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Maximal Lattice-Free Polyhedra in Mixed-Integer Cutting Plane Theory

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Maximal Lattice-Free Polyhedra in Mixed-Integer Cutting Plane Theory

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ZUSAMMENFASSUNG

Die vorliegende Dissertationsschrift befasst sich mit der Gewinnung, Bewertung und Analyse von Schnittebenen für *gemischt-ganzzahlige lineare Optimierungsprobleme*, kurz GLO's. Bei Optimierungsproblemen dieses Typs möchte man eine lineare Zielfunktion über einer endlichen Menge von linearen Gleichungs- und Ungleichungsbedingungen unter der Ganzzahligkeitsforderung an alle oder nur einen Teil der endlich vielen Variablen maximieren oder minimieren.

Viele praktische Probleme lassen sich mathematisch als GLO modellieren. Das gleichzeitige Vorhandensein diskreter und kontinuierlicher Variablen führt dazu, dass GLO's algorithmisch schwer zu lösen sind. Oft lassen sich Lösungen für praktisch relevante Probleme einer gewissen Grösse mit den heute bekannten Lösungsverfahren nur heuristisch oder überhaupt nicht berechnen. Deshalb ist man daran interessiert, neue Lösungstechniken zu entwickeln.

Eine häufig verwendete Methode zum Lösen von GLO's ist der Einsatz von *Schnittebenenverfahren*. Die zentrale Idee solcher Verfahren ist es, aus dem gegebenen GLO eine Folge von linearen Optimierungsproblemen zu konstruieren, deren Formulierung sich durch das Hinzufügen linearer Nebenbedingungen – sogenannter *Schnittebenen* – sukzessive verbessert, bis schliesslich eine optimale Lösung für eines der linearen Optimierungsprobleme die Ganzzahligkeitsforderung für die betreffenden Variablen erfüllt.

Für eine Vielzahl kombinatorischer Probleme lassen sich direkt ganze Familien von Schnittebenen ableiten, die von der kombinatorischen Struktur der Probleme herrühren. Im Gegensatz dazu lässt sich für ein *allgemeines* GLO

keine strukturelle Eigenschaft ausnutzen. Deshalb kann sich die Erzeugung von Schnittebenen nur auf die Zielfunktion und das gegebene, unstrukturierte System linearer Gleichungen und Ungleichungen stützen. Einerseits hat dies zur Folge, dass die Gewinnung von starken Schnittebenen für allgemeine GLO's gegenüber strukturierten Problemen schwieriger ist. Andererseits macht gerade dieser Aspekt die Analyse der Schnittebenengenerierung für allgemeine GLO's mathematisch interessant.

In dieser Dissertationsschrift wird ein Ansatz vorgestellt, der es erlaubt, mit Hilfe von *gitterpunktfreien Polyedern*, dass heisst Polyedern ohne inneren, ganzzahligen Punkt, Schnittebenen für ein allgemeines GLO zu gewinnen. Ausgangspunkt ist eine optimale Lösung der linearen Relaxierung des GLO und ein mit ihr assoziiertes, optimales Simplextableau. Durch Betrachtung von mehreren Zeilen dieses Simplextableaus wird eine weitere Relaxierung gewonnen.

Der erste Teil der Dissertationsschrift widmet sich der Analyse dieser Relaxierung und bespricht, wie Schnittebenen für das allgemeine GLO aus der untersuchten Relaxierung hergeleitet werden können. Es zeigt sich, dass die generierten Schnittebenen im Raum der diskreten Variablen eine geometrische Deutung besitzen und dass die stärksten aus der verwendeten Relaxierung ableitbaren Schnittebenen zu *maximal gitterpunktfreien Polyedern* korrespondieren. Damit lassen sich Fragestellungen über Schnittebenen in Fragestellungen über maximal gitterpunktfreie Polyeder übersetzen.

Der zweite Teil der Arbeit beschäftigt sich mit der Bewertung der generierten Schnittebenen. Im Ergebnis sind vor allem die Schnittebenen wichtig, die besonders schwer zu gewinnen sind, weil es zu ihrer Generierung sehr komplexer, maximal gitterpunktfreier Polyeder bedarf. Es wird ausserdem gezeigt, dass unter bestimmten Annahmen an das Gleichungs- und Ungleichungssystem des zugrundeliegenden GLO wichtige Schnittebenen durch Schnittebenen approximiert werden können, die aus weniger komplexen, maximal gitterpunktfreien Polyedern herleitbar sind. Mithilfe eines wahrscheinlichkeitstheoretischen Modells wird die Bewertung der Schnittebenen ergänzt und das Ergebnis geometrisch interpretiert.

Der dritte Teil der Dissertationsschrift konzentriert sich auf die Analyse von gitterpunktfreien Polyedern und untersucht die für einen Schnittebenenalgorithmus wichtige Klasse von gitterpunktfreien, *ganzzahligen Polyedern*. Zwei verschiedene Maximalitätsbegriffe werden eingeführt und die zugehörigen Klassen von Polyedern definiert. Konkret wird unterschieden in gitterpunktfreie, ganzzahlige Polyeder, die in keinem anderen gitterpunktfreien, ganzzahligen Polyeder echt enthalten sind; und in gitterpunktfreie, ganzzahlige Polyeder, die in keiner anderen *gitterpunktfreien, konvexen Menge* echt enthalten sind. Im Anschluss findet eine Analyse der beiden Klassen statt, die im Wesentlichen auf die Eigenschaften der Vertreter der Klassen und die Beziehung der Klassen zueinander fokussiert. Die Ergebnisse zeigen, dass beide Klassen sehr viele und sehr grosse Elemente beinhalten.

Sowohl für den zweiten als auch den dritten Teil der Arbeit werden Aussagen über zweidimensionale, gitterpunktfreie, konvexe Mengen benötigt. Deshalb ist ein vierter Teil der Arbeit für die Herleitung dieser Aussagen reserviert.

ABSTRACT

This thesis deals with the generation, evaluation, and analysis of cutting planes for *mixed-integer linear programs* (MILP's). Such optimization problems involve finitely many variables, some of which are required to be integer. The aim is to maximize or minimize a linear objective function over a set of finitely many linear equations and inequalities.

Many industrial problems can be formulated as MILP's. The presence of both, discrete and continuous variables, makes it difficult to solve MILP's algorithmically. The currently available algorithms fail to solve many real-life problems in acceptable time or can only provide heuristic solutions. As a consequence, there is an ongoing interest in novel solution techniques.

A standard approach to solve MILP's is to apply *cutting plane methods*. Here, the underlying MILP is used to construct a sequence of linear programs whose formulations are improved by successively adding linear constraints – so-called *cutting planes* – until one of the linear programs has an optimal solution which satisfies the integrality conditions on the integer constrained variables.

For many combinatorial problems, it is possible to immediately deduce several families of cutting planes by exploiting the inherent combinatorial structure of the problem. However, for *general* MILP's, no structural properties can be used. The generation of cutting planes must rather be based on the objective function and the given, unstructured set of linear equations and inequalities. On the one hand, this makes the derivation of strong cutting planes for general MILP's more difficult than the derivation of cutting planes for structured problems. On the other hand, for this very reason, the anal-

ysis of cutting plane generation for general MILP's becomes mathematically interesting.

This thesis presents an approach to generate cutting planes for a general MILP. The cutting planes are obtained from *lattice-free polyhedra*, that is polyhedra without interior integer point. The point of departure is an optimal solution of the linear programming relaxation of the underlying MILP. By considering multiple rows of an associated simplex tableau, a further relaxation is derived.

The first part of this thesis is dedicated to the analysis of this relaxation and it is shown how cutting planes for the general MILP can be deduced from the considered relaxation. It turns out that the generated cutting planes have a geometric interpretation in the space of the discrete variables. In particular, it is shown that the strongest cutting planes which can be derived from the considered relaxation correspond to *maximal lattice-free polyhedra*. As a result, problems on cutting planes are transferable into problems on maximal lattice-free polyhedra.

The second part of this thesis addresses the evaluation of the generated cutting planes. It is shown that the cutting planes which are important, are at the same time the cutting planes which are difficult to derive in the sense that they correspond to highly complex maximal lattice-free polyhedra. In addition, it is shown that under certain assumptions on the underlying system of linear equations and inequalities, the important cutting planes can be approximated with cutting planes which correspond to less complex maximal lattice-free polyhedra. A probabilistic model is used to complement the analysis. Moreover, a geometric interpretation of the results is given.

The third part of this thesis focuses on the analysis of lattice-free polyhedra. In particular, the class of lattice-free *integral polyhedra* is investigated, a class which is important within a cutting plane framework. Two different notions of maximality are introduced. It is distinguished into the class of lattice-free integral polyhedra which are not properly contained in another lattice-free integral polyhedron, and the class of lattice-free integral polyhedra which are not properly contained in another *lattice-free convex set*. Both classes are analyzed, especially with respect to the properties of their representatives and the relation between the two classes. It is shown that both classes are of large cardinality and that they contain very large elements.

For the second as well as the third part of this thesis, statements about two-dimensional lattice-free convex sets are needed. For that reason, the fourth part of this thesis is devoted to the derivation of these results.

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CHAPTER 1

INTRODUCTION

The aim of this chapter is to provide an overview of the agenda of this thesis. We introduce the underlying optimization problem and explain step by step our motivation for choosing the research questions that are studied in this thesis.

We assume that a general mixed-integer linear program (MILP) is given in the form

$$\begin{aligned} \max\ c^\top x \quad \text{s.t. } Ax &= b, \\ x &\geq 0, \\ x_i &\in \mathbb{Z} \quad \text{for } i \in \mathcal{I}, \\ x_i &\in \mathbb{R} \quad \text{for } i \in \mathcal{C}, \end{aligned} \tag{1.1}$$

where A, b, and c are rational and $\mathcal{I}$, resp. $\mathcal{C}$, is a set of integer constrained, resp. continuous, variables. The linear programming relaxation of (1.1) is the optimization problem (1.1) where the condition $x_i \in \mathbb{Z}$ is replaced by the weaker condition $x_i \in \mathbb{R}$ for all $i \in \mathcal{I}$. To avoid trivial cases we assume that the feasible region of (1.1) is non-empty and that its linear programming relaxation is bounded. Solving the linear programming relaxation yields an optimal vertex x^* with corresponding sets B and N of basic and non-basic variables which satisfy

$$x_i = f_i + \sum_{j \in N} r_i^j x_j \quad \forall i \in B,$$

where $f_i \in \mathbb{Q}_+$ and $r_i^j \in \mathbb{Q}$ for all $i \in B$ and all $j \in N$. We assume that x^* is not feasible for (1.1), otherwise we have already found an optimal solution.

Our aim is to generate cutting planes (or *cuts* for short) which cut off x^*, i.e. inequalities which are valid for every feasible point of (1.1), but violated by x^*.

