=(60,80,100,120) \mathrm{GeV}
$$

compared to the analytic estimate from $\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}$ as well as $\tilde{C}_{V / A}^{0,0}(\Lambda)$, $\tilde{C}_{V / A}^{0,1}(\Lambda)$. Their relative size grows for smaller values of $E\left(\right.$ as $\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1}$


Figure 4.14: Summation of the Coulomb rungs: $\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)$ (black curve), $\left.\operatorname{Im} G^{0}\left(a, v, m_{t}, \nu\right)\right|_{\nu=1,(1 / \epsilon)=0}+\tilde{C}^{\text {int }, 0}(\Lambda)+\tilde{C}^{\text {int, } 1}(\Lambda)$ (blue curve) and the difference of both (red curve) as functions of the kinetic energy $E$ for various values of the cut $\Lambda$. Here, $\tilde{C}^{\text {int, } 0 / 1}(\Lambda)$ denote the corrections to $\operatorname{Re} G^{0}\left(a, v, m_{t}, \nu\right)$ due to $\tilde{C}_{V / A}^{\text {int } 0 / 1}(\Lambda)$. The relation between $\Lambda$ and $\Lambda_{c}$ is given in Eq. (4.37).
drops down below threshold) and goes down for larger values of $E$. By excluding the range $\Lambda<80 \mathrm{GeV}$ it is therefore possible to keep the contributions beyond NNLO below a $2 \%$ level, if one is interested only in the energy region above threshold. For the region below threshold, the NLO and NNLO contributions are not sufficient, if a precision better than $2 \%$ is desired.

In this case, one can extract the phase space effects from the exact numerical resummation using the quantity $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right](\Lambda)$ that was defined in Sec. 4.2. From this, we obtain the $\mathcal{O}\left(\left(\Lambda^{2} / m_{t}^{2}\right)^{0}\right)$ contributions to the cross section given in Tabs. 4.2, 4.3, 4.4 and 4.5 that are denoted as "Summed Coul." The value denoted as "Diff." is the difference of the "Summed Coul." value and the sum of the analytic phase space effects from $\tilde{C}_{V / A}^{0,0}, \tilde{C}_{V / A}^{0,1}$ and $\delta \tilde{c}_{1}^{c}$ given in the tables. We find that the numerical size of the exact resummed result exceeds the contributions from NLO and NNLO (neglecting the $\mathrm{N}^{3} \mathrm{LO}$ corrections that are still contained in the tables) by an amount that is always smaller than the NNLO contribution. For instance, in the case $\Lambda=100 \mathrm{GeV}$ and $E=0 \mathrm{GeV}$ we have -23.57 fb and around -13.01 fb from the analytic NLO and NNLO result, re-
spectively. The sum differs from the resummed result by around 8 fb , which is smaller than the absolute value of the NNLO contribution. Apart from that we observe that the "Diff." value goes down for larger values of $\Lambda$. These facts shows that the $\left(\pi a m_{t} / \Lambda\right)^{n}$ summation converges reliably. Yet, if one aims at an accuracy of at least 5 fb for the cross section, it is necessary to take the "Diff." values into account.

The comparison of Figs. 4.13 and 4.14 suggests that phase space contributions beyond NNLO contained in $\left[\operatorname{Re} G_{v, m_{t}}^{0}\right](\Lambda)$ are by far more relevant than those in $\left[\operatorname{Im} G_{v, m_{t}}^{0}\right](\Lambda)$. This is due to the rather large deviation of the analytic estimate up to NNLO from the resummation of all Coulomb rungs. However, the quantity $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right](\Lambda)$, defined in Sec. 4.2, is suppressed with respect to $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right](\Lambda)$ by the factor

$$
\frac{2 C_{V, 1}^{\mathrm{born}} C_{V, 1}^{\mathrm{int}}+2 C_{A, 1}^{\mathrm{born}} C_{A, 1}^{\mathrm{intt}}}{\left(C_{V, 1}^{\mathrm{born}}\right)^{2}+\left(C_{A, 1}^{\mathrm{born}}\right)^{2}} \approx-0.037
$$

when it enters the cross section. This corresponds to the usual $\alpha \sim v^{2}$-suppression of the interference effects. Therefore we find only the moderate corrections "Diff." at $\mathcal{O}\left(\Lambda^{2} / m_{t}^{2}\right)$ given in the tables. For instance, for $\Lambda=100 \mathrm{GeV}$ and $E=$ 0 GeV we have the values -18.96 fb and around 14.32 fb for the analytic NLO and NNLO result, respectively. The sum deviates from the "Summed Coul." result by around -6 fb , which is comparable to the above correction coming from $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right](\Lambda)$ and has to be taken into account for a 5 fb accuracy. At this point we note that instead of the contributions $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right](\Lambda)$ and $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right](\Lambda)$, as defined by the LO Coulomb Green function $\tilde{G}_{v, m_{t}, \nu=1}^{0}(0, \mathbf{p})$, it is reasonable to use the according contributions associated with the NLO or even the NNLO Coulomb Green function $\tilde{G}_{v, m_{t}, \nu=1}^{0}(0, \mathbf{p})$ for numerical applications.

It remains to discuss higher order Coulomb corrections to $\mathbf{p}^{2} / m_{t}^{2}$-suppressed phase space effects other than interference effects. Since these effects are numerically suppressed by a factor of at most 0.25 with respect to the interference effects at NLO and NNLO, as we can extract from the tables, we expect their contributions beyond NNLO to the cross section to be at most of 3 fb for $\Lambda \gtrsim 60 \mathrm{GeV}$ and therefore below the 5 fb precision threshold.

The above given analysis covers essentially two different parametric expansions of the phase space effects, namely the $\left(\Lambda^{2} / m_{t}^{2}\right)^{n}$ (or $\left.\left(\mathbf{p}^{2} / m_{t}^{2}\right)^{n}\right)$ expansion related to insertions of higher dimensional operators and the $\left(\pi a m_{t} / \Lambda\right)^{n}$ expansion related to insertions of Coulomb interactions. In addition to these, there are also the $\left(m_{t} E / \Lambda^{2}\right)^{n}$ and the $\left(m_{t} \Gamma_{t} / \Lambda^{2}\right)^{n}$ expansion, which were not discussed explicitly. We find that the quality of the latter expansions is rather good, i.e. for $-5 \mathrm{GeV} \lesssim E \lesssim 5 \mathrm{GeV}$ and $\Lambda \gtrsim 80 \mathrm{GeV}$ it is sufficient to take only the linear contributions $m_{t} E / \Lambda^{2}$ and $m_{t} \Gamma_{t} / \Lambda^{2}$ (these already contribute beyond NNLO) into account to obtain a precision better than 5 fb . Apart from that, the higher order $\left(m_{t} E / \Lambda^{2}\right)^{n}$ and $\left(m_{t} \Gamma_{t} / \Lambda^{2}\right)^{n}$ terms related to the dominant contributions
are contained in $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right]$ and $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right]$. We even find that it is possible to completely neglect the energy-dependence of the phase space effects and assume $E=0$ and keep the 5 fb accuracy.

From the above discussion we conclude that to achieve a 5 fb accuracy of phase space effects in the threshold cross section $(-5 \mathrm{GeV} \lesssim E \lesssim 5 \mathrm{GeV})$ it is sufficient to take into account the NLO, NNLO and $\mathrm{N}^{3} \mathrm{LO}$ phase space effects given in an analytic form in Secs. 4.4 and 4.5 in combination with the numerical contributions from $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right]$ and $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right]$ for the dominant phase space effects. This is true, if the value of $\Lambda$ is restricted to the interval

$$
80 \mathrm{GeV} \lesssim \Lambda \lesssim 120 \mathrm{GeV}
$$

which roughly corresponds to the interval

$$
20 \mathrm{GeV} \lesssim \Lambda_{c} \lesssim 45 \mathrm{GeV}
$$

for the experimental cut $\Lambda_{c}$ on the invariant masses of the $b W^{+}$and $\bar{b} W^{-}$pairs that will be introduced in the next section. We emphasize, however, that we have not included phase space effects originating from ultrasoft gluon radiation. We will give a short estimate of their size in the outlook of this work.

### 4.7 Kinematical Variables of the Top Decay Products and Relativistic Corrections to the Cut

At the end of Sec. 4.1 we gave the physical interpretation of $\Lambda^{2}$ as a cut on the invariant masses of the detected $b W^{+}$and $\bar{b} W^{-}$pairs. In this section we refine this analysis and compute relativistic corrections to the cut.

Let us first make a general remark. The effective theory we use was constructed such that it can describe only top-antitop pairs produced in the threshold energy region $\sqrt{s} \approx 2 m_{t}$. This means that it describes only "resonant" top-antitop production, corresponding to interferences of two double-resonant $e^{+} e^{-} \rightarrow t \bar{t} \rightarrow b W^{+} \bar{b} W^{-}$diagrams, shown in Fig. 3.1 (a), and interferences of a double resonant diagram and a single resonant diagram, $e^{+} e^{-} \rightarrow t \bar{b} W^{-} \rightarrow$ $b W^{+} \bar{b} W^{-}$or $e^{+} e^{-} \rightarrow \bar{t} b W^{+} \rightarrow b W^{+} \bar{b} W^{-}$, shown in Figs. 3.1 (b) - (i). In all these cases of interference, in which at least one double-resonant diagram is present, the $b W^{+}$pair can be assigned uniquely to the top quark and the $\bar{b} W^{-}$pair to the antitop quark. The other cases of interference of diagrams with the $b W^{+} \bar{b} W^{-}$ final state are not described by the effective theory and will be referred to as irreducible background diagrams. Their numerical contributions to the cross section will be estimated in Sec. 4.8 using the programs MadGraph and MadEvent.

The invariant masses of $b W^{+}$and $\bar{b} W^{-}$are defined as

$$
\begin{equation*}
M_{t}^{2}=\left(k_{1}+q_{1}\right)^{2}, \quad M_{\bar{t}}^{2}=\left(k_{2}+q_{2}\right)^{2}, \tag{4.33}
\end{equation*}
$$

where $k_{1}^{\mu}, k_{2}^{\mu}$ are the 4 -momenta of $b$ and $\bar{b}$ and $q_{1}^{\mu}, q_{2}^{\mu}$ are the 4 -momenta of $W^{+}$ and $W^{-}$, respectively. Due to the association of $b W^{+}$with the top and $\bar{b} W^{-}$with the antitop, we can write down the 4 -momentum conservation

$$
\tilde{p}_{t}=k_{1}+q_{1}, \quad \tilde{p}_{\bar{t}}=k_{2}+q_{2},
$$

where $\tilde{p}_{t}^{\mu}$ and $\tilde{p}_{t}^{\mu}$ are top and antitop 4 -momentum, respectively. Thus, for simplicity $M_{t}^{2}$ and $M_{t}^{2}$ are also referred to as the invariant masses of top and antitop. We define the cross section $\sigma_{\mathrm{thr}}\left(\Lambda_{c}\right)$ of threshold top pair production such that it corresponds to taking into account only those measured $b W^{+}$and $\bar{b} W^{-}$events that fulfill the conditions

$$
\begin{equation*}
\left(m_{t}-\Lambda_{c}\right)^{2} \leq M_{t, \bar{t}} \leq\left(m_{t}+\Lambda_{c}\right)^{2} . \tag{4.34}
\end{equation*}
$$

Therefore our requirement to experiment is to detect and count only a distinguished set of $b W$ pairs or their decay products, respectively.

On the theory side, by these conditions the original phase space integration over all momenta $k_{1}, k_{2}, q_{1}, q_{2}$ (that are in accord with 4 -momentum conservation) is restricted to the region of top pair threshold events. Due to the given kinematical situation (i.e. the use of the c.m. frame) the 4 -momenta of top and antitop have the form

$$
\begin{align*}
& k_{1}^{\mu}+q_{1}^{\mu}=\left(k_{1}^{0}+q_{1}^{0}, \mathbf{k}_{1}+\mathbf{q}_{1}\right)=\left(\tilde{p}_{t}^{0}, \tilde{\mathbf{p}}_{t}\right)=\left(\frac{\sqrt{s}}{2}+p_{0}, \mathbf{p}\right), \\
& k_{2}^{\mu}+q_{2}^{\mu}=\left(k_{2}^{0}+q_{2}^{0}, \mathbf{k}_{2}+\mathbf{q}_{2}\right)=\left(\tilde{p}_{t}^{0}, \tilde{\mathbf{p}}_{\bar{t}}\right)=\left(\frac{\sqrt{s}}{2}-p_{0}, \mathbf{p}\right), \tag{4.35}
\end{align*}
$$

where $p^{\mu}$ is the momentum associated with the phase space integration that remains after $b W$ subintegrations have been carried out, see Eq. (4.3) and the related description.

By the combination of Eqs. $(4.33,4.34,4.35)$ and the definitions of $t_{1,2}$ in Eq. (4.4) we obtain the restrictions

$$
\begin{aligned}
& -2 m_{t} \Lambda_{c}+\Lambda_{c}^{2} \leq t_{1}+\frac{1}{4 m_{t}^{2}}\left(m_{t} E+\frac{1}{2}\left(t_{1}-t_{2}\right)\right)^{2} \leq 2 m_{t} \Lambda_{c}+\Lambda_{c}^{2} \\
& -2 m_{t} \Lambda_{c}+\Lambda_{c}^{2} \leq t_{2}+\frac{1}{4 m_{t}^{2}}\left(m_{t} E+\frac{1}{2}\left(t_{2}-t_{1}\right)\right)^{2} \leq 2 m_{t} \Lambda_{c}+\Lambda_{c}^{2}
\end{aligned}
$$

In order to solve this system of inequalities for $t_{1}$ and $t_{2}$ we use the fact that the phase space regions where $t_{1}$ or $t_{2}$ are far away from zero (that means top or antitop are far off-shell) give negligibly small contributions. Therefore we determine $t_{1}$ from the inequalities in the first line where we may set $t_{2}=0$ and determine $t_{2}$ from the inequalities in the second line where we may set $t_{1}=0$. An
additional expansion in $\Lambda_{c} / m_{t}$ and $E / m_{t}$ up to $\mathcal{O}\left(\Lambda_{c}^{3} / m_{t}^{3}\right)$ and $\mathcal{O}\left(E^{2} / m_{t}^{2}\right)$ yields

$$
\begin{align*}
&-2 m_{t} \Lambda_{c}\left(1-\frac{E}{4 m_{t}}\right)+\frac{3 \Lambda_{c}^{2}}{4}+\frac{3 \Lambda_{c}^{3}}{16 m_{t}}-\frac{E^{2}}{4} \leq t_{1,2} \\
& t_{1,2} \leq 2 m_{t} \Lambda_{c}\left(1-\frac{E}{4 m_{t}}\right)+\frac{3 \Lambda_{c}^{2}}{4}-\frac{3 \Lambda_{c}^{3}}{16 m_{t}}-\frac{E^{2}}{4} \tag{4.36}
\end{align*}
$$

By a comparison of the lower bound to the one in Eq. (4.7) using Eq. (4.5), we find ${ }^{5}$

$$
\begin{equation*}
\Lambda^{2}=2 m_{t} \Lambda_{c}\left(1-\frac{E}{4 m_{t}}\right)-\frac{3 \Lambda_{c}^{2}}{4}-\frac{3 \Lambda_{c}^{3}}{16 m_{t}}+\frac{E^{2}}{4} \tag{4.37}
\end{equation*}
$$

Here, $2 m_{t} \Lambda_{c}$ is the leading term and the all the other terms are relativistic corrections.

### 4.8 Comparison to MadGraph and the Irreducible Background

In this section we compare the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ NRQCD prediction for the $t \bar{t}$ threshold production cross section to the $b W^{+} \bar{b} W^{-}$production cross section generated by the programs MadGraph and MadEvent [112-114], where the decay width of the top quark is included correctly. Since our NRQCD prediction describes only resonant $t \bar{t}$ production as was argued in the last section, it is necessary to subtract from the MadGraph/MadEvent result the contribution coming from background diagrams in order to have the same starting point for both approaches. This was done for the most part in the evaluation given in the following, however, it must be mentioned that there are some background diagrams that could not be subtracted for technical reasons related to the use of the MadGraph program. Yet our evaluation is applicable because their contributions can be estimated to be negligible. Since the accuracy of our comparison is better than the NNLO level, we include on the NRQCD side also $\mathrm{N}^{3} \mathrm{LO}$ effects, for example $\mathrm{N}^{3} \mathrm{LO}$ phase space contributions or the energy-dependence of the intermediate photon and $Z$ boson, which is described at NNLO by means of the operators $\mathcal{O}_{V / A, \mathbf{p}, 1}^{(1)}$.

[^10]The uppermost graph in Fig. 4.15 shows the cross sections without phase space cuts applied. We observe a clear discrepancy between the two predictions. While the MadGraph/MadEvent prediction drops down to zero when the c.m. energy goes down below threshold (which is located at 344 GeV ), the standard NRQCD prediction for the cross section not containing phase space cuts assumes a constant but finite value, which is unphysical. By the inclusion of the NLO and $\mathrm{N}^{3} \mathrm{LO}$ phase space corrections determined in Sec. 4.4 also the NRQCD prediction drops down to zero. In addition it leads to the agreement between the MadGraph/MadEvent and the NRQCD prediction. This can be seen also from Fig. 4.16, where the MadGraph/MadEvent prediction for the cross section is plotted in the various panels referring to different values of the cut. While the red curves correspond to the MadGraph/MadEvent prediction itself, the black lines that are symmetric around zero indicate its statistical error, multiplied by 50 . The difference between the NRQCD result containing the phase space corrections and the MadGraph/MadEvent result, multiplied by 50, is represented by red dots and blue dots. The blue dots correspond to the exact (numerical) integration of the phase space, whereas the red dots correspond to the analytic expressions of the operator product expansion at NLO given in Sec. 4.4. We find that almost all blue dots are within the statistical MadGraph/MadEvent error and therefore both approaches are in very good agreement. The deviation of red and blue dots for energies far above and far below threshold and for small values of the cut is due to the missing higher order terms in the $E / \Lambda$ expansion. Fig. 4.17 shows the same graphs except for the fact that here also the $\mathrm{N}^{3} \mathrm{LO}$ phase space contributions are taken into account, leading to a reduction of the difference between the exact result and the one obtained by the operator product expansion.

The irreducible background to resonant $t \bar{t}$ threshold production mentioned in the last section contains interferences of two single-resonant diagrams, a singleand a non-resonant diagram (which does not contain a top/antitop line) and two non-resonant diagrams. Since none of these diagrams is enhanced by Coulomb singularities, we estimate their contributions using the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ prediction by MadGraph/MadEvent. These contributions (except for the few diagrams that could not be subtracted as mentioned above), multiplied by 50, are plotted in Figs. 4.16 and 4.17 as the straight black lines. Their numerical size is at the level of $5-10 \mathrm{fb}$ for $\Lambda_{c}=50 \mathrm{GeV}$ and can be reduced by a variation of the cut. For $\Lambda_{c}=20 \mathrm{GeV}$ their contribution is below the 5 fb level.