Virtually all traditional cutting planes that are used by general-purpose MILP solvers, most notably lift-and-project cuts (see, for instance, [BCC93]), Gomory mixed-integer cuts (see, for instance, [Gom60]), or mixed-integer rounding cuts (see, for instance, [NW90]), are derived by considering only one equation. Normally, the strategy is to generate a linear combination of the original constraints $Ax = b$. Then one applies integrality arguments to the resulting equation. Cuts obtained in this way are *split cuts* (see, for instance, [CKS90]). Unfortunately, an approach that is based on such cuts alone does not give rise to a finite cutting plane algorithm. In [CKS90], an instance in only three variables is presented and it is shown that a cutting plane algorithm based on split cuts does not converge finitely.

Example 1.1. Consider the following MILP.

$$\begin{aligned} \max\ t \quad \text{s.t.} \qquad -x_1 + t &\leq 0, \\ -x_2 + t &\leq 0, \\ x_1 + x_2 + t &\leq 2, \\ x_1, x_2 &\in \mathbb{Z}, \\ t &\in \mathbb{R}_+. \end{aligned}$$

The cut needed to solve this problem is $t \leq 0$. However, in [CKS90] it is shown that this cut cannot be obtained by applying split cuts only. $\diamond$

In [ALWW07], Andersen et al. initiated a new approach for cutting plane generation by considering two rows of a simplex tableau simultaneously. This approach allows to deduce cutting planes that cannot be obtained by considering one single equation. In particular, the desired cut $t \leq 0$ in Example 1.1 can be derived immediately.

Meanwhile, the two-row case has been analyzed quite exhaustively, most notably due to Andersen et al. [ALWW07], Borozan and Cornuéjols [BC07], Cornuéjols and Margot [CM08], and Basu et al. [BBCM11]. However, the basic idea of the two-row approach can be generalized to the case of multiple rows in a straightforward way. For that, the point of departure is an optimal vertex x^* of the linear programming relaxation of (1.1). We assume that $m := |B \cap \mathcal{I}| \geq 2$ and $f_i \notin \mathbb{Z}$ for at least one $i \in B \cap \mathcal{I}$. We consider the set

$$P_I := \left\{ (x, s) \in \mathbb{Z}^m \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j \right\},$$

where $N := \{1, \ldots, n\}$ represents the non-basic variables and f, resp. r^j, is the vector consisting of all f_i's, resp. r^j_i's, such that $i \in B \cap \mathcal{I}$.

The set P_I is the underlying mixed-integer set in this thesis. Our motivation for analyzing P_I is that it can be obtained as a relaxation of the feasible region of a general MILP. Therefore, valid inequalities for P_I give rise to cutting planes for the original mixed-integer set. Consequently, our aim is to derive valid inequalities for P_I, or equivalently, for $\mathrm{conv}(P_I)$.

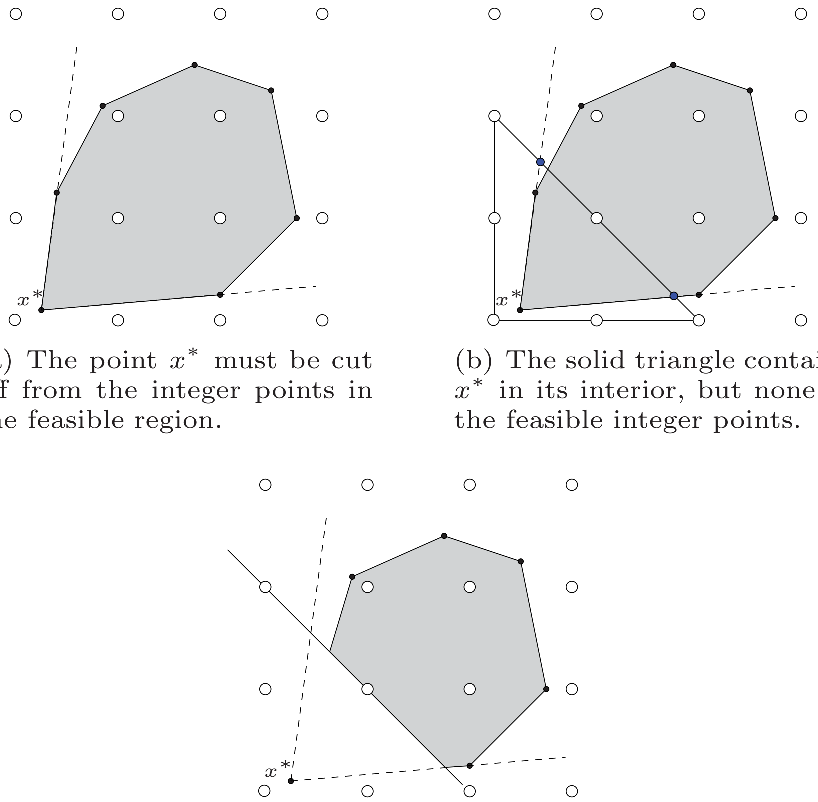
In Chapter 3, we show that valid inequalities for $\mathrm{conv}(P_I)$ correspond to combinatorial objects in the space of the discrete variables. More precisely, they correspond to *lattice-free* polyhedra, i.e. polyhedra that do not contain an interior integer point. The basic properties of the set $\mathrm{conv}(P_I)$ are summarized in Section 3.1, and the relation between lattice-free polyhedra and the facet-defining inequalities for $\mathrm{conv}(P_I)$ is presented in Section 3.2.

By considering $\mathrm{conv}(P_I)$, the feasible region of the original MILP (1.1) is relaxed in two ways. First, we drop all integrality conditions on the non-basic variables. Second, the non-negativity restrictions on all basic variables are ignored. The latter relaxation has been introduced by Gomory [Gom69] and is known as the classical group relaxation. The first relaxation, however, is the great novelty in the new approach. It preserves much of the complexity of the original model, but keeps it sufficiently simple to analyze it.

The following example illustrates the cutting plane approach that we have in mind.

Example 1.2. Fig. 1.1 exemplifies our intended approach to generate cutting planes. For simplicity, let $m = 2$. The gray regions in Fig.s 1.1(a) and 1.1(b) represent the projection of the linear programming relaxation onto the space of the x-variables. After relaxing the integrality conditions on the non-basic variables and the non-negativity restrictions on the basic variables, we obtain a *corner polyhedron* (see, for instance, [Gom69]). The convex hull of the two dashed half-lines in Fig.s 1.1(a)–1.1(c) is the projection of the corner polyhedron onto the space of the x-variables. Fig. 1.1(b) shows how the solid lattice-free triangle is used to cut off x^*. The intersection points of the triangle and the two dashed half-lines determine the cutting plane. After adding the cutting plane, the feasible region of the linear programming relaxation becomes smaller. Its projection onto the space of the x-variables is the gray region in Fig. 1.1(c). $\diamond$

Since, by assumption, x^* is not feasible for (1.1), we aim at generating cutting planes that are violated by the basic solution $x_i^* = f_i$ for all $i \in B$ and $x_j^* = 0$ for all $j \in N$. For that, we look for valid inequalities for $\mathrm{conv}(P_I)$ which cut off the point (f, o). It turns out that the non-trivial facet-defining inequalities for $\mathrm{conv}(P_I)$, i.e. the strongest inequalities that we can derive from our relaxation, do perform this task: all of them are violated by (f, o). This implies that we can focus our attention on non-trivial facet-defining inequalities for $\mathrm{conv}(P_I)$. At this point the enormous power of the applied



(a) The point x^* must be cut off from the integer points in the feasible region.

(b) The solid triangle contains x^* in its interior, but none of the feasible integer points.

(c) The solid line separates x^* from the integer points in the feasible region.

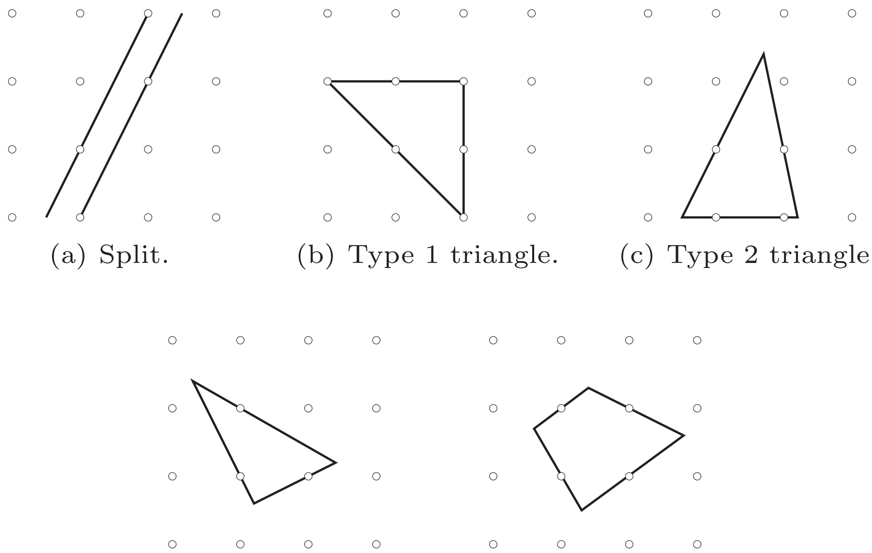
Figure 1.1: Derivation of a cutting plane.

relaxation comes into play, because all these non-trivial facet-defining inequalities correspond to lattice-free polyhedra which possess beautiful geometrical properties. The exact relation is stated in Theorem 3.9 where we show that every non-trivial facet-defining inequality for $\text{conv}(P_I)$ can be derived from a lattice-free polyhedron which has a representation as the sum of a polytope and a linear space. It follows that strongest cutting planes are associated with *maximal lattice-free polyhedra*, i.e. lattice-free polyhedra which are not properly contained in another lattice-free polyhedron. Structural properties of maximal lattice-free polyhedra entail information on the corresponding cutting planes and therefore, instead of analyzing cutting planes, we can equivalently analyze maximal lattice-free polyhedra. As a result of this, several questions related to these polyhedra and their associated cutting planes arise.

Certainly, the aim in cutting plane generation should not be to produce a bulk of cuts which just cut off the current optimal linear programming solution, but rather to identify a (preferably small) set of well-chosen cuts which are "important" in some sense. Here, "important" is difficult to define. There are several approaches to evaluate cutting planes, for instance with respect to the volume which is cut off, a comparison of the cut coefficients, or the improvement of the objective function value after adding a cut or a set of cuts. The choice of the measure is highly dependent on the particular structure of the problem. Since we start from a general MILP it is simply not possible to say which measure is most suitable. In this thesis, we use a strength measure of Goemans [Goe95] to evaluate non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$. Every such inequality corresponds to a maximal lattice-free polyhedron in the m-dimensional space of the x-variables. And each such polyhedron P can be represented as $P = \mathcal{P} + \mathcal{L}$, where $\mathcal{P}$ is a polytope and $\mathcal{L}$ is a linear space. The codimension of $\mathcal{L}$ is called the *split-dimension of* P. In turn, the split-dimension of a non-trivial facet-defining inequality I for $\operatorname{conv}(P_I)$ is defined to be the smallest split-dimension of a maximal lattice-free polyhedron P such that P can be used to derive an inequality for $\operatorname{conv}(P_I)$ which is equal to or which dominates the inequality I.