Figure 4.15: Our predictions of the $\mathcal{O}\left(\alpha_{s}^{0}\right)$ cross section (red lines) compared to MadGraph/MadEvent results (black lines). The cross section plotted in the uppermost panel is obtained without phase space cuts. The other panels refer to phase space cuts of $\Lambda_{c}=50, \Lambda_{c}=30, \Lambda_{c}=10$ and $\Lambda_{c}=5 \mathrm{GeV}$.


Figure 4.16: Illustration of the agreement between the MadGraph/MadEvent and the NRQCD cross section prediction including phase space effects at NLO for various values of the cut $\Lambda_{c}$.


Figure 4.17: Illustration of the agreement between the MadGraph/MadEvent and the NRQCD cross section prediction including phase space effects at NLO and $\mathrm{N}^{3} \mathrm{LO}$ for various values of the cut $\Lambda_{c}$.

## Chapter 5

## Numerical Summary

In this chapter we give a numerical summary of the contributions to the threshold cross section obtained in the previous calculations. In Figs. 5.1 we have plotted the cross section $\sigma_{\mathrm{thr}}^{\text {NNLL,coul }}$ originating from the NNLL order Coulomb Green function $G^{c}\left(a, v, m_{t}, \nu\right)$ (uppermost curve). For this and the following curves we used the numerical techniques and codes of the TOPPIC program developed in Ref. [80] (see also Ref. [79]). We employed the $1 S$ mass scheme [40,63] and used four-loop renormalization group running $[115,116]$ for the strong coupling $\alpha_{s}$. We also applied the electromagnetic coupling $\alpha^{n_{f}=8}\left(m_{t}\right)$ defined in Sec. 2.3, which includes one-loop fermionic vacuum polarization effects. In addition, we have plotted the sum $\sigma_{\mathrm{thr}}^{\mathrm{NNLL}, \text { coul }}+\Delta \sigma_{\mathrm{thr}}^{\mathrm{ew}}$, where $\Delta \sigma_{\mathrm{thr}}^{\mathrm{ew}}=\Delta^{\mathrm{ew}, \overline{\mathrm{MS}}} \cdot \sigma_{\mathrm{thr}}^{\mathrm{LL}}$ is the NNLL order contribution to the cross section containing electroweak effects, but no fermionic vacuum polarization effects according to Eqs. $(1.29,2.27)$. This curve is slightly below $\sigma_{\text {thr }}^{\text {NNL,coul }}$. The next curve plotted is the sum $\sigma_{\text {thr }}^{\text {NNLL,coul }}+\Delta \sigma_{\text {thr }}^{\text {ew }}+$ $\left(\Delta \sigma_{\text {thr }}^{\Gamma, 1}+\Delta \sigma_{\text {thr }}^{\Gamma, 2}\right)$, where $\Delta \sigma_{\text {thr }}^{\Gamma, 1}$ is the NNLL order correction from interference and time dilatation effects given in Eq. (3.6) and $\Delta \sigma_{\text {thr }}^{\Gamma, 2}$ contains the NLL evolution of the forward scattering operators, see Eq. (3.11). Finally, the lowest curve is the sum $\sigma_{\text {thr }}^{\mathrm{NNL}, \text { coul }}+\Delta \sigma_{\text {thr }}^{\mathrm{ew}}+\left(\Delta \sigma_{\text {thr }}^{\Gamma, 1}+\Delta \sigma_{\text {thr }}^{\Gamma, 2}\right)+\Delta \sigma_{\text {thr }}^{\mathrm{psm}}\left(\Lambda_{c}\right)$. The quantity $\Delta \sigma_{\text {thr }}^{\mathrm{psm}}\left(\Lambda_{c}\right)$ contains phase space effects that are described by the combination of the NLO, NNLO and $\mathrm{N}^{3} \mathrm{LO}$ analytic terms given in Secs 4.4 and 4.5 and the numerical contributions from $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right]$ and $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right]$ for the dominant phase space effects. We neglected the energy-dependence of phase space effects beyond NNLO and assumed $E=0$. We also neglected the phase space corrections that are obtained by means of $\delta^{\mathrm{PS}}\left[\operatorname{Im} G^{0}\right]$ and $\delta^{\mathrm{PS}}\left[\operatorname{Re} G^{0}\right]$, if the NLO or NNLO Coulomb Green function is used.

For the experimental cut $\Lambda_{c}$ on the $b W$ invariant masses we chose the values $\Lambda_{c}=35 \mathrm{GeV}$ and $\Lambda_{c}=20 \mathrm{GeV}$, referring to the upper and lower panel in Fig. 5.1. All curves also include the energy-dependence originating from the intermediate photon and $Z$ boson that is formally described by the operators $C_{V / A, 1}^{(1), \text { born }}$ up to NNLL in the effective theory. As input parameters we chose $M_{1 S}=172 \mathrm{MeV}$, $\Gamma_{t}=1.36 \mathrm{GeV}, \alpha=1 / 137.036, \alpha_{s}\left(M_{Z}\right)=0.118$, a Higgs mass of $M_{H}=130 \mathrm{GeV}$
and the values given in Eqs. (2.28). As the velocity renormalization scale we used $\nu=0.2$.

For $\Lambda_{c}=35 \mathrm{GeV}$ we find that the sum of all effects leads to a shift of around -75 fb below the peak, -90 fb close to the peak and -65 fb above the peak. This corresponds to relative shift of around $-35 \%,-8 \%$ and $-7 \%$, respectively. For $\Lambda_{c}=20 \mathrm{GeV}$ we find shifts of around -90 fb below the peak, -100 fb close to the peak and -80 fb above the peak, corresponding to relative shifts of around $-45 \%,-9 \%$ and $-8 \%$. We note that the small value of the cross section below the peak position leads to rather large relative uncertainties coming from phase space effects in that region. Assuming an absolute uncertainty of 5 fb , the relative uncertainty at $\sqrt{s} \approx 340$ is at the level of a few percent. On the other hand, for values $\sqrt{s} \gtrsim 342$ the precision is already better than $2 \%$.


Figure 5.1: Summations of the various contributions to the threshold cross section. The black curves correspond to the NNLL order cross section for the Coulomb potential. The blue line includes NNLL usual electroweak corrections, the red line in addition NNLL interference and lifetime dilatation and the brown line includes also phase space effects at NLO, NNLO and beyond. The upper and the lower panel refers to invariant mass cuts of $\Lambda_{c}=35 \mathrm{GeV}$ and $\Lambda_{c}=20 \mathrm{GeV}$, respectively.

## Chapter 6

## Conclusions and Outlook

## Conclusions

In this work we have determined several electroweak corrections to the top-antitop threshold production rate in electron-positron annihilation that is determined using the effective theory NRQCD. They are important ingredients for an accurate theoretical prediction of the threshold cross section required for a precise extraction of prominent Standard Model parameters at a future Linear Collider.

The effects of initial state polarization have been implemented into the NRQCD treatment, in particular for the electroweak effects beyond LL order.

The real parts of the NNLL order electroweak matching conditions for the Wilson coefficients of the dominant NRQCD operators have been determined. The results include the contributions of all electroweak one-loop effects that are integrated out when NRQCD is matched to the Standard Model except for pure QED corrections. We have pointed out discrepancies with respect to earlier work.

A systematic procedure for the effective theory treatment of the effects arising from the finite lifetime of the top quark up to NNLL order has been developed. It involves complex Wilson coefficients and anomalous dimensions. The imaginary parts are associated with experimental cuts that are imposed to define the cross section. Interference and time dilatation effects, which were considered in detail, lead to absorptive NNLL order corrections of the level of several percent and in particular change the line-shape of the cross section such that they imply a top mass shift of $30-50 \mathrm{MeV}$. We have shown that the effective theory treatment of finite lifetime effects leads to ultraviolet phase space divergences, which renormalize $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$forward scattering operators. Due to a mixing effect the Wilson coefficients of these operators obtain a running, which gives rise to an energy-independent correction to the cross section contributing already at NLL order. We have determined corrections that are required for a proper description of the NRQCD top-antitop phase space in the presence of finite lifetime effects at NLO. It has been shown that these phase space effects can be included systematically into the effective theory by means of an operator product expansion that
involves a phase space cut at the hard scale. A power counting for these effects is provided by the scaling of the cut. The phase space effects lead to a finite imaginary renormalization of the hard-scale Wilson coefficients of $\left(e^{+} e^{-}\right)\left(e^{+} e^{-}\right)$ forward scattering operators at NLO and to other effective theory couplings beyond NLO. This renormalization procedure guarantees that the contributions to the cross section are obtained correctly from the $e^{+} e^{-}$forward scattering amplitude according to the usual rule provided by the optical theorem. It has been shown that the quality of the operator product expansion is very good at NLO and sufficient beyond NLO for a prediction of the cross section with a 5 fb accuracy, provided that the experimental cut $\Lambda_{c}$ on the top/antitop invariant masses is in the range $20 \mathrm{GeV} \lesssim \Lambda_{c} \lesssim 45 \mathrm{GeV}$. The comparison of our $\mathcal{O}\left(\alpha_{s}^{0}\right)$ results with the those obtained from the programs MadGraph and MadEvent leads to a very good agreement for the allowed values of the cut. Phase space effects at NLO result in an energy-independent additive correction to the cross section prediction, which compensates the unphysical behaviour of previous NRQCD predictions in the region below threshold. The contributions to the cross section are at the level of -50 fb , which corresponds to a relative shift of the cross section at the level of five to ten percent (depending on the c. m. energy point that is considered). Since they are sensitive to experimental cuts, these corrections offer the possibility to reduce uncertainties induced by background events.

## Outlook

Concerning pure QED effects in top-antitop threshold production in electronpositron annihilation a coherent treatment beyond the LL level has not yet been achieved and is expected to be an important ingredient for the theoretical prediction.