In Chapter 4, we investigate which of the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ are needed to approximate $\operatorname{conv}(P_I)$ sufficiently well with respect to the strength measure of Goemans. In Theorem 4.4, we show that, in general, good approximations for $\operatorname{conv}(P_I)$ can be expected only by having available all the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ of split-dimension m. This result is clearly unsatisfactory since the complexity of the corresponding maximal lattice-free polyhedra increases with increasing split-dimension. Consequently, inequalities of split-dimension m are difficult to generate. In contrast to this negative result on the strength, in Theorem 4.7, we show that by restricting the size of the data, inequalities of split-dimension m can be approximated using inequalities of split-dimension one (i.e. split cuts). This is a positive message since split cuts are the easiest objects in terms of complexity. In particular, we show that, given the dimension m of the x-variable space, the fractionality of the current optimal solution (f, o), and the *max-facet-width* of a lattice-free polyhedron P of split-dimension m, then the inequality corresponding to P can be approximated to within a constant factor which involves only these three quantities. For the special case where P is a *regular lattice-free simplex* (RLS), in Theorem 4.8, we even state a constant which involves only the dimension m. This raises hope that cuts with low split-dimension perform well in practice.

In Chapter 6, we address the case $m = 2$ in order to obtain deeper results on the approximability of inequalities of split-dimension two by split cuts. As pointed out, the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ are associated with maximal lattice-free polyhedra. In dimension two, these polyhedra can be partitioned into five types which are shown in Fig. 1.2 (see Proposition 5.3 on p. 37 for the precise definition of each type).



(a) Split. (b) Type 1 triangle. (c) Type 2 triangle.

(d) Type 3 triangle. (e) Quadrilateral.

Figure 1.2: All types of two-dimensional maximal lattice-free polyhedra.

Since every non-trivial facet-defining inequality for $\operatorname{conv}(P_I)$ corresponds to one of the above maximal lattice-free sets, they are called split, type 1, type 2, type 3, or quadrilateral inequalities. In [BBCM11] it has been shown that the closures of split and type 1 inequalities may produce an arbitrarily bad approximation of $\operatorname{conv}(P_I)$, whereas the closures of type 2 or type 3 or quadrilateral inequalities deliver good approximations in terms of the strength measure of Goemans. More concretely, in [BBCM11] sequences of examples are constructed in which cuts from triangles of types 2 and 3, and quadrilaterals cannot be approximated to within a constant factor by using split and type 1 inequalities only. The approximation becomes worse as the triangles and quadrilaterals converge towards a split. We think that this is geometrically counterintuitive. Therefore, in Chapter 6, we refine the argument by taking into consideration the probability that such a situation emerges when f is uniformly distributed in the interior of a given maximal lattice-free triangle of type 2, type 3, or quadrilateral. The precise model is explained in Section 6.2. Our main result of the probabilistic analysis in

Chapter 6 is stated in Theorem 6.2, where we show that the addition of a single type 2 inequality to the split closure becomes less likely to be beneficial the closer the type 2 triangle looks like a split. Our analysis in Chapter 6 suggests that this is true for type 3 and quadrilateral inequalities as well.

The performance of cuts may be evaluated in two different ways. In dimension two, if one considers only *one round* of cuts, then – using the strength measure of Goemans – split and type 1 inequalities can be arbitrarily bad in approximating $\operatorname{conv}(P_I)$. On the other hand, within a cutting plane framework where *several rounds* of cuts are considered, it is enough to add split and type 1 inequalities iteratively, in order to terminate with an optimal mixed-integer point after a finite number of applied rounds (see [DL09] and also [BCM11] and [DPW11] for a generalization of the results in [DL09]). Using the correspondence between the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ and maximal lattice-free polyhedra this insight leads to a natural question: Which maximal lattice-free polyhedra are important in a cutting plane framework? Admittedly, this question is too general to be answered completely within this thesis. Nevertheless, the answer must have to do with the integer points on the boundary of the maximal lattice-free polyhedra. Since rationality of the input data is assumed we only need to consider maximal lattice-free rational polyhedra.

In Chapter 7, we show in Theorem 7.2 that, given the dimension (i.e. the number of simplex tableau rows from which a non-trivial facet-defining inequality for $\operatorname{conv}(P_I)$ is derived) and the rationality of a corresponding maximal lattice-free polyhedron P, then only finitely many different shapes are possible for P, provided we identify any two polyhedra which coincide up to a transformation which preserves the integer lattice. Unfortunately, "finitely many" does not mean "few". Indeed, in Section 7.3 we provide an upper bound on the volume of such a polytope. Our bound is by far not best possible, but suggests that the number of potential shapes may explode dramatically with increasing dimension. This makes clear that there is no chance to enumerate all shapes based on a computer code, even for small dimensions.

The fact that in dimension two only split and type 1 inequalities are needed within a cutting plane framework is not just coincidence, but rather has to do with the integer points on the boundary. The *X-body*[1] of a lattice-free polyhedron is the convex hull of the integer points on its boundary. In particular, a lattice-free polyhedron coincides with its X-body if and only if it is an *integral polyhedron* in the sense that every minimal (non-empty) face contains integer points. In Fig. 1.2, only the split and the type 1 triangle

[1] The notion "X-body" is quite unintuitive, but we use it for historical reasons.

are integral polyhedra. Recently, it has been proved by Del Pia and Weismantel [DPW11] that the X-body of a lattice-free polyhedron is connected with the importance of the polyhedron in a cutting plane procedure. To be precise, within a cutting plane framework, only *lattice-free integral polyhedra* are needed. Thus, a characterization of maximal lattice-free integral polyhedra is desired. In dimensions one and two, all shapes of maximal lattice-free integral polyhedra are known. On the other hand, their number is expected to be huge in dimensions beyond three.

In Chapter 8, we classify all three-dimensional maximal lattice-free integral polyhedra. We first show that we can restrict our attention to polytopes. Then, in Theorem 8.1, we enumerate all three-dimensional maximal lattice-free polytopes with integer vertices.

Theorems 6.2 and 8.1 are proved by intensively using two-dimensional tools which cannot be deduced offhand. Therefore, we dedicate an extra chapter to the two-dimensional relation between the area and the *lattice width* of lattice-free convex sets. In Chapter 5, we prove several inequalities which involve the area and the lattice width in the plane. In Theorem 5.6, we present our results for arbitrary lattice-free convex sets and in Theorem 5.9 we present our results for centrally symmetric ones. We further characterize the extreme lattice-free convex sets and relate our results to the *covering minima* introduced in [KL86]. Moreover, in Theorem 5.10 we rectify a result of [KL88] with a new proof.

CHAPTER 2

NOTATION AND FOUNDATIONS

We assume that the reader is familiar with the basic notions and concepts in linear algebra, discrete optimization, and convex geometry. Information on linear algebra can be found in the book of Stroth [Str95]. For information on lattices and convexity, in particular with respect to polyhedra, we refer to the books of Barvinok [Bar02], Gruber [Gru07], Gruber and Lekkerkerker [GL87], Rockafellar [Roc72], and Schneider [Sch93a]. The book of Schrijver [Sch86, Chapter 23] contains useful information on cutting plane theory. At the beginning of each chapter or section we will provide the reader with the relevant material. In this chapter, we briefly outline tools which are permanently used throughout this thesis.

In this thesis, all vectors are usually considered to be column vectors. The transposition of a vector or a matrix is denoted by $(\cdot)^\top$. Furthermore, we denote by e_j the j-th unit vector, by o the origin, and by $\mathbb{1}$ the vector whose entries are 1. These vectors are assumed to have suitable dimension, depending on the context. For $x, y \in \mathbb{R}^d$, we denote by $[x, y]$ the line segment with endpoints x and y, and by $[x, y\rangle$ the half-line emanating from x and passing through y. The largest integer less than or equal to $x \in \mathbb{R}$ is denoted by $\lfloor x \rfloor$, and the smallest integer greater than or equal to x by $\lceil x \rceil$. For $x \in \mathbb{R}$ we denote by $\lfloor x \rceil$ the *nearest integer* to x, i.e. $\lfloor x \rceil := \lfloor x \rfloor$ if $0 \leq x - \lfloor x \rfloor \leq \frac{1}{2}$ and $\lfloor x \rceil := \lceil x \rceil$ if $\frac{1}{2} < x - \lfloor x \rfloor < 1$. Moreover, $\operatorname{sgn}(x)$ denotes the *signum function* and $|x|$ the *absolute value* of $x \in \mathbb{R}$. The *maximum norm* of a vector $x \in \mathbb{R}^d$ is denoted by $\|x\|_\infty := \max_{i=1,\ldots,d} |x_i|$, and the *support* of x by $\operatorname{supp}(x) := \{i \in \{1, \ldots, d\} : x_i \neq 0\}$. By $\gcd(x_1, \ldots, x_d)$, we denote the *greatest common divisor* of integers $x_1, \ldots, x_d$. If $M \in \mathbb{R}^{d \times d}$ is a matrix, then

$\det(M)$ denotes the *determinant* of M. If $x \in \mathbb{Q}^d \setminus \mathbb{Z}^d$, then the *precision of* x is the smallest integer $q \in \mathbb{Z}_+$ such that x has a representation $x = (\frac{p_1}{q}, \ldots, \frac{p_d}{q})$, where $p_j \in \mathbb{Z}$ for all $j = 1, \ldots, d$.

Given a set $K \subseteq \mathbb{R}^d$, we use the functionals $\operatorname{conv}(K)$ (convex hull of K), $\operatorname{aff}(K)$ (affine hull of K), $\operatorname{lin}(K)$ (linear hull of K), $\operatorname{int}(K)$ (interior of K), $\operatorname{relint}(K)$ (relative interior of K), $\operatorname{bd}(K)$ (boundary of K), $\operatorname{relbd}(K)$ (relative boundary of K), and $\operatorname{vert}(K)$ (set of vertices of K). For $K \subseteq \mathbb{R}^d$, $\operatorname{vol}(K)$ denotes the volume of K measured in $\operatorname{aff}(K)$. If $K \subseteq \mathbb{R}^2$, then we use $A(K)$ rather than $\operatorname{vol}(K)$ to denote the area of K. If K is a set of objects (for instance, points in $\mathbb{R}^d$ or facets of a polyhedron), then $|K|$ denotes the cardinality of K.

If $K, L \subseteq \mathbb{R}^d$ are two sets, then we use $K + L := \{x + y : x \in K,\ y \in L\}$ to denote the *Minkowski addition* of K and L. Often, one of the two sets, K or L, will be a single vector, say $c \in \mathbb{R}^d$. In this case, we write $c + K := \{c + x : x \in K\}$. The *scalar multiple*[1] *of* K *by the scalar* $s \in \mathbb{R}$ is denoted by $sK := \{sx : x \in K\}$. Usually, it will be assumed that $0 < s < +\infty$.

The intersection of finitely many closed half-spaces is said to be a *polyhedron*. By $\mathcal{P}^d$ we denote the set of polyhedra in $\mathbb{R}^d$ (where the elements of $\mathcal{P}^d$ do not have to be full-dimensional). If a polyhedron is bounded, then we call it a *polytope*. A polyhedron $P \in \mathcal{P}^d$ is said to be *integral* if $P = \operatorname{conv}(P \cap \mathbb{Z}^d)$, and P is said to be *rational* if sP is an integral polyhedron for some finite integer $s \geq 1$. The *precision of a rational polyhedron* $P \in \mathcal{P}^d$ is the smallest integer $s \geq 1$ such that sP is an integral polyhedron. For a polyhedron $P \in \mathcal{P}^d$, the set $\operatorname{conv}(P \cap \mathbb{Z}^d)$ is called the *X-body* of P.

A polytope $S \in \mathcal{P}^d$ is said to be a *simplex* if S is the convex hull of finitely many affinely independent points. If S is a simplex in $\mathbb{R}^d$ with vertices $p_0, \ldots, p_k$ $(0 \leq k \leq d)$ and p is a point in $\operatorname{aff}(S)$, then p can be uniquely represented by $p = \sum_{j=0}^{k} \lambda_j p_j$, where $\lambda_j \in \mathbb{R}$ for all $j = 0, \ldots, k$ and $\lambda_0 + \cdots + \lambda_k = 1$. The multipliers $\lambda_0, \ldots, \lambda_k$ are called the *barycentric coordinates* of p with respect to the simplex S. The point p lies in the relative interior of S if and only if $\lambda_0, \ldots, \lambda_k > 0$.

An additive subgroup Λ of $\mathbb{R}^d$ is said to be a *lattice* if the intersection of Λ with every compact set of $\mathbb{R}^d$ is finite. In this thesis, only full-dimensional lattices $\Lambda \subseteq \mathbb{Z}^d$ are considered. Let $\operatorname{Aff}(\Lambda)$ denote the group of all affine transformations T in $\mathbb{R}^d$ with $T(\Lambda) = \Lambda$. It holds $\operatorname{Aff}(\Lambda) \subseteq \operatorname{Aff}(\mathbb{Z}^d)$. Henceforth, the transformations in $\operatorname{Aff}(\Lambda)$ are called Λ-*preserving*, while the transformations in $\operatorname{Aff}(\mathbb{Z}^d)$ are called *unimodular*. If P and Q are two polyhedra in $\mathbb{R}^d$ which coincide up to a Λ-preserving transformation, then we simply say that

[1] There is only one exception where we deviate from this definition of the scalar multiple: if K is a polyhedron of covering type (this notion will be defined in Chapter 4). However, it will always be clear from the context which definition is used.

both polyhedra are *equivalent*, and if no such transformation exists, then we call P and Q *distinct*. The *dual lattice* of Λ is denoted by Λ^*. We denote by $\mathcal{K}^d$ the class of closed convex sets in $\mathbb{R}^d$ with a non-empty interior. Bounded elements of $\mathcal{K}^d$ are referred to as *convex bodies*. For $K \in \mathcal{K}^d$ the *lattice width* of K with respect to the lattice Λ is defined by

$$w_\Lambda(K) := \inf_{u \in \Lambda^* \setminus \{o\}} w(K, u),$$

where $w(K, u)$ is the *width* of K along the vector $u \in \mathbb{R}^d$ and given by

$$w(K, u) := \sup_{x \in K} u^\top x - \inf_{x \in K} u^\top x.$$

If $K \in \mathcal{K}^d$ is a convex body, then "inf" and "sup" in the above definitions become "min" and "max". We say that $w_\Lambda(K)$ *is attained by* $u \in \Lambda^* \setminus \{o\}$ if $w_\Lambda(K) = w(K, u)$. The lattice width of K with respect to Λ can be seen as the smallest number of "lattice slices" of K along any non-zero vector in Λ^*. If $\Lambda = \mathbb{Z}^d$, then we write $w(K)$ instead of $w_{\mathbb{Z}^d}(K)$ to denote the lattice width of K with respect to the lattice $\mathbb{Z}^d$. We note that the lattice width is invariant with respect to Λ-preserving transformations. For $K \in \mathcal{K}^d$, the *support function* of K is defined by $h(K, u) := \sup\{u^\top x : x \in K\}$ (with "max" instead of "sup" for convex bodies) and satisfies $w(K, u) = h(K, u) + h(K, -u)$, where $u \in \mathbb{R}^d$.

In the special case where $P \in \mathcal{P}^d$ is a full-dimensional rational polyhedron, the maximum width of P along all its facet directions gives a measure of how wide P is: let $\mathcal{F}$ index the facet-defining inequalities of P and assume that for each facet-defining inequality $(v^i)^\top x \le v_0^i$, $i \in \mathcal{F}$, the coefficients of v^i are integer and satisfy $\gcd(v_1^i, \ldots, v_d^i) = 1$. Then we call $\max\{w(P, v^i) : i \in \mathcal{F}\}$ the *max-facet-width* of P.

For $K \in \mathcal{K}^d$, the set $DK := \{x - y : x, y \in K\}$ is called the *difference set* of K (resp. *difference body* if K is a convex body). If $K \in \mathcal{K}^d$ contains the origin in its interior, then the *Minkowski functional* of K is defined by $\|u\|_K := \inf\{\lambda \ge 0 : u \in \lambda K\}$, where $u \in \mathbb{R}^d$ (with "min" instead of "inf" if K is a convex body). The set $K^* := \{u \in \mathbb{R}^d : h(K, u) \le 1\}$ is referred to as the *polar set* of $K \in \mathcal{K}^d$ (resp. *polar body*, if K is a convex body).

If Λ is a (full-dimensional) lattice in $\mathbb{R}^d$, then a set $K \in \mathcal{K}^d$ is said to be Λ-*free* if the interior of K is disjoint with Λ, i.e. if $\text{int}(K) \cap \Lambda = \emptyset$. Moreover, a Λ-free set $K \in \mathcal{K}^d$ is said to be *maximal* Λ-*free* if K is not properly contained in another Λ-free set from $\mathcal{K}^d$. For $\Lambda = \mathbb{Z}^d$, we say *(maximal) lattice-free* rather than (maximal) Λ-free. We point out that our definition of (maximal) lattice-freeness is suitable for the applications in mixed-integer cutting plane theory which we have in mind. For different definitions of lattice-freeness see, for instance, [Rez86], [Sca85], or [Seb99]. In this thesis, we fix our underlying lattice to be $\mathbb{Z}^d$, though, due to affine invariance, the obtained results are independent of the concrete choice of the lattice Λ. Thus, if not explicitly stated otherwise, we will restrict our attention to the standard lattice $\Lambda = \mathbb{Z}^d$.

It is known that every maximal lattice-free $K \in \mathcal{K}^d$ is a polyhedron with an integer point in the relative interior of each facet of K (see, for instance, [Lov89, Propositions 3.2 and 3.3]). These polyhedra can be both rational or irrational. The set $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 1, \ -x_2 \leq 0, -x_1 + \sqrt{2}x_2 \leq \sqrt{2}\}$ is an example of an irrational maximal lattice-free polyhedron in $\mathbb{R}^2$. In this thesis, however, we usually consider maximal lattice-free rational polyhedra since we aim at algorithmic applications. The only place where irrationality is allowed will be in Chapter 5 where we derive theoretical results for general lattice-free convex sets in the two-dimensional case. The following proposition summarizes known properties of maximal lattice-free rational polyhedra that we need in this thesis.

Proposition 2.1 (see Lovász [Lov89])**.** *Let $K \in \mathcal{K}^d$ be a maximal lattice-free rational polyhedron. Then the following statements hold.*

I. K is full-dimensional and has at most 2^d facets.

II. Each facet of K contains an integer point in its relative interior.

III. K has a representation $K = \mathcal{P} + \mathcal{L}$, where $\mathcal{P}$ is a polytope and $\mathcal{L}$ is a linear space.

We refer to [BCCZ10, Section 2] for further details and a proof of the above proposition. If $K \in \mathcal{K}^d$ is a maximal lattice-free rational polyhedron such that $K = \mathcal{P} + \mathcal{L}$, then the codimension of the linear space $\mathcal{L}$ is said to be the *split-dimension of* K. The split-dimension of K is a measure for the complexity of K. A maximal lattice-free rational polyhedron with split-dimension equal to one is called a *split* and is a set of the form $\{x \in \mathbb{R}^d : \pi_0 \leq \pi^\top x \leq \pi_0 + 1\}$, where $(\pi, \pi_0) \in \mathbb{Z}^{d+1}$ and $\gcd(\pi_1, \ldots, \pi_d) = 1$ (see Fig. 5.1(a) for an example of a split in dimension two). Cutting planes which can be obtained from such a set are called *split cuts* (see Cook et al. [CKS90]). The larger the split-dimension of a maximal lattice-free rational polyhedron $K \subseteq \mathbb{R}^d$ is, the more complex is K. For instance, if $K \subseteq \mathbb{R}^3$ has split-dimension equal to one, then K is just a split. If the split-dimension of $K \subseteq \mathbb{R}^3$ is equal to two, then K is a triangle or a quadrilateral plus the span of a single vector.

CHAPTER 3

FROM CUTTING PLANES TO LATTICE-FREE POLYHEDRA

In this chapter, we explore links between cutting planes for general mixed-integer linear programs (MILP's) and lattice-free convex sets which have a representation as the sum of a polytope and a linear space. The split-dimension of a maximal lattice-free rational polyhedron which corresponds to a cutting plane turns out to play a significant role.

Our point of departure is the mixed-integer set

$$P_I = \left\{ (x,s) \in \mathbb{Z}^m \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j \right\},$$

where $m \geq 2$, $N := \{1, \ldots, n\}$, $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$, and $r^j \in \mathbb{Q}^m \setminus \{o\}$ for all $j \in N$. We refer to f as the *root vertex* and to the vectors r^j as *rays*. By scaling, we can assume that $r^j \in \mathbb{Z}^m$ and $\gcd(r^j_1, \ldots, r^j_m) = 1$ for all $j \in N$. The linear programming relaxation of P_I is denoted by

$$P_{LP} := \left\{ (x,s) \in \mathbb{R}^m \times \mathbb{R}^n_+ : x = f + \sum_{j \in N} r^j s_j \right\}.$$

It is straightforward to verify that P_I is empty if and only if the set $f + \text{lin}(\{r^j\}_{j \in N})$ does not contain any integer points. In the remainder of this thesis we assume that $\text{cone}(\{r^j\}_{j \in N}) = \mathbb{R}^m$ (we will explain later why this assumption is natural) which ensures that $P_I \neq \emptyset$.

Our aim is to relate the facet-defining inequalities for $\operatorname{conv}(P_I)$ to geometrical objects in the space of the x-variables. More precisely, we show that all non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ are *intersection cuts* (see Balas [Bal71]) in the sense that they are deducible from a convex set that has no integer point in its interior. By a *non-trivial inequality* we mean an inequality which is not the conic combination of non-negativity restrictions.