As to the finite lifetime effects at present only the NLL running of the forward scattering operators is known. The NNLL running is obtained from $\mathcal{O}\left(\alpha_{s}\right)$ corrections to the absorptive parts of the matching conditions for the production currents and the quark bilinear operators because these can give rise to ultraviolet divergences at the order $\mathrm{N}^{3} \mathrm{LL}$. These include corrections to the static potential originating from the insertion of a quark loop into the gluon line of the according full QCD diagram.

Concerning contributions to the cross section that involve a phase space cut, the impact of ultrasoft gluon radiation has to be analyzed in a future work. Ultrasoft gluon interactions can effectively shift the value of the cut by an amount of the ultrasoft scale, $\Delta^{\mathrm{us}} \sim 3 \mathrm{GeV}$. The resulting ambiguity in the phase space corrections can be estimated by the consideration of the leading term in Eq. (4.19). A shift $\Delta^{\text {us }}$ of the kinetic energy by an ultrasoft gluon leads to a phase space
correction

$$
\sim \frac{\Gamma_{t}}{\sqrt{2 m_{t}\left[\Lambda_{c}+\Delta^{\mathrm{us}}\right]}}=\frac{\Gamma_{t}}{\sqrt{2 m_{t} \Lambda_{c}}}\left[1-\frac{\Delta^{\mathrm{us}}}{2 \Lambda_{c}}+\mathcal{O}\left(\left(\frac{\Delta^{\mathrm{us}}}{\Lambda_{c}}\right)^{2}\right)\right] .
$$

For values of the cut above 20 GeV this results in a correction of less than $10 \%$ compared to the leading phase space effect and therefore to a contribution to the cross section of less than 5 fb . Thus, according to this simple analysis phase space corrections associated with ultrasoft gluons lead to negligible contributions.

Finally, we would like to point out that the effective theory approach for the treatment of finite lifetime effects presented in this work is not limited to the case of top-antitop production. It is applicable also to problems that involve unstable particles in related fields, for example to the description of squark pair production.

## Appendix A

## Scalar n-point Functions

In this appendix we calculate scalar $n$-point functions needed for the calculation of one-loop diagrams in Sec. 2.2.

## Definitions

The scalar $n$-point functions for $n \leq 4$ are defined as

$$
\begin{aligned}
& A_{0}(m)=-16 i \pi^{2} \tilde{\mu}^{2 \epsilon} \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left[q^{2}-m^{2}+i \epsilon\right]}, \\
& B_{0}\left(p, m_{1}, m_{2}\right)=-16 i \pi^{2} \tilde{\mu}^{2 \epsilon} \int \frac{d^{d} q}{(2 \pi)^{d}} \frac{1}{\left[q^{2}-m_{1}^{2}+i \epsilon\right]\left[(q+p)^{2}-m_{2}^{2}+i \epsilon\right]}, \\
& C_{0}\left(p_{1}, p_{2}, m_{1}, m_{2}, m_{3}\right)=-16 i \pi^{2} \tilde{\mu}^{2 \epsilon} \int \frac{d^{d} q}{(2 \pi)^{d}}\left\{\left[q^{2}-m_{1}^{2}+i \epsilon\right]\left[\left(q+p_{1}\right)^{2}-m_{2}^{2}+i \epsilon\right]\right. \\
& \left.\times\left[\left(q+p_{2}\right)^{2}-m_{3}^{2}+i \epsilon\right]\right\}^{-1}, \\
& D_{0}\left(p_{1}, p_{2}, p_{3}, m_{1}, m_{2}, m_{3}, m_{4}\right)= \\
& =-16 i \pi^{2} \tilde{\mu}^{2 \epsilon} \int \frac{d^{d} q}{(2 \pi)^{d}}\left\{\left[q^{2}-m_{1}^{2}+i \epsilon\right]\left[\left(q+p_{1}\right)^{2}-m_{2}^{2}+i \epsilon\right]\right. \\
& \left.\times\left[\left(q+p_{2}\right)^{2}-m_{3}^{2}+i \epsilon\right]\left[\left(q+p_{3}\right)^{2}-m_{4}^{2}+i \epsilon\right]\right\}^{-1},
\end{aligned}
$$

where $d=4-2 \epsilon$ is the number of space-time dimensions, $\epsilon$ being a complex number, and $\tilde{\mu}^{2}=\mu^{2} e^{\gamma_{\mathrm{E}}-\ln (4 \pi)}$ is a mass parameter. For symmetry reasons $B_{0}\left(p, m_{1}, m_{2}\right)$ does not depend on all four $p$ components but on the square $p^{2}$ only. In analogy $C_{0}\left(p_{1}, p_{2}, m_{1}, m_{2}, m_{3}\right)$ depends only on the Lorentz invariant combinations of $p_{1}$ and $p_{2}$. Therefore we also use the sufficient notation

$$
\begin{aligned}
B_{0}\left(p^{2}, m_{1}^{2}, m_{2}^{2}\right) & \equiv B_{0}\left(p, m_{1}, m_{2}\right) \\
C_{0}\left(p_{1}^{2},\left(p_{1}-p_{2}\right)^{2}, p_{2}^{2}, m_{1}^{2}, m_{2}^{2}, m_{3}^{2}\right) & \equiv C_{0}\left(p_{1}, p_{2}, m_{1}, m_{2}, m_{3}\right) .
\end{aligned}
$$

## Evaluation

First we have $A_{0}(m)=B_{0}(0, m, 0)$. The $B_{0}$ function can be expressed in terms of the logarithm and trivial functions. We find (here $m_{1}^{2}, m_{2}^{2}, m^{2}>0$ and $p^{2} \neq 0$ )

$$
\begin{align*}
B_{0}\left(0, m^{2}, m^{2}\right)= & \frac{1}{\epsilon}-\ln \frac{m^{2}}{\mu^{2}}+\mathcal{O}(\epsilon), \\
B_{0}\left(0, m^{2}, 0\right)= & \frac{1}{\epsilon}+1-\ln \frac{m^{2}}{\mu^{2}}+\mathcal{O}(\epsilon), \\
B_{0}\left(0, m_{1}^{2}, m_{2}^{2}\right)= & \frac{1}{\epsilon}+1-\frac{m_{1}^{2} \ln \frac{m_{1}^{2}}{\mu^{2}}-m_{2}^{2} \ln \frac{m_{2}^{2}}{\mu^{2}}}{m_{1}^{2}-m_{2}^{2}}+\mathcal{O}(\epsilon), \\
B_{0}\left(p^{2}, 0,0\right)= & \frac{1}{\epsilon}+2-\ln \left|\frac{p^{2}}{\mu^{2}}\right|+\theta\left(\frac{p^{2}}{\mu^{2}}\right) i \pi+\mathcal{O}(\epsilon), \\
B_{0}\left(p^{2}, m^{2}, 0\right)= & \frac{1}{\epsilon}+2+\frac{m^{2}-p^{2}}{p^{2}} \ln \left|\frac{m^{2}-p^{2}}{m^{2}}\right|-\ln \frac{m^{2}}{\mu^{2}} \\
& +\theta\left(\frac{p^{2}-m^{2}}{\mu^{2}}\right) \frac{p^{2}-m^{2}}{p^{2}} i \pi+\mathcal{O}(\epsilon), \\
B_{0}\left(p^{2}, m_{1}^{2}, m_{2}^{2}\right)= & \frac{1}{\epsilon}+2-\frac{1}{2} \ln \frac{m_{1}^{2} m_{2}^{2}}{\mu^{4}}+\frac{m_{1}^{2}-m_{2}^{2}}{2 p^{2}} \ln \frac{m_{2}^{2}}{m_{1}^{2}} \\
& -R+\mathcal{O}(\epsilon), \tag{A.1}
\end{align*}
$$

where
$R=\left\{\begin{array}{ll}\operatorname{sgn}(A) \frac{\sqrt{|A||B|}}{p^{2}}\left(\ln \left[\operatorname{sgn}(A) \frac{\sqrt{|B|}+\sqrt{|A|}}{\sqrt{|B|}-\sqrt{|A|}}\right]-\theta(A) i \pi\right) \quad & \text { if } p^{2} \leq\left(m_{1}-m_{2}\right)^{2} \\ \text { or } p^{2} \geq\left(m_{1}+m_{2}\right)^{2}\end{array}\right\} \begin{array}{ll}2 \frac{\sqrt{|A||B|}}{p^{2}} & \arctan \left(\frac{\sqrt{|B|}}{\sqrt{|A|}}\right) \quad \text { if }\left(m_{1}-m_{2}\right)^{2}<p^{2}<\left(m_{1}+m_{2}\right)^{2}\end{array}$
with $A=p^{2}-\left(m_{1}+m_{2}\right)^{2}$ and $B=p^{2}-\left(m_{1}-m_{2}\right)^{2}$. The other relevant cases follow from the symmetry with respect to exchange of the second and third argument. $B_{0}(0,0,0)$ is not defined.