Section 3.1 reviews the structure of the set $\operatorname{conv}(P_I)$ and its facet-defining inequalities. In Section 3.2, we establish the link between facet-defining inequalities for $\operatorname{conv}(P_I)$ and lattice-free polyhedra.

3.1 Properties of $\operatorname{conv}(P_I)$

In this section, we summarize properties of the set $\operatorname{conv}(P_I)$. All statements are generalizations of the results in Andersen et al. [ALWW07] and can be proved in a similar way. We refer to [Wag08, Chapter 2] for the proofs.

Lemma 3.1. *Let P_I be defined as above. Then the following statements hold.*

I. The dimension of $\operatorname{conv}(P_I)$ *is n.*

II. The extreme rays of $\operatorname{conv}(P_I)$ *are (r^j, e_j) for $j \in N$.*

III. Every vertex $(\bar{x}, \bar{s})$ of $\operatorname{conv}(P_I)$ *satisfies $\bar{x} \in \mathbb{Z}^m$ and $1 \leq |\operatorname{supp}(\bar{s})| \leq m$.*

Our aim is to generate valid inequalities for $\operatorname{conv}(P_I)$ that are violated by (f, o). The next corollary shows that non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ always cut off the point (f, o). Therefore, the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ are the strongest cutting planes which can be derived from the set P_I. Throughout this thesis we will write them in the form given by the following corollary.

Corollary 3.2. *Every non-trivial valid inequality for* $\operatorname{conv}(P_I)$ *that is satisfied with equality at a point in P_I can be written in the form*

$$\sum_{j \in N} \psi(r^j) s_j \geq 1, \tag{3.1}$$

where $\psi(r^j) \geq 0$ for all $j \in N$.

A non-trivial valid inequality for $\operatorname{conv}(P_I)$ which is written in the form (3.1) is said to be in *standard form*. In the remainder of this thesis, we discuss only non-trivial valid inequalities for $\operatorname{conv}(P_I)$ in standard form. Therefore, we will simply say *valid inequality for* $\operatorname{conv}(P_I)$ and mean that this inequality is non-trivial and given in standard form.

A valid inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ is entirely determined by its coefficients $\psi(r^j)$, $j\in N$. For $\psi := (\psi(r^1),\dots,\psi(r^n))$ we denote by $N^0_\psi := \{j\in N : \psi(r^j)=0\}$ the set of variables with coefficient zero and by $N^{\neq 0}_\psi := N\setminus N^0_\psi$ the remainder of the variables. This allows us to introduce an object which is associated with the inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$.

Lemma 3.3. *Let $\sum_{j\in N}\psi(r^j)s_j \geq 1$ be a valid inequality for $\operatorname{conv}(P_I)$. Furthermore, define the points $v^j := f + \frac{1}{\psi(r^j)}r^j$ for all $j\in N^{\neq 0}_\psi$ and consider the polyhedron*

$$B^\psi := \left\{ x\in\mathbb{R}^m : \exists s\in\mathbb{R}^n_+ \text{ s.t. } (x,s)\in P_{LP} \text{ and } \sum_{j\in N}\psi(r^j)s_j \leq 1 \right\}.$$

Then the following statements hold.

I. $B^\psi = \operatorname{conv}\left(\{f\}\cup\{v^j\}_{j\in N^{\neq 0}_\psi}\right) + \operatorname{cone}\left(\{r^j\}_{j\in N^0_\psi}\right).$

II. The interior of B^ψ does not contain any integer points.

Example 3.4. Let the following MILP be given:

$$\begin{array}{llrrrrl}
\max\ x_4 & \text{s.t.} & -2x_1 & & & +3x_4 \leq 0, & \\
 & & & -2x_2 & & +3x_4 \leq 0, & \\
 & & & & -2x_3 & +3x_4 \leq 0, & \\
 & & 2x_1 & +2x_2 & +2x_3 & +3x_4 \leq 4, & \\
 & & & & & x_i \geq 0, & i=1,2,3,4, \\
 & & & & & x_i \in \mathbb{Z}, & i=1,2,3.
\end{array} \tag{3.2}$$

After introducing slack variables t_1,t_2,t_3,t_4, we solve the linear programming relaxation. The optimal simplex tableau provides the following constraints:

$$\begin{array}{lcl}
x_1 & = & \frac{1}{2} - \frac{1}{8}t_1 + \frac{3}{8}t_2 - \frac{1}{8}t_3 - \frac{1}{8}t_4, \\
x_2 & = & \frac{1}{2} - \frac{1}{8}t_1 - \frac{1}{8}t_2 + \frac{3}{8}t_3 - \frac{1}{8}t_4, \\
x_3 & = & \frac{1}{2} - \frac{1}{8}t_1 - \frac{1}{8}t_2 - \frac{1}{8}t_3 + \frac{3}{8}t_4, \\
x_4 & = & \frac{1}{3} - \frac{1}{12}t_1 - \frac{1}{12}t_2 - \frac{1}{12}t_3 - \frac{1}{12}t_4.
\end{array} \tag{3.3}$$

Problem (3.2) requires x_1, x_2, and x_3 to be integer, but they are fractional in the optimal solution $x^* = (\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{3})$. Thus, we take the first three rows of the optimal simplex tableau and rescale the slack variables by $s_i := \frac{t_i}{8}$ for $i=1,2,3,4$ to obtain $f=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, $r^1=(-1,-1,-1)$, $r^2=(3,-1,-1)$, $r^3=(-1,3,-1)$, and $r^4=(-1,-1,3)$. This yields the set

$$P_I = \left\{ (x,s) \in \mathbb{Z}^3 \times \mathbb{R}^4_+ : x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} s_1 + \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} s_2 + \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} s_3 + \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} s_4 \right\}.$$

The inequality

$$2s_1 + 2s_2 + 2s_3 + 2s_4 \geq 1 \tag{3.4}$$

is facet-defining for $\operatorname{conv}(P_I)$. With $\psi = (2,2,2,2)$ we obtain $v^1 = o$, $v^2 = 2e_1$, $v^3 = 2e_2$, and $v^4 = 2e_3$. The corresponding polyhedron B^ψ, the tetrahedron $B^\psi = \operatorname{conv}(\{f, v^1, v^2, v^3, v^4\})$, is shown in Fig. 3.1. Note that f is contained in the interior of B^ψ.

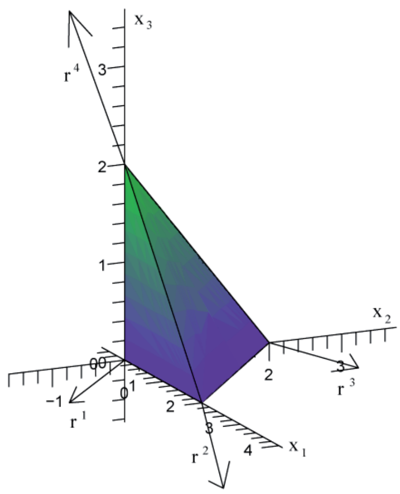


Figure 3.1: The polyhedron B^ψ for the inequality (3.4).

By resubstituting $s_i = \frac{t_i}{8}$ for $i = 1,2,3,4$ and scaling, we obtain from (3.4) the inequality

$$\frac{1}{12}t_1 + \frac{1}{12}t_2 + \frac{1}{12}t_3 + \frac{1}{12}t_4 \geq \frac{1}{3}.$$

Using this, the last equation in (3.3) results in the cut $x_4 \leq 0$. In a cutting plane algorithm, this cut would lead to immediate termination. $\diamond$

The lattice-free polyhedron B^ψ is an m-dimensional representation of the valid inequality $\sum_{j \in N} \psi(r^j) s_j \geq 1$ for $\operatorname{conv}(P_I)$. Each such inequality maps to a unique polyhedron B^ψ which is given by the coefficients $\psi(r^j)$, $j \in N$, and can be computed with the help of Lemma 3.3 I. Furthermore, the coefficient $\psi(r^j)$, $j \in N_\psi^{\neq 0}$, is equal to the ratio between the norm of r^j and the distance between f and v^j.

In Lemma 3.3, we assume that a valid inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ is given and we construct a lattice-free polyhedron $B^\psi \subseteq \mathbb{R}^m$ which represents the inequality in the space of the discrete variables. However, in algorithmic applications one may want to do the converse. One would like to start with a lattice-free polyhedron $B \subseteq \mathbb{R}^m$ and then to construct a valid inequality $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ with the help of B. This is indeed possible: Lemma 3.3 implies that every valid inequality for $\operatorname{conv}(P_I)$ can be obtained from a lattice-free polyhedron $B \subseteq \mathbb{R}^m$ with f in its interior by defining $\psi^B : \mathbb{R}^m \mapsto \mathbb{R}_+$ to be the Minkowski functional of $B - f$ (see [BC09] for details). Then the inequality $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ is called the *cut associated with* B. Hence, every valid inequality for $\operatorname{conv}(P_I)$ is an intersection cut. Obviously, we have $B^{\psi^B} \subseteq B$ and the inclusion can be strict, for instance when B is a polytope which has a vertex w such that $w \neq f + \frac{1}{\psi^B(r^j)} r^j$ for every $j \in N$.

We are interested in the inequalities $\sum_{j\in N} \psi(r^j)s_j \geq 1$ which are facet-defining for $\operatorname{conv}(P_I)$. Since the root vertex f and all the rays r^j, $j \in N$, are assumed to be rational, it follows that $\operatorname{conv}(P_I)$ is a rational polyhedron (see Meyer [Mey74, Theorem 3.9]). Thus, every facet-defining inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ has rational coefficients $\psi(r^j)$ for all $j \in N$. From Lemma 3.3 I, it follows that only rational polyhedra B^ψ are needed. We may therefore consider in the following only cuts $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ which are associated with rational polyhedra B. Indeed, if $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ is the cut associated with an irrational polyhedron $B \subseteq \mathbb{R}^m$, then there exists a rational polyhedron $\bar{B} \subseteq \mathbb{R}^m$ such that the cut associated with $\bar{B}$ is exactly (or dominates) the cut associated with B (see [CM09, Section 3.2.2] for a detailed discussion of the case $m = 2$).

Assumption 3.5. *Let $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ be the cut associated with a lattice-free polyhedron $B \subseteq \mathbb{R}^m$ with f in its interior. Then B is rational.*

For any $(\bar{x}, \bar{s}) \in P_{LP}$ such that $\sum_{j\in N} \psi(r^j)\bar{s}_j < 1$, we have $\bar{x} \in \operatorname{int}(B^\psi)$. Therefore, adding the inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ to P_{LP} would cut off the point $(\bar{x}, \bar{s})$. The interior of B^ψ is therefore an m-dimensional representation of all points in the x-variable space that are affected by the addition of the inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ to P_{LP}. In particular, we always have $f \in \operatorname{int}(B^\psi)$ since f can be expressed using $s = o$. Geometrically, this is clear from the fact that $\operatorname{cone}(\{r^j\}_{j\in N}) = \mathbb{R}^m$, by assumption. We point out that it has already been shown by Zambelli [Zam09, Theorem 1] in a more general context that all non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ can be obtained from a maximal lattice-free polyhedron which contains f in its interior. Therefore, our assumption $\operatorname{cone}(\{r^j\}_{j\in N}) = \mathbb{R}^m$ just simplifies the analysis, but does not restrict the general case.

Given a facet-defining inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$, there exist n affinely independent points $(x^i, s^i) \in P_I$, $i = 1, \ldots, n$, such that $\sum_{j\in N} \psi(r^j)s^i_j = 1$. The integer points x^i, $i = 1, \ldots, n$, are on the boundary of B^ψ, i.e. they belong to the X-body of B^ψ, which is

$$X^\psi := \left\{ x \in \mathbb{Z}^m : \exists s \in \mathbb{R}^n_+ \text{ s.t. } (x, s) \in P_{LP} \text{ and } \sum_{j\in N} \psi(r^j)s_j = 1 \right\}.$$

Obviously, $X^\psi = B^\psi \cap \mathbb{Z}^m$, and $X^\psi \neq \emptyset$ whenever $\sum_{j\in N} \psi(r^j)s_j \geq 1$ defines a facet of $\operatorname{conv}(P_I)$. The next corollary illustrates the relation between non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ and the polyhedra B^ψ.

Corollary 3.6. *A valid inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ is not facet-defining for $\operatorname{conv}(P_I)$ if there exists a valid inequality $\sum_{j\in N} \bar{\psi}(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ such that $B^\psi \subsetneq B^{\bar{\psi}}$.*

Let $\mathcal{B}(P_I)$ be the set of all polyhedra B^ψ arising from a valid inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$. An interesting implication of Corollary 3.6 is that polyhedra B^ψ which correspond to non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ are inclusion-maximal within $\mathcal{B}(P_I)$. However, the opposite does not hold true. There exist polyhedra B^ψ which are inclusion-maximal within $\mathcal{B}(P_I)$, but which do not correspond to non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$.

We emphasize that Corollary 3.6 does not necessarily imply that the polyhedron B^ψ of a non-trivial facet-defining inequality $\sum_{j\in N} \psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ is maximal lattice-free since this property depends on the rays r^j, $j \in N$, which stem from the simplex tableau of an optimal vertex of the linear programming relaxation of (1.1). This situation changes if the investigation is not based on a finite number of rays r^j, $j \in N$, but an infinite number. Indeed, relaxing the n-dimensional space of the s-variables to an infinite-dimensional space, where a variable s_r is defined for any $r \in \mathbb{Q}^m \setminus \{o\}$, yields a bijection between certain maximal lattice-free rational polyhedra in $\mathbb{R}^m$ and the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$. However, the infinite model is not considered in this thesis. We refer to Basu et al. [BBCM11], Borozan and Cornuéjols [BC07], and Cornuéjols and Margot [CM08] for a thorough elaboration of the case $m = 2$.

3.2 Facets of $\operatorname{conv}(P_I)$ and lattice-free polyhedra

In this section, we analyze non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ and we present their geometric characterization in the x-variable space. More precisely, we show that the polyhedron B^ψ arising from a facet-defining inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ is contained in a polyhedron which has a representation as the sum of a polytope and a linear space.

In preparation for the main theorem of this section, we review a basic tool from linear algebra. By I_m we denote the identity matrix in $\mathbb{R}^m$, i.e. the $m \times m$ matrix with entries one on the main diagonal and zeros elsewhere.

Observation 3.7. *Let $R \in \mathbb{R}^{d\times m}$ be a matrix of full row rank and let $L = \{x \in \mathbb{R}^m : Rx = 0\}$. Then, for any $x \in \mathbb{R}^m$, the vector $\bar{x} := (I_m - R^\top(RR^\top)^{-1}R)x$ is the* orthogonal projection *of x to L, i.e. $\bar{x}$ satisfies $\bar{x} \in L$ and $(x-\bar{x})^\top\bar{x} = 0$.*

We need to adapt Observation 3.7 to affine subspaces. For that, we call a vector $\bar{x} \in \mathbb{R}^m$ the *affine orthogonal projection* of $x \in \mathbb{R}^m$ to the affine subspace $v + L \subseteq \mathbb{R}^m$ if it satisfies $\bar{x} \in v + L$ and $(x-\bar{x})^\top(\bar{x}-v) = 0$.

Corollary 3.8. *Let $R \in \mathbb{R}^{d\times m}$ be a matrix of full row rank and let $v + L$ be an affine subspace, where $v \in \mathbb{R}^m$ and $L = \{x \in \mathbb{R}^m : Rx = 0\}$. Then, for any $x \in \mathbb{R}^m$, the vector*

$$\bar{x} := v + (I_m - R^\top(RR^\top)^{-1}R)(x-v)$$

is the affine orthogonal projection of x to the affine subspace $v + L$.

In the following theorem we give a complete characterization of the facet-defining inequalities $\sum_{j\in N}\psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ with $N_\psi^0 \neq \emptyset$. We note that Theorem 3.9 is implied by recent results in [BCCZ10].

Theorem 3.9. *Let $\sum_{j\in N}\psi(r^j)s_j \geq 1$ be a facet-defining inequality for $\operatorname{conv}(P_I)$ with $N_\psi^0 \neq \emptyset$. Moreover, let $S_\psi := \operatorname{lin}(\{r^j\}_{j\in N_\psi^0})$ and let $\dim(S_\psi) = d$. Then the following statements hold.*

I. There exists a lattice-free rational polytope $P_\psi \subseteq \mathbb{R}^{m-d}$ such that $B^\psi \subseteq P_\psi + S_\psi$.

II. If $d = m-1$, then there exists $(\pi, \pi_0) \in \mathbb{Z}^{m+1}$ with $B^\psi \subseteq \{x \in \mathbb{R}^m : \pi_0 \leq \pi^\top x \leq \pi_0 + 1\}$.

Proof. Part I. First, observe that $1 \leq d \leq m-1$. Since $\dim(S_\psi) = d$, there exist $j_1, \ldots, j_d \in N_\psi^0$ such that the set of vectors $\{r^{j_1}, \ldots, r^{j_d}\}$ is a linear basis of S_ψ. For simplicity, we assume that $S_\psi = \operatorname{lin}(\{r^1, \ldots, r^d\})$.

Let $f + S_\psi^\perp = \{x \in \mathbb{R}^m : R^\top(x - f) = o\}$, where $R := [r^1, \ldots, r^d]$ is the matrix with columns r^j, $j = 1, \ldots, d$, and where $S_\psi^\perp$ denotes the orthogonal complement of S_ψ. From Lemma 3.3 I, it follows that $B^\psi = \operatorname{conv}(\{f\} \cup \{v^j\}_{j\in N_\psi^{\neq 0}}) + \operatorname{cone}(\{r^j\}_{j\in N_\psi^0})$, where $v^j = f + \frac{1}{\psi(r^j)} r^j$ for all $j \in N_\psi^{\neq 0}$. Applying Corollary 3.8, the affine orthogonal projection of the points v^j, $j \in N_\psi^{\neq 0}$, to the affine subspace $f + S_\psi^\perp$ yields the corresponding points $p^j := v^j - (R(R^\top R)^{-1}R^\top)\frac{1}{\psi(r^j)} r^j = v^j - \sum_{i=1}^d \eta_i^j r^i$, where $\eta^j = (\eta_1^j, \ldots, \eta_d^j)^\top := (R^\top R)^{-1} R^\top (\frac{1}{\psi(r^j)} r^j) \in \mathbb{R}^d$ for all $j \in N_\psi^{\neq 0}$. Let $P_\psi := \operatorname{conv}(\{f\} \cup \{p^j\}_{j\in N_\psi^{\neq 0}})$. Then P_ψ is rational since f is rational and all the projection points p^j, $j \in N_\psi^{\neq 0}$, are also rational. By construction, $P_\psi \subseteq \mathbb{R}^{m-d}$. From Lemma 3.3 II, it follows that B^ψ is lattice-free. Thus, the set $M_\psi := \operatorname{conv}(\{f\} \cup \{v^j\}_{j\in N_\psi^{\neq 0}}) + \operatorname{lin}(\{r^j\}_{j\in N_\psi^0})$ is also lattice-free. Since $v^j - p^j = \sum_{i=1}^d \eta_i^j r^i \in \operatorname{lin}(\{r^j\}_{j\in N_\psi^0})$ for all $j \in N_\psi^{\neq 0}$ we have $B^\psi \subseteq M_\psi = \operatorname{conv}(\{f\} \cup \{p^j\}_{j\in N_\psi^{\neq 0}}) + \operatorname{lin}(\{r^j\}_{j\in N_\psi^0}) = P_\psi + S_\psi$.

Part II. Let $d = m-1$. Then S_ψ is a hyperplane and there exists an integral vector $\pi \in \mathbb{Z}^m$ such that $\gcd(\pi_1, \ldots, \pi_m) = 1$ and $S_\psi = \{x \in \mathbb{R}^m : \pi^\top x = 0\}$. From Part I, it follows that $P_\psi = [p, q]$ with $p, q \in \mathbb{Q}^m$ being two distinct points. We obtain $B^\psi \subseteq \{x \in \mathbb{R}^m : \pi^\top p \le \pi^\top x \le \pi^\top q\}$. Since $f \in \operatorname{int}(B^\psi)$ we have $\pi^\top p < \pi^\top f < \pi^\top q$. Let $\Theta_0 := \lfloor \pi^\top p \rfloor$ and $\Theta_1 := \lceil \pi^\top q \rceil$ and consider the set $S_\pi := \{x \in \mathbb{R}^m : \Theta_0 \le \pi^\top x \le \Theta_1\}$. If $\Theta_0 + 1 < \Theta_1$, then it follows from the assumption $\gcd(\pi_1, \ldots, \pi_m) = 1$ that there exists $x^I \in \{x \in \mathbb{Z}^m : \pi^\top x = \Theta_0 + 1\} \subseteq \operatorname{int}(S_\pi)$. In addition, x^I is in the interior of B^ψ which is a contradiction to the fact that B^ψ is lattice-free. Hence, we must have $\Theta_1 = \Theta_0 + 1$. This implies $B^\psi \subseteq S_\pi = \{x \in \mathbb{R}^m : \Theta_0 \le \pi^\top x \le \Theta_0 + 1\}$. $\square$