The $C_{0}$ function can be expressed in terms of the logarithm, the dilogarithm $\mathrm{Li}_{2}$, defined as

$$
\mathrm{Li}_{2}(z)=-\int_{0}^{z} \frac{\ln (1-t)}{t} d t
$$

and trivial functions. For the momentum configurations appearing in our calcu-
lation we find the results (omitting $\mathcal{O}(\epsilon)$ contributions)

$$
\begin{aligned}
& C_{0}\left(p^{2}, 0,0,0,0, m^{2}\right)=\frac{1}{p^{2}}\left(\frac{\pi^{2}}{6}-\mathrm{Li}_{2}\left(1+\frac{p^{2}}{m^{2}}\right)-i \pi \ln \left(1+\frac{p_{1}^{2}}{m_{3}^{2}}\right)\right) \text { if } p^{2}>0, \\
& C_{0}\left(0, k^{2}, 0,0, m^{2}, m^{2}\right)=\frac{1}{k^{2}}\left(\ln \frac{1+\sqrt{1-r}}{1-\sqrt{1-r}}-i \pi\right)^{2} \text { if } r=\frac{4 m^{2}}{k^{2}} \leq 1 \\
& C_{0}\left(-p^{2}, 0, p^{2}, 0,0, m^{2}\right)=-\frac{1}{2 p^{2}}\left(\frac{\pi^{2}}{4}-\operatorname{Li}_{2}\left(1-\frac{2 p^{2}}{m^{2}}\right)+\mathrm{Li}_{2}\left(\frac{2 p^{2}}{m^{2}}-1+i \epsilon\right)\right) \\
& \text { if } 2 p^{2}>m^{2} \\
& C_{0}\left(-m^{2}, m^{2}, 0,0, m^{2}, m_{3}^{2}\right)=-\frac{1}{4 m^{2}}\left(\frac{\pi^{2}}{6}-\mathrm{Li}_{2}\left(1-\frac{4 m^{2}}{m_{3}^{2}}\right)\right) \text { if } 4 m^{2}>m_{3}^{2} .
\end{aligned}
$$

## Degenerate three- and four-point function

In the case where $p_{1}$ and $p_{2}$ in $C_{0}\left(p_{1}, p_{2}, m_{1}, m_{2}, m_{3}\right)$ are linearly dependent, one can use simple linear algebra to express $C_{0}$ in terms of $B_{0}$ functions. In analogy, $D_{0}$ can be expressed in terms of $C_{0}$ functions, if its three momenta are linearly dependent. Such linear dependency occurs in our calculation since we assume the produced top quarks to be at rest. The appearing momentum configurations involve the degenerate $C_{0}$ and $D_{0}$ functions

$$
\begin{aligned}
C_{0}\left(k,-k, m_{1}, m_{2}, m_{3}\right)= & a B_{0}\left(2 k, m_{2}, m_{3}\right)+b B_{0}\left(k, m_{1}, m_{3}\right) \\
& +c B_{0}\left(k, m_{1}, m_{2}\right) \\
D_{0}\left(k,-k, l, m_{1}, m_{2}, m_{3}, m_{4}\right)= & a C_{0}\left(2 k, k+l, m_{3}, m_{2}, m_{4}\right) \\
& +b C_{0}\left(k,-l, m_{1}, m_{3}, m_{4}\right)+c C_{0}\left(k, l, m_{1}, m_{2}, m_{3}\right)
\end{aligned}
$$

where

$$
a=\frac{2}{-2 k^{2}-2 m_{1}^{2}+m_{2}^{2}+m_{3}^{2}} \quad \text { and } \quad b=c=-\frac{1}{2} a .
$$

## Appendix B

## Electroweak One-Loop Top-Antitop Vertex Corrections

In this section we give the explicit expressions obtained for the one-loop $t \bar{t}$ vertex corrections described in Sec. 2.2. They read

$$
\begin{align*}
& a(W)=\binom{Q_{t}-2 t_{3}^{t}}{\beta_{L}^{t}-2 t_{3}^{t} \frac{c_{w}}{s_{w}}} \frac{1}{12 s_{w}^{2}}\left\{-5-\frac{M_{W}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& +\left(3+\frac{M_{W}^{2}}{m_{t}^{2}}-\frac{8 m_{t}^{2}}{m_{t}^{2}+M_{W}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right) \\
& \left.+\frac{8 m_{t}^{2}}{m_{t}^{2}+M_{W}^{2}} B_{0}\left(4 m_{t}^{2}, 0,0\right)\right\},  \tag{B.1}\\
& a(W, W)=\binom{2 t_{3}^{t}}{2 t_{3}^{t} \frac{c_{w}}{s_{w}}} \frac{1}{12 s_{w}^{2}}\left\{-1-\frac{M_{W}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& \left.-\left(1-\frac{M_{W}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)+10 B_{0}\left(4 m_{t}^{2}, M_{W}^{2}, M_{W}^{2}\right)\right\},  \tag{B.2}\\
& a_{\mathrm{twr}}(W)=\binom{Q_{t}}{\beta_{L}^{t}} \frac{1}{4 s_{w}^{2}}\left\{3+\frac{3 M_{W}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& \left.-\left(1+\frac{3 M_{W}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)\right\} \\
& +\binom{0}{\frac{2 t_{3}^{t}}{2 s_{w} c_{w}}} \frac{1}{4 s_{w}^{2}}\left\{-1-\frac{2 M_{W}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& \left.\left.-B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)\right)\right\}, \\
& a_{\mathrm{bwchr}}=\binom{-t_{3}^{t}}{-\frac{c_{w}}{s_{w}} t_{3}^{t}} \frac{1}{s_{w}^{2}} B_{0}\left(0, M_{W}^{2}, M_{W}^{2}\right), \tag{B.3}
\end{align*}
$$

$$
\begin{align*}
& a(\phi)=\binom{Q_{t}-2 t_{3}^{t}}{\beta_{L}^{t}-2 t_{3}^{t} \frac{c_{w}}{s_{w}}} \frac{1}{24 s_{w}^{2}}\left\{-\frac{2 m_{t}^{2}}{M_{W}^{2}}-B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& +\left(1-\frac{5 m_{t}^{2}}{M_{W}^{2}}+\frac{8 m_{t}^{2}}{m_{t}^{2}+M_{W}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right) \\
& \left.+\frac{8 m_{t}^{2}}{M_{W}^{2}} \frac{m_{t}^{2}}{m_{t}^{2}+M_{W}^{2}} B_{0}\left(4 m_{t}^{2}, 0,0\right)\right\},  \tag{B.4}\\
& a(\phi, \phi)=\binom{2 t_{3}^{t}}{2 t_{3}^{t} \frac{c_{w}-s_{w}^{2}}{2 s_{w} c_{w}}} \frac{1}{24 s_{w}^{2}} \\
& \times\left\{\frac{2 m_{t}^{2}}{M_{W}^{2}}-B_{0}\left(0, M_{W}^{2}, 0\right)+\left(1-\frac{m_{t}^{2}}{M_{W}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)\right. \\
& \left.+\frac{4 m_{t}^{2}}{M_{W}^{2}} B_{0}\left(4 m_{t}^{2}, M_{W}^{2}, M_{W}^{2}\right)\right\},  \tag{B.5}\\
& a_{\mathrm{twr}}(\phi)=\binom{Q_{t}}{\beta_{L}^{t}} \frac{1}{8 s_{w}^{2}}\left\{\frac{2 m_{t}^{2}}{M_{W}^{2}}+3 B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& \left.-\left(3+\frac{m_{t}^{2}}{M_{W}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)\right\} \\
& +\binom{0}{\frac{2 t_{3}^{t}}{2 s_{w} c_{w}}} \frac{1}{8 s_{w}^{2}}\left\{-\frac{m_{t}^{2}}{M_{W}^{2}}-B_{0}\left(0, M_{W}^{2}, 0\right)\right. \\
& \left.+\left(1+\frac{m_{t}^{2}}{M_{W}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{W}^{2}, 0\right)\right\},  \tag{B.6}\\
& a(W, \phi)=\binom{0}{0},  \tag{B.7}\\
& a(Z)=\binom{Q_{t}\left(\left(\beta_{R}^{t}\right)^{2}+\left(\beta_{L}^{t}\right)^{2}\right)}{\left(\beta_{R}^{t}\right)^{3}+\left(\beta_{L}^{t}\right)^{3}} \frac{1}{6}\left\{-5+B_{0}\left(0, m_{t}^{2}, 0\right)\right. \\
& \left.-\frac{M_{Z}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{Z}^{2}, 0\right)+\left(2+\frac{M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{Z}^{2}, m_{t}^{2}\right)\right\} \\
& +\binom{Q_{t}}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)}\left(\left(\beta_{R}^{t}\right)^{2}+\left(\beta_{L}^{t}\right)^{2}\right) \\
& \times\left\{\frac{2 m_{t}^{2}}{M_{Z}^{2}}\left(B_{0}\left(4 m_{t}^{2}, m_{t}^{2}, m_{t}^{2}\right)-B_{0}\left(m_{t}^{2}, m_{Z}^{2}, m_{t}^{2}\right)\right)\right\}, \tag{B.8}
\end{align*}
$$

$$
\left.\begin{array}{rl}
a_{\mathrm{twr}}(Z)= & \binom{Q_{t}}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)}\left[\frac { 1 } { 2 } ( \beta _ { R } ^ { 2 } + \beta _ { L } ^ { 2 } ) \left\{\left(-3+\frac{4 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}}\right) B_{0}\left(0, m_{t}^{2}, m_{t}^{2}\right)\right.\right. \\
& +\left(\frac{3 M_{Z}^{2}}{m_{t}^{2}}-\frac{8 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}}\right) B_{0}\left(0, M_{Z}, 0\right) \\
& \left.+\left(2-\frac{3 M_{Z}^{2}}{m_{t}^{2}}+\frac{4 m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}}\right) B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)\right\} \\
+ & \frac{1}{8} \beta_{R}^{t} \beta_{L}^{t} \frac{m_{t}^{2}}{4 m_{t}^{2}-M_{Z}^{2}}\left\{B_{0}\left(0, m_{t}^{2}, m_{t}^{2}\right)+\left(2-\frac{M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(0, M_{Z}^{2}, 0\right)\right. \\
& \left.\left.-\left(3-\frac{M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)\right\}\right] \\
+ & \left(\begin{array}{c}
\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)
\end{array}\right)\left(\beta_{R}^{t}-\beta_{L}^{t}\right)\left\{1+B_{0}\left(0, m_{t}^{2}, 0\right)-\frac{M_{Z}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{Z}^{2}, 0\right)\right. \\
& \left.-\left(2-\frac{M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{Z}^{2}, m_{t}^{2}\right)\right\}, \\
a(\chi)=( & Q_{t} \\
\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right) \tag{B.10}
\end{array}\right) \frac{\left(t_{3}^{t}\right)}{6 s_{w}^{2}}\left\{-\frac{2 m_{t}^{2}}{M_{W}^{2}}+\frac{m_{t}^{2}}{M_{W}^{2}} B_{0}\left(0, m_{t}^{2}, 0\right)\right\}, ~(\mathrm{~B} .1
$$