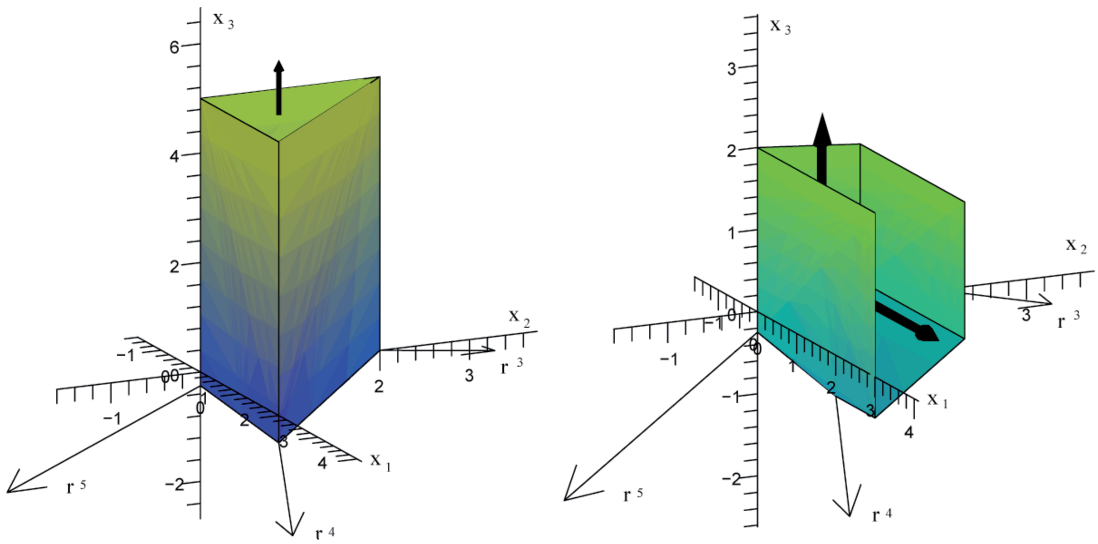
Example 3.10. Consider the set

$$P_I = \left\{ (x, s) \in \mathbb{Z}^3 \times \mathbb{R}^5_+ : x = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} s_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} s_2 + \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} s_3 + \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} s_4 + \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} s_5 \right\}.$$

The vector $\psi_1 = (1, 0, 2, 2, 4)$ represents the inequality

$$s_1 + 2s_3 + 2s_4 + 4s_5 \ge 1 \tag{3.5}$$

which is facet-defining for $\operatorname{conv}(P_I)$. Its corresponding set $B^{\psi_1} = \operatorname{conv}(\{f, v_1^1, v_1^3, v_1^4, v_1^5\}) + \operatorname{cone}(\{r^2\})$, where $v_1^1 = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2})$, $v_1^3 = (0, 2, 0)$, $v_1^4 = (2, 0, -\frac{1}{2})$, and $v_1^5 = (0, 0, -\frac{1}{4})$, is shown in Fig. 3.2(a). From Theorem 3.9 I, it follows that there exist a lattice-free polytope P_{ψ_1} and a linear subspace S_{ψ_1} such that $B^{\psi_1} \subseteq P_{\psi_1} + S_{\psi_1}$. Applying the construction in the proof of Theorem 3.9 yields $P_{\psi_1} = \operatorname{conv}(\{(0, 0, \frac{1}{2}), (2, 0, \frac{1}{2}), (0, 2, \frac{1}{2})\})$ and $S_{\psi_1} = \operatorname{lin}(\{(0, 0, 1)\})$. Note that there is no pair $(\pi, \pi_0) \in \mathbb{Z}^4$ such that $B^{\psi_1} \subseteq \{x \in \mathbb{R}^3 : \pi_0 \leq \pi^\top x \leq \pi_0 + 1\}$.



(a) The polyhedron B^{ψ_1} for the inequality (3.5).

(b) The polyhedron B^{ψ_2} for the inequality (3.6).

Figure 3.2: Illustration of Example 3.10.

The inequality

$$6s_3 + 2s_4 + 4s_5 \geq 1 \tag{3.6}$$

is facet-defining for $\operatorname{conv}(P_I)$, too. Here, the associated vector is $\psi_2 = (0, 0, 6, 2, 4)$ and corresponds to $B^{\psi_2} = \operatorname{conv}(\{f, v_2^3, v_2^4, v_2^5\}) + \operatorname{cone}(\{r^1, r^2\})$, where $v_2^3 = (\frac{1}{3}, 1, \frac{1}{3})$, $v_2^4 = (2, 0, -\frac{1}{2})$, and $v_2^5 = (0, 0, -\frac{1}{4})$. The set B^{ψ_2} is shown in Fig. 3.2(b). In this case, the corresponding sets are $P_{\psi_2} = \operatorname{conv}(\{(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})\})$ and $S_{\psi_2} = \operatorname{lin}(\{(1, 0, 0), (0, 0, 1)\})$. The pair $\pi = (0, 1, 0)$ and $\pi_0 = 0$ satisfies $B^{\psi_2} \subseteq \{x \in \mathbb{R}^3 : \pi_0 \leq \pi^\top x \leq \pi_0 + 1\} = \{x \in \mathbb{R}^3 : 0 \leq x_2 \leq 1\}$. $\diamond$

Theorem 3.9 allows us to classify the non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$ with respect to their split-dimension. Let $\sum_{j \in N} \psi(r^j) s_j \geq 1$ be a valid inequality for $\operatorname{conv}(P_I)$. We define the *split-dimension of the inequality* $\sum_{j \in N} \psi(r^j) s_j \geq 1$ to be the smallest split-dimension of a maximal lattice-free rational polyhedron B such that $B^\psi \subseteq B$. By Proposition 2.1 III,

B is the sum of a polytope $\mathcal{P}_B$ and a linear space $\mathcal{S}_B$. Hence, the inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ is (or is dominated by) the cut associated with B. The polyhedron B is the more complex the lower the dimension of the linear space $\mathcal{S}_B$ is, since the complexity comes from the polytope $\mathcal{P}_B$, not from the linear space $\mathcal{S}_B$. We note that using the split-dimension as a measure of the complexity of an inequality is rather theoretical and it is not clear how to compute the split-dimension when a particular inequality is given. An inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ has split-dimension at most $m-\dim(S_\psi)$, where $S_\psi = \operatorname{lin}(\{r^j\}_{j\in N^0_\psi})$. In particular, the inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ is a split cut whenever $\dim(S_\psi) = m-1$ and therefore easy to generate because only the normal vector of the corresponding split needs to be determined. On the other hand, an inequality $\sum_{j\in N}\psi(r^j)s_j \geq 1$ for $\operatorname{conv}(P_I)$ with $\dim(S_\psi) \leq m-2$ is not a split cut.

The split-dimension is the basic geometric difference between non-trivial facet-defining inequalities for $\operatorname{conv}(P_I)$. It is natural to ask whether this fact is connected to the significance of an inequality for $\operatorname{conv}(P_I)$. One might think that a higher split-dimension means a higher "value" of the corresponding inequality. In the next chapter, we will shed light on this relationship by showing that inequalities with split-dimension equal to m are needed in order to approximate $\operatorname{conv}(P_I)$ closely.

CHAPTER 4

EVALUATION OF CUTTING PLANES

In this chapter, we focus on the evaluation of facet-defining inequalities for $\text{conv}(P_I)$. The evaluation is based on a strength measure introduced by Goemans [Goe95]. Using this strength measure, we show in Section 4.1 that only inequalities with full split-dimension (i.e. split-dimension equal to m) give rise to a good approximation of $\text{conv}(P_I)$, whereas inequalities with low split-dimension (i.e. split-dimension less than m) might approximate $\text{conv}(P_I)$ arbitrarily badly, in general. However, in Section 4.2, we also show that ordinary split cuts approximate $\text{conv}(P_I)$ to within a constant factor when the size of the input data is given.

Let us first introduce the applied measure. In [Goe95], Goemans proposed a measure for evaluating the strength of valid inequalities for polyhedra of covering type. A non-empty polyhedron $P = \{x \in \mathbb{R}^n_+ : Ax \geq b\}$ is said to be *of covering type* if all entries in A and b are non-negative and $o \notin P$. Let $\alpha \in (0, +\infty)$ and let P be a polyhedron of covering type. Then we denote by $\alpha P := \{x \in \mathbb{R}^n_+ : \alpha x \in P\}$ the *dilation of P by the factor α* and we define $\alpha P := \mathbb{R}^n_+$ if $\alpha = +\infty$. Note that $P \subseteq \alpha P$ for every $\alpha \geq 1$. Let $Q \subseteq \mathbb{R}^n_+$ be any convex set such that $P \subseteq Q$. The *strength of P with respect to Q*, denoted by $t(P, Q)$, is defined to be the minimum value of $\alpha \geq 1$ such that $Q \subseteq \alpha P$. Geometrically, $t(P, Q)$ is the smallest positive number α needed to inflate P such that the convex set Q is contained in αP. Therefore, $t(P, Q)$ measures the proximity of P to its relaxation Q. For our purposes, the underlying idea is that "a high strength means high benefit". In other words, let P and Q be two relaxations of a set M such that $M \subseteq P \subseteq Q$. The larger $t(P, Q)$ the better is the gap to M closed by P compared to that of Q.

The following lemma is essentially due to Goemans and has been generalized by Basu et al.

Lemma 4.1. ([Goe95, Theorem 3] and [BBCM11, Theorem 1.3].) *Let* $P := \{x \in \mathbb{R}^n_+ : a_i^\top x \geq b_i$ for all $i = 1, \dots, m\}$ *be a polyhedron of covering type and let* $Q \subseteq \mathbb{R}^n_+$ *be any convex set such that* $P \subseteq Q$. *Then*

$$t(P, Q) = \max_{i=1,\dots,m} \left\{ \frac{b_i}{\inf\{a_i^\top x : x \in Q\}} : b_i > 0 \right\}.$$

If $\inf\{a_i^\top x : x \in Q\} = 0$ *for some* $i \in \{1, \dots, m\}$ *with* $b_i > 0$*, then* $t(P, Q)$ *is defined to be* $+\infty$.

Lemma 4.1 reduces the computation of $t(P, Q)$ to the case where only a single inequality of P is added to Q. We note that only facet-defining inequalities for P need to be considered.

By Corollary 3.2, every non-trivial facet-defining inequality for $\mathrm{conv}(P_I)$ can be written in the form $\sum_{j \in N} \psi(r^j) s_j \geq 1$. Thus, in order to investigate these inequalities it is enough to consider the space of the s-variables only. This makes the analysis much easier from a notational point of view. Instead of $\mathrm{conv}(P_I)$ we consider in the following the set

$$R_f(r^1, \dots, r^n) := \mathrm{conv}\left(\left\{ s \in \mathbb{R}^n_+ : f + \sum_{j \in N} r^j s_j \in \mathbb{Z}^m \right\} \right)$$

which is the projection of $\mathrm{conv}(P_I)$ onto the space of the s-variables. Note that $R_f(r^1, \dots, r^n)$ is a polyhedron (as it is the projection of a polyhedron) of covering type. In the following we use R^n_f instead of $R_f(r^1, \dots, r^n)$ for simplicity. For $i = 1, \dots, m$ we denote by $\mathcal{S}^i(R^n_f) \subseteq \mathbb{R}^n_+$ the *i-dimensional split closure*, i.e. the intersection of the trivial inequalities $s_j \geq 0$ for all $j = 1, \dots, n$ and all valid inequalities $\sum_{j \in N} \psi(r^j) s_j \geq 1$ for R^n_f of split-dimension at most i. We note that $\mathcal{S}^1(R^n_f)$ is the usual split closure, whereas $\mathcal{S}^m(R^n_f) = R^n_f$.