$$
a_{\mathrm{twr}}(\chi)=\binom{Q_{t}}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)} \frac{\left(t_{3}^{t}\right)^{2}}{2 s_{w}^{2}} \frac{m_{t}^{2} M_{Z}^{2}}{M_{W}^{2}\left(4 m_{t}^{2}-M_{Z}^{2}\right)}
$$

$$
\times\left\{-2+\left(3-\frac{4 m_{t}^{2}}{M_{Z}^{2}}\right) B_{0}\left(0, m_{t}^{2}, 0\right)+\left(8-\frac{3 M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(0, M_{Z}^{2}, 0\right)\right.
$$

$$
\begin{equation*}
\left.-\left(10-\frac{3 M_{Z}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)\right\} \tag{B.11}
\end{equation*}
$$

$$
a(H)=\binom{Q_{t}}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)} \frac{1}{24 s_{w}^{2}} \frac{m_{t}^{2}}{M_{W}^{2}}\left\{-2+B_{0}\left(0, m_{t}^{2}, 0\right)-\frac{M_{H}^{2}}{m_{t}^{2}} B_{0}\left(0, M_{H}^{2}, 0\right)\right.
$$

$$
+\left(2+\frac{M_{H}^{2}}{m_{t}^{2}}-\frac{24 m_{t}^{2}}{M_{H}^{2}}\right) B_{0}\left(m_{t}^{2}, M_{H}^{2}, m_{t}^{2}\right)
$$

$$
\begin{equation*}
\left.+\frac{24 m_{t}^{2}}{M_{H}^{2}} B_{0}\left(4 m_{t}^{2}, m_{t}^{2}, m_{t}^{2}\right)\right\} \tag{B.12}
\end{equation*}
$$

$$
a_{\mathrm{twr}}(H)=\binom{Q_{t}}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)} \frac{1}{8 s_{w}^{2}} \frac{m_{t}^{2}}{M_{W}^{2}}\left\{2-\left(4-\frac{3 M_{H}^{2}}{m_{t}^{2}}\right) B_{0}\left(0, M_{H}^{2}, 0\right)\right.
$$

$$
\begin{equation*}
\left.-3 B_{0}\left(0, m_{t}^{2}, 0\right)+\left(6-\frac{3 M_{H}^{2}}{m_{t}^{2}}\right) B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{H}^{2}\right)\right\} \tag{B.13}
\end{equation*}
$$

$$
\begin{align*}
\begin{aligned}
a(Z, H)= & \binom{0}{\frac{1}{2}\left(\beta_{R}^{t}+\beta_{L}^{t}\right)} \frac{1}{2 s_{w}^{2} c_{w}^{2}} \frac{1}{4 m_{t}^{2}-M_{Z}^{2}-M_{H}^{2}} \\
& \times\left\{-M_{Z}^{2} B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{Z}^{2}\right)-\left(4 m_{t}^{2}-M_{H}^{2}\right) B_{0}\left(m_{t}^{2}, m_{t}^{2}, M_{H}^{2}\right)\right. \\
& \left.\quad+\left(4 m_{t}^{2}+M_{Z}^{2}-M_{H}^{2}\right) B_{0}\left(4 m_{t}^{2}, M_{Z}^{2}, M_{H}^{2}\right)\right\},
\end{aligned} \\
a(\chi, H)=\binom{0}{0} .
\end{align*}
$$

## Appendix C

## Electroweak One-Loop Diagrams

In this chapter we show electroweak one-loop diagrams that are computed in the c. m. frame at the $t \bar{t}$ production threshold in Sec. 2.2.

(a)

(b)

(c)

(d)

Figure C.1: Pure QED diagrams, which are not computed in this work.


Figure C.2: Boson self-energy diagrams that are zero. The symbol $f_{i}$ refers to one of the massive fermions (including the electron). The symbols $u_{+}$and $u_{-}$ denote ghost fields.

T1 C1 N1 T1 C1 N1
T1 C1 N1
T1 C4 N4

T1 C1 N7
T2 C1 N11
T2 C4 N14
T2 C1 N16

T2 C2 N17
T2 C1 N24
T2 C1 N28
T2 C2 N29

T2 C7 N34

Figure C.3: Boson self-energy diagrams. The symbol $f_{i}$ refers to one of the massive fermions (including the electron). The symbols $u_{+}$and $u_{-}$denote ghost fields.


Figure C.4: $t \bar{t}$ vertex correction diagrams. Sum of diagrams (j) to (m) is zero.

(a)

(b)

(c)

(d)

(e)

Figure C.5: $e^{+} e^{-}$vertex correction diagrams. Diagrams (d) and (e) are zero.


Figure C.6: Box diagrams.

## Appendix D

## Non-Relativistic Green Functions

We present explicit analytic expressions for zero-distance Green functions associated with the non-relativistic top-antitop dynamics in the cases of ${ }^{3} S_{1}$ and ${ }^{3} P_{1}$ production obtained in dimensional regularization in Refs. [22, 43]. Our calculations in Chap. 3 partly rely on them and the calculations in Chap. 4 partly reproduce them. We use the abbreviations $a \equiv-\mathcal{V}_{c}(\nu) /(4 \pi)=C_{F} \alpha_{s}\left(m_{t} \nu\right)$ and $G^{j} \equiv G^{j}\left(a, v, m_{t}, \nu\right)$ where $j=0,1, k, \delta, r$, kin, dil.

$$
\begin{align*}
G^{0}= & \frac{m_{t}^{2}}{4 \pi}\left\{i v-a\left[\ln \left(\frac{-i v}{\nu}\right)-\frac{1}{2}+\ln 2+\gamma_{E}+\psi\left(1-\frac{i a}{2 v}\right)\right]\right\} \\
& +\frac{m_{t}^{2} a}{4 \pi} \frac{1}{4 \epsilon},  \tag{D.1}\\
G^{1}= & \frac{m_{t}^{4}}{4 \pi}\left\{i v^{3}-a v^{2}\left[\ln \left(\frac{-i v}{\nu}\right)-1+\ln 2+\gamma_{E}+\Psi\left(1-\frac{i a}{2 v}\right)\right]\right. \\
& \left.+i \frac{v a^{2}}{4}-\frac{a^{3}}{4}\left[\ln \left(\frac{-i v}{\nu}\right)-\frac{7}{4}+\ln 2+\gamma_{E}+\Psi\left(1-\frac{i a}{2 v}\right)\right]\right\} \\
& +\frac{m_{t}^{4}}{16 \pi}\left(\frac{1}{\epsilon}+\frac{2}{3}\right)\left(v^{2} a+\frac{a^{3}}{8}\right),  \tag{D.2}\\
G^{k}= & -\frac{m_{t}^{2}}{8 \pi a}\left\{i v-a\left[\ln \left(\frac{-i v}{\nu}\right)-2+2 \ln 2+\gamma_{E}+\psi\left(1-\frac{i a}{2 v}\right)\right]\right\}^{2} \\
& +\frac{m_{t}^{2}}{8 \pi a}\left[-v^{2}+\frac{a^{2}}{4}\left(\frac{1}{3 \epsilon^{2}}-\frac{2}{\epsilon}\left(1-\frac{2}{3} \ln 2\right)+\frac{4}{3}\right.\right. \\
& \left.\left.-8 \ln 2+\frac{8}{3} \ln ^{2} 2+\frac{\pi^{2}}{9}\right)\right], \\
G^{\delta}= & -\frac{m_{t}^{2}}{16 \pi^{2}}\left\{i v-a\left[\ln \left(\frac{-i v}{\nu}\right)-\frac{1}{2}+\ln 2+\gamma_{E}+\psi\left(1-\frac{i a}{2 v}\right)\right]\right\}^{2} \\
& +\frac{m_{t}^{2} a^{2}}{256 \pi^{2}} \frac{1}{\epsilon},
\end{align*}
$$

$$
\begin{align*}
G^{r}= & -\frac{m_{t}^{2}}{16 \pi^{2}}\left\{i v-a\left[\ln \left(\frac{-i v}{\nu}\right)-\frac{3}{2}+\ln 2+\gamma_{E}+\psi\left(1-\frac{i a}{2 v}\right)\right]\right\}^{2} \\
& +\frac{m_{t}^{2}}{16 \pi^{2}}\left[-v^{2}+\frac{a^{2}}{4}\left(\frac{1}{4 \epsilon^{2}}-\frac{1}{\epsilon}-2\right)\right]-\frac{v^{2}}{4 \pi} \frac{\partial}{\partial a} G^{0}\left(a, v, m_{t}, \nu\right), \\
G^{\mathrm{kin}}= & \frac{a m_{t}^{2}}{16 \pi}\left\{i v-a\left[\ln \left(\frac{-i v}{\nu}\right)-\frac{3}{2}+\ln 2+\gamma_{E}+\psi\left(1-\frac{i a}{2 v}\right)\right]\right\}^{2} \\
& -\frac{m_{t}^{2}}{16 \pi}\left[-a v^{2}+\frac{a^{3}}{4}\left(\frac{1}{4 \epsilon^{2}}-\frac{1}{\epsilon}-2\right)\right] \\
& +\frac{v^{2}}{2}\left[1+a \frac{\partial}{\partial a}+\frac{v}{4} \frac{\partial}{\partial v}\right] G^{0}\left(a, v, m_{t}, \nu\right),  \tag{D.3}\\
G^{\mathrm{dil}}= & -i \frac{\Gamma_{t}}{2 m_{t}}\left[1+\frac{v}{2} \frac{\partial}{\partial v}+a \frac{\partial}{\partial a}\right] G^{0}\left(a, v, m_{t}, \nu\right) . \tag{D.4}
\end{align*}
$$

The function $\psi(z)$ is the digamma function, $\psi(z) \equiv(d / d z) \ln \Gamma(z)$.

## Appendix E

## Basic Phase Space Integrals

In the following we present analytic expressions for the integrals that are defined in Eqs. (4.16), which are needed for the one- and two-loop integrations carried out in Secs. 4.4 and 4.5. We do not give formulae for integrals that can be obtained from others using symmetry relations related to the exchange $t_{1} \leftrightarrow t_{2}$ or complex conjugation. The results given are true only up to the highest order of the $E / m_{t}$ and $\Gamma_{t} / m_{t}$ expansion that is written down. We use the abbreviations $e \equiv E / m_{t}$, $g \equiv \Gamma_{t} / m_{t}$ and $m \equiv m_{t}$.

$$
\begin{aligned}
I_{0}^{2000}= & m\left[\frac{1}{6} \lambda(4+3 i \sqrt{2} \pi-6 \sqrt{2} \operatorname{arsinh}(1))+\frac{e(-4+i \sqrt{2} \pi-2 \sqrt{2} \operatorname{arsinh}(1))}{2 \lambda}\right. \\
& \left.-\frac{g(28 i+3 \sqrt{2} \pi+6 i \sqrt{2} \operatorname{arsinh}(1))}{12 \lambda}\right], \\
I_{0}^{1100}= & m\left[\frac{\sqrt{2} \pi \lambda^{3}}{3 g}+\frac{\sqrt{2} e \pi \lambda}{g}+\frac{1}{3} \lambda(2-3 \sqrt{2} \operatorname{arsinh}(1))+\frac{e^{2} \pi}{\sqrt{2} g \lambda}-\frac{g \pi}{4 \sqrt{2} \lambda}\right. \\
& \left.-\frac{e(2+\sqrt{2} \operatorname{arsinh}(1))}{\lambda}\right], \\
I_{0}^{1010}= & m\left[2 \lambda(4+i \sqrt{2} \pi-2 \sqrt{2} \operatorname{arsinh}(1))-2 \sqrt{e+i g} \pi^{2}\right. \\
& \left.+\frac{2 \sqrt{2} e(-i \pi+2 \operatorname{arsinh}(1))}{\lambda}+\frac{g(-4 i+3 \sqrt{2} \pi+6 i \sqrt{2} \operatorname{arsinh}(1))}{\lambda}\right], \\
I_{0}^{1001}= & m\left[\lambda(8-\sqrt{2} \ln (17+12 \sqrt{2}))-\frac{\sqrt{2} g \pi}{\lambda}+\frac{\sqrt{2} e \ln (17+12 \sqrt{2})}{\lambda}\right],
\end{aligned}
$$

$$
\begin{aligned}
& I_{0}^{2010}=\frac{1}{m}\left[\frac{\pi^{2}}{2 \sqrt{e+i g}}+\frac{4+3 i \sqrt{2} \pi-6 \sqrt{2} \operatorname{arsinh}(1)}{2 \lambda}\right. \\
& \left.+\frac{g(4 i-11 \sqrt{2} \pi-22 i \sqrt{2} \operatorname{arsinh}(1))}{12 \lambda^{3}}-\frac{e(4-i \sqrt{2} \pi+2 \sqrt{2} \operatorname{arsinh}(1))}{6 \lambda^{3}}\right], \\
& I_{0}^{2001}=\frac{1}{m}\left[\frac{4-i \sqrt{2} \pi-6 \sqrt{2} \operatorname{arsinh}(1)}{2 \lambda}+\frac{g(12 i+3 \sqrt{2} \pi-2 i \sqrt{2} \operatorname{arsinh}(1))}{4 \lambda^{3}}\right. \\
& \left.-\frac{e(4+3 i \sqrt{2} \pi+2 \sqrt{2} \operatorname{arsinh}(1))}{6 \lambda^{3}}\right], \\
& I_{0}^{1101}=\frac{1}{m}\left[-\frac{\sqrt{2} \pi \lambda}{g}+\frac{i \sqrt{e-i g} \pi^{2}}{g}+\frac{\sqrt{2} e \pi}{g \lambda}+\frac{2-2 i \sqrt{2} \pi-3 \sqrt{2} \operatorname{arsinh}(1)}{\lambda}\right. \\
& -\frac{e^{2} \pi}{3 \sqrt{2} g \lambda^{3}}-\frac{e(2+2 i \sqrt{2} \pi+\sqrt{2} \operatorname{arsinh}(1))}{3 \lambda^{3}} \\
& \left.+\frac{g(-11 \sqrt{2} \pi+16 i(2+\sqrt{2} \operatorname{arsinh}(1)))}{24 \lambda^{3}}\right], \\
& I_{0}^{1111}=\frac{1}{m_{t}^{3}}\left[\frac{\pi^{2} \operatorname{Re}[\sqrt{e+i g}]}{g^{2}}-\frac{2 \sqrt{2} \pi}{g \lambda}-\frac{2 \sqrt{2} e \pi}{3 g \lambda^{3}}+\frac{2(2+\sqrt{2} \operatorname{arsinh}(1))}{3 \lambda^{3}}\right. \\
& \left.+\frac{\sqrt{2} e^{2} \pi}{5 g \lambda^{5}}+\frac{7 g \pi}{10 \sqrt{2} \lambda^{5}}+\frac{e(4-2 \sqrt{2} \operatorname{arsinh}(1))}{5 \lambda^{5}}\right], \\
& I_{1}^{2000, x}=-2 \pi+i \pi^{2}-\frac{2 \sqrt{2} \sqrt{e+i g x}(\pi+2 i(\sqrt{2}+\operatorname{arsinh}(1)))}{\lambda}+\frac{4 i g \pi(-1+x)}{\lambda^{2}}, \\
& I_{1}^{0000, x}=-2 \pi-i \pi^{2}+\frac{2 \sqrt{2} \sqrt{e+i g x}(\pi-2 i(\sqrt{2}+\operatorname{arsinh}(1)))}{\lambda}+\frac{4 i g \pi(1+x)}{\lambda^{2}}, \\
& I_{1}^{1010, x}=-\frac{1}{2} \pi^{2}(3 \pi+4 i \ln 2+8 i \ln (\sqrt{e+i g}+\sqrt{e+i g x})-8 i \ln (\lambda)) \\
& +\frac{8 \sqrt{2} \sqrt{e+i g x}(\pi+2 i \operatorname{arsinh}(1))}{\lambda}+\frac{2 g \pi(\pi-2 i(-1+x))}{\lambda^{2}}, \\
& I_{1}^{0101, x}=\frac{1}{2} \pi^{2}(-3 \pi+4 i \ln 2+8 i \ln (\sqrt{e-i g}-\sqrt{e+i g x})-8 i \ln (\lambda)) \\
& -\frac{8 \sqrt{2} \sqrt{e+i g x}(\pi-2 i \operatorname{arsinh}(1))}{\lambda}+\frac{2 g \pi(\pi-2 i(1+x))}{\lambda^{2}}, \\
& I_{1}^{1001, x}=\frac{\pi^{3}}{2}+\frac{16 i \sqrt{2} \sqrt{e+i g x} \operatorname{arsinh}(1)}{\lambda}-\frac{2 g \pi(\pi+2 i x)}{\lambda^{2}},
\end{aligned}
$$

$$
\begin{aligned}
& I_{1}^{2010, x}=\frac{1}{m^{2}}\left[-\frac{\pi^{2}}{g(x-1)}\left(1-\frac{\sqrt{e+i g x}}{\sqrt{e+i g}}\right)+\frac{i \pi(2 i+\pi)}{\lambda^{2}}\right. \\
& \left.-\frac{2 \sqrt{2} \sqrt{e+i g x}(\pi+2 i(\sqrt{2}+\operatorname{arsinh}(1)))}{3 \lambda^{3}}-\frac{g \pi(2 i+\pi)}{\lambda^{4}}\right], \\
& I_{1}^{0201, x}=\frac{1}{m^{2}}\left[\frac{\pi^{2}}{g(x+1)}\left(1+\frac{\sqrt{e+i g x}}{\sqrt{e-i g}}\right)-\frac{i \pi(-2 i+\pi)}{\lambda^{2}}\right. \\
& \left.+\frac{2 \sqrt{2} \sqrt{e+i g x}(\pi-2 i(\sqrt{2}+\operatorname{arsinh}(1)))}{3 \lambda^{3}}-\frac{g \pi(-2 i+\pi)}{\lambda^{4}}\right], \\
& I_{1}^{2001, x}=\frac{1}{m^{2}}\left[-\frac{i \pi(-2 i+\pi)}{\lambda^{2}}+\frac{2 \sqrt{2} \sqrt{e+i g x}(3 \pi-2 i(\sqrt{2}+\operatorname{arsinh}(1)))}{3 \lambda^{3}}\right. \\
& \left.+\frac{g \pi(2 i+\pi)}{\lambda^{4}}\right], \\
& I_{1}^{0210, x}=\frac{1}{m^{2}}\left[\frac{i \pi(2 i+\pi)}{\lambda^{2}}-\frac{2 i \sqrt{2} \sqrt{e+i g x}(-3 i \pi+2(\sqrt{2}+\operatorname{arsinh}(1)))}{3 \lambda^{3}}\right. \\
& \left.+\frac{g \pi(-2 i+\pi)}{\lambda^{4}}\right], \\
& I_{1}^{1011, x}=\frac{1}{m^{2}}\left[\frac{\pi^{2}(-i \pi+\ln 2+2 \ln (\sqrt{e+i g}+\sqrt{e+i g x})-2 \ln (\lambda))}{g}\right. \\
& \left.+\frac{4 i \sqrt{2} \pi \sqrt{e+i g x}}{g \lambda}+\frac{-2 \pi+2 i \pi^{2}}{\lambda^{2}}\right], \\
& I_{1}^{1111, x}=\frac{1}{m^{4}}\left[\frac{\pi^{3}}{g^{2}}-\frac{i \pi^{2} \ln (\sqrt{e-i g}-\sqrt{e+i g x})}{g^{2}}\right. \\
& \left.+\frac{i \pi^{2} \ln (\sqrt{e+i g}+\sqrt{e+i g x})}{g^{2}}-\frac{2 \pi^{2}}{g \lambda^{2}}-\frac{8 i \sqrt{2} \pi \sqrt{e+i g x}}{3 g \lambda^{3}}+\frac{2 \pi}{\lambda^{4}}\right], \\
& I_{2}^{1111}=\frac{1}{m^{5}}\left[-\frac{i \sqrt{e-i g} \pi^{2}}{8 g^{3}}-\frac{i e \pi^{2}}{8 \sqrt{e+i g} g^{3}}+\frac{2 \sqrt{2} \pi}{3 g \lambda^{3}}+\frac{2 \sqrt{2} e \pi}{5 g \lambda^{5}}\right. \\
& \left.+\frac{-128+128 i \sqrt{2} \pi+54 \sqrt{2} \operatorname{arsinh}(1)+5 \sqrt{2} \log (3+2 \sqrt{2})}{160 \lambda^{5}}\right] .
\end{aligned}
$$

## Appendix F

## Constants and Abbreviations

In this appendix we give expressions for constants appearing in the main part of this work.

The $\mathcal{O}\left(\alpha_{s}^{2}\right)$ and $\mathcal{O}\left(\alpha_{s}^{3}\right)$ coefficients of the QCD $\beta$-function

$$
\beta\left(\alpha_{s}\right)=-\frac{\alpha_{s}^{2}}{4 \pi} \beta_{0}-\frac{\alpha_{s}^{3}}{(4 \pi)^{2}} \beta_{1}+\mathcal{O}\left(\alpha_{s}^{3}\right)
$$

read

$$
\begin{aligned}
& \beta_{0}=\frac{11}{3} C_{A}-\frac{4}{3} T_{F} n_{l}, \\
& \beta_{1}=\frac{34}{3} C_{A}^{2}-\frac{20}{3} C_{A} T_{F} n_{l}-4 C_{F} T_{F} n_{l} .
\end{aligned}
$$

The color factors of $\mathrm{SU}(3)_{c}$ are

$$
\begin{array}{ll}
T_{F}=\frac{1}{2}, & C_{A}=N_{c}=3,
\end{array} C_{F}=\frac{N_{c}^{2}-1}{2 N_{c}}=\frac{4}{3}, ~ 子 C_{d}=8 C_{F}-3 C_{A}, ~ l
$$

and the number of light quarks is $n_{l}=5$.
The constants that appear in the effective Coulomb potential $\tilde{V}_{c}(\mathbf{p}, \mathbf{q})$ read

$$
\begin{aligned}
a_{1}= & \frac{31}{9} C_{A}-\frac{20}{9} T_{F} n_{l} \\
a_{2}= & \left(\frac{4343}{162}+4 \pi^{2}-\frac{\pi^{4}}{4}+\frac{22}{3} \zeta_{3}\right) C_{A}^{2}-\left(\frac{1798}{81}+\frac{56}{3} \zeta_{3}\right) C_{A} T_{F} n_{l} \\
& -\left(\frac{55}{3}-16 \zeta_{3}\right) C_{F} T_{F} n_{l}+\left(\frac{20}{9} T_{F} n_{l}\right)^{2} .
\end{aligned}
$$

Here, $\zeta$ denotes the Riemann $\zeta$-function.

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[^0]:    ${ }^{1}$ An alternative method for the extraction of the top Yukawa coupling is the measurement of Higgs associated top pair production rates, see e. g. Refs. [9, 10].

[^1]:    ${ }^{1}$ Here, $a_{\tau}(\mathbf{k})$ and $a_{\tau^{\prime}}^{c}\left(\mathbf{k}^{\prime}\right)$ are operators for the annihilation of an electron and a positron with spin $\tau, \tau^{\prime}$ and 3 -momentum $\mathbf{k}, \mathbf{k}^{\prime}$, respectively, and $|0\rangle$ is the vacuum state.

[^2]:    ${ }^{2}$ At LL order one usually uses the zero-distance Green function $G^{0}\left(a, v, m_{t}, \nu\right)$ instead of $\mathcal{A}_{1}\left(v, m_{t}, \nu\right)$, which leads to

    $$
    \begin{aligned}
    \sigma^{\mathrm{LL}}\left(P_{-}, P_{+}\right)=2 N_{c} \operatorname{Im}\{ & {\left[\left(1-P_{-} P_{+}\left(\left(C_{V, 1}^{\text {born }}\right)^{2}+\left(C_{A, 1}^{\text {born }}\right)^{2}\right)\right.\right.} \\
    & \left.\left.+\left(P_{-}-P_{+}\right) 2 C_{V, 1}^{\text {born }} C_{A, 1}^{\text {born }}\right] G^{0}\left(a, v, m_{t}, \nu\right)\right\} .
    \end{aligned}
    $$

    according to Eq. (1.28).

[^3]:    ${ }^{3}$ Of course there are also hard $\mathcal{O}\left(\alpha_{s}\right)$ and $\mathcal{O}\left(\alpha_{s}^{2}\right) \mathrm{QCD}$ corrections to the coefficients of the operators $\mathcal{O}_{V / A, \mathbf{p}, \sigma}$. Although these corrections actually constitute interference effects between QCD and the electroweak theory, they are conventionally considered simply as QCD effects. We do not take QCD effects into account in this discussion.

[^4]:    ${ }^{1}$ In a more complete treatment one would also involve the instability of the $W$ boson. The effects of this additional decay should be suppressed at least by a factor $\Gamma_{W} / E_{\text {kin }}^{W}$ and are neglected in this work. Here, $\Gamma_{W}$ and $E_{\text {kin }}^{W}$ denote decay width and kinetic energy of the $W$ boson, respectively.

[^5]:    ${ }^{2}$ There are also other interferences, namely those of single- or non-resonant diagrams with the final state $b W^{+} \bar{b} W^{-}$. These background diagrams are not described by the effective theory. Their numerical contributions will be estimated in Sec. 4.8.

[^6]:    ${ }^{3}$ In a complete analysis of electroweak effects the top quark width depends on the input parameters given above and is not an independent parameter. For the purpose of the numerical analysis in this work, however, our treatment is sufficient.

[^7]:    ${ }^{1}$ The obvious differences from the technical point of view are the following. Since diagrams (c) and (d) have insertions in the uncut loops, one has to replace the function $D^{c}(p)$ in Eq. (4.30) by the corresponding one containing the kinetic energy insertion and similarly other insertions for the remaining corrections. In the cases of the kinetic energy and the time dilatation correction this leads to expressions that involve the basic integral $I_{2}^{1111}$ in addition to the integrals $I_{1}^{p_{1} q_{1} p_{2} q_{2}, 1}$. For diagrams (f) - (j) one has to include correction terms in the phase space integration similarly to the one-loop calculation, the difference being the use of $I_{1}^{p_{1} q_{1} p_{2} q_{2}, 1}$ instead of $I_{0}^{p_{1} q_{1} p_{2} q_{2}, 1}$.

[^8]:    ${ }^{2}$ Since $\mathcal{V}_{s}^{(s)}$ does not depend on the momentum $\mathbf{p}$, any cut one-loop subgraph (associated with a phase space integration over $\mathbf{p}$ ) that contains only the $\mathcal{V}_{s}^{(s)}$ potential factorizes from the rest of the diagram. At two-loop level this subgraph is contracted to $\delta \tilde{c}_{1}^{s, 1}$.

[^9]:    ${ }^{3}$ From the formal point of view, Coulomb resonances contained in the zero-distance Green functions in fact correspond to a summation of $(\pi a / v)^{n}$ terms. However, the numerical coefficients lead to a suppression of higher order terms so that one can consider the $(\pi a / v)^{n}$ expansion as an $\left(\alpha_{s} / v\right)^{n}$ expansion from the numerical point of view.
    ${ }^{4}$ From the parametric structure of all these phase space contributions one can easily guess that the correction due to the potential

[^10]:    ${ }^{5}$ Because the $t_{1,2}$ range in Eq. (4.36) is not symmetric around zero, the comparison of the upper bounds yields a result for $\Lambda^{2}$ that deviates from this by an addend (3/2) $\Lambda_{c}^{2}-E^{2} / 2$. However, this deviation is negligible because the additional restriction

    $$
    0 \leq \mathbf{p}^{2}=m_{t} E-\frac{1}{2}\left(t_{1}+t_{2}\right)
    $$

    cuts off the upper edges of the $t_{1,2}$ range anyway (except for the numerically irrelevant regions where $t_{1} \approx \Lambda^{2}, t_{2} \approx-\Lambda^{2}$ and $t_{1} \approx-\Lambda^{2}, t_{2} \approx \Lambda^{2}$ in the case $E>0$ ), see Fig 4.2.