Let $i \in \{1, \dots, m-1\}$ and let $\sum_{j \in N} \psi(r^j) s_j \geq 1$ be a valid inequality for R^n_f of split-dimension greater than i. Let $\mathcal{F}(R^n_f)$ be the set obtained by adding to $\mathcal{S}^i(R^n_f)$ the inequality $\sum_{j \in N} \psi(r^j) s_j \geq 1$. The following observation follows directly from Lemma 4.1.

Observation 4.2.

$$t(\mathcal{F}(R^n_f), \mathcal{S}^i(R^n_f)) = \frac{1}{\min\{\sum_{j \in N} \psi(r^j) s_j : s \in \mathcal{S}^i(R^n_f)\}}.$$

Let $B \subseteq \mathbb{R}^m$ be a lattice-free rational polyhedron with the root vertex f in its interior and let $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ be the cut associated with B. If its split-dimension is equal to m (implying that B is a polytope), then, since B, f, and $r^1, \dots, r^n$ are rational, we can assume that the rays $r^1, \dots, r^n$ are scaled such that the points $f + r^j$, $j \in N$, are on the boundary of B (see [BBCM11, Assumption 5.1] and the subsequent paragraph therein for an explanation why this assumption is feasible). In this case, we define a *corner ray* to be a ray r^j where the point $f + r^j$ is a vertex of B. In the course of this thesis we will have to compute a strength $t(\mathcal{F}(R_f^n), \mathcal{S}^i(R_f^n))$ as in Observation 4.2 several times. Thus, we will often deal with optimization problems of the following type:

$$\min \sum_{j\in N} \psi^B(r^j)s_j \quad \text{s.t.} \quad s \in \mathcal{S}^i(R_f^n) \tag{4.1}$$

for some (facet-defining) inequality $\sum_{j\in N} \psi^B(r^j)s_j \geq 1$ of split-dimension m. Scaling the rays as described above implies that the objective function becomes $\sum_{j=1}^n s_j$. Under the assumption of scaled rays, it is shown in [BBCM11, Theorem 4.2][1] that (4.1) reduces to the problem where only those rays are present which cannot be written as a convex combination of other rays. In particular, if the set of rays $\{r^1, \dots, r^n\}$ contains all the corner rays of the polyhedron B, say $\{r^1, \dots, r^k\}$, then the optimal objective value of (4.1) is equal to the optimal objective value of the following minimization problem:

$$\begin{array}{lrl} \min & \sum_{j=1}^k s_j & \\ \text{s.t.} & \sum_{j=1}^k \psi^l(r^j)s_j \geq 1, & \text{for } l = 1, \dots, p, \\ & s_j \geq 0, & \text{for } j = 1, \dots, k, \end{array}$$

where the inequalities $\sum_{j\in N} \psi^l(r^j)s_j \geq 1$, $l = 1, \dots, p$, are facet-defining for $\mathcal{S}^i(R_f^n)$. In the remainder of this thesis, when we consider problems of the above type, we assume that the set of rays consists only of rays which cannot be written as a convex combination of other rays.

For the case $m = 2$ it has already been shown by Basu et al. [BBCM11, Theorem 1.8] that $t(R_f^n, \mathcal{S}^1(R_f^n))$ can be arbitrarily large and thus the split closure may produce an arbitrarily bad approximation of R_f^n. In the next section, we generalize this result to dimensions $m \geq 2$, i.e. we show that using only facet-defining inequalities for R_f^n of split-dimension less than m will lead to an arbitrarily bad approximation of R_f^n, in general.

[1] The proof in [BBCM11, Theorem 4.2] deals with the case $m = 2$, but it is straightforward to show that the statement holds true for any $m \geq 2$.

4.1 A negative result on the strength

In this section, we show that – in terms of our strength measure – any lower-dimensional split closure $\mathcal{S}^i(R_f^n)$, $i = 1, \dots, m-1$, does not provide a good approximation of R_f^n, in general. For the case $m = 2$ this result follows from Basu et al. [BBCM11, Theorem 1.8] who show that inequalities of split-dimension equal to one do not suffice to give a constant approximation. We first state a technical lemma which is needed for the proof of the main theorem in this section.

Lemma 4.3. *Let $c \in \mathbb{Q}\backslash\mathbb{Z}$ and let $M := \{(x,c) \in \mathbb{R}^m : x_j \in \{0,1\}$ for all $j = 1, \dots, m-1\}$. If a convex set $S \subseteq \mathbb{R}^m$ satisfies $\operatorname{conv}(M) \subseteq \operatorname{relint}(S)$, then for any $v \in \mathbb{Z}^m$ such that $v_m \neq 0$ the interior of $S + \operatorname{lin}(\{v\})$ contains an integer point.*

Proof. By the definition of M, we have $\operatorname{conv}(M) = \{(x,c) \in \mathbb{R}^m : x = \sum_{j=1}^{m-1} \lambda_j e_j,$ with $0 \le \lambda_j \le 1$ for all $j = 1, \dots, m-1\}$.

Without loss of generality assume $v_m > 0$ and consider the set $M_{\mathbb{Z}} := \{x + \frac{\lceil c \rceil - c}{v_m} v : x \in M\}$. Observe that the last coordinate of all points in $\operatorname{conv}(M_{\mathbb{Z}})$ is integer. Furthermore, we have $\operatorname{conv}(M_{\mathbb{Z}}) \subseteq \operatorname{int}(S + \operatorname{lin}(\{v\}))$, by construction. We show that $\operatorname{conv}(M_{\mathbb{Z}})$ contains an integer point. Note that $\operatorname{conv}(M_{\mathbb{Z}})$ and $\operatorname{conv}(M)$ are translates. Since in both sets, $\operatorname{conv}(M_{\mathbb{Z}})$ and $\operatorname{conv}(M)$, the last coordinate of all points is constant, their projections to the hyperplane $H := \{x \in \mathbb{R}^m : x_m = 0\}$, denoted by $\operatorname{proj}_H(\operatorname{conv}(M_{\mathbb{Z}}))$ and $\operatorname{proj}_H(\operatorname{conv}(M))$, are also translates. The set $\operatorname{proj}_H(\operatorname{conv}(M))$ is generated by the vectors e_j for $j = 1, \dots, m-1$ and therefore contains an integer point. Note that any translate $\theta + \operatorname{proj}_H(\operatorname{conv}(M))$ with $\theta \in \mathbb{R}^m$ and $\theta_m = 0$ contains an integer point: if θ is an integer vector, then $\theta + \operatorname{proj}_H(\operatorname{conv}(M))$ has an integer vertex; if θ is not an integer vector, then $\theta + \operatorname{proj}_H(\operatorname{conv}(M))$ contains an integer point in its relative interior (if all components $\theta_1, \dots, \theta_{m-1}$ are fractional) or on its relative boundary (if not all components $\theta_1, \dots, \theta_{m-1}$ are fractional). Thus, $\operatorname{proj}_H(\operatorname{conv}(M_{\mathbb{Z}}))$ contains an integer point. The statement follows from the fact that $\operatorname{conv}(M_{\mathbb{Z}})$ is a translate of $\operatorname{proj}_H(\operatorname{conv}(M_{\mathbb{Z}}))$ with an integer vector. ☐

We are now prepared to prove our main theorem in this section.

Theorem 4.4. *For any $\alpha > 1$ there exists a choice of a root vertex $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$ and rays $r^1, \dots, r^n \in \mathbb{Q}^m$ such that $\mathcal{S}^{m-1}(R_f^n) \nsubseteq \alpha \mathcal{S}^m(R_f^n)$.*

Proof. To prove our assertion we construct a maximal lattice-free polyhedron P of split-dimension m. Then we show that the gap between the cut associated with P and the $(m-1)$-dimensional split closure $\mathcal{S}^{m-1}(R_f^n)$ can become arbitrarily large. In other words, we show that the optimization problem

$$\min \sum_{j=1}^{n} \psi^P(r^j)s_j \quad \text{s.t.} \quad s \in \mathcal{S}^{m-1}(R_f^n) \tag{4.2}$$

can have an arbitrarily small positive objective value for a particular choice of f and $r^1, \ldots, r^n$. The theorem then follows from Lemma 4.1 and Observation 4.2.

Let $\delta \in \mathbb{Z}_+$ be sufficiently large and let $f := (\frac{1}{2}, \ldots, \frac{1}{2}, 1 + \frac{1}{\delta} - \frac{1}{\delta^m})$, $v^a := (\frac{1}{2}, \ldots, \frac{1}{2}, 1 + \frac{1}{\delta})$, and

$$E := \left\{ \left(\frac{1}{2}, \ldots, \frac{1}{2}, 0\right) + w : w \in \left\{ \pm(\delta+1)\frac{m-1}{2} e_i : i = 1, \ldots, m-1 \right\} \right\}.$$

Let $P := \operatorname{conv}(\{v^a\} \cup E)$. Then P is a maximal lattice-free rational polyhedron of split-dimension m. Every line segment $[v^a, e]$, $e \in E$, contains one of the points of the set

$$K := \left\{ \left(\frac{1}{2}, \ldots, \frac{1}{2}, 1\right) + w : w \in \left\{ \pm\frac{m-1}{2} e_i : i = 1, \ldots, m-1 \right\} \right\}.$$

We define the (scaled corner) rays to be $r^a := \frac{1}{\delta^m} e_m$ together with the members of the set

$$R := \left\{ \pm(\delta+1)\frac{m-1}{2} e_i - \frac{\delta^m + \delta^{m-1} - 1}{\delta^m} e_m : i = 1, \ldots, m-1 \right\}.$$

The inequality associated with P is $s_a + \sum_{r \in R} s_r \geq 1$. Let L be any maximal lattice-free rational polyhedron of split-dimension less than or equal to $m-1$ and let $\psi^L(r^a)s_a + \sum_{r \in R} \psi^L(r)s_r \geq 1$ be the cut associated with L. We first show that $\psi^L(r) \geq \delta + 1$ for at least one $r \in R$. Assume the opposite, i.e. assume $\psi^L(r) < \delta + 1$ for all $r \in R$. Then f and the points $f + \frac{1}{\delta+1} r$, $r \in R$, belong to the interior of L. Thus,

$$C := \operatorname{conv}\left(f \cup \left\{ \left(\frac{1}{2}, \ldots, \frac{1}{2}, \frac{\delta^{m+1} + \delta^m - \delta}{\delta^m(\delta+1)}\right) + w : \right.\right.$$
$$\left.\left. w \in \left\{ \pm\frac{m-1}{2} e_i : i = 1, \ldots, m-1 \right\} \right\} \right) \subseteq \operatorname{int}(L).$$

Since L is rational and of split-dimension at most $m-1$, there exists $\pi \in \mathbb{Z}^m \setminus \{o\}$ such that $C + \operatorname{lin}(\{\pi\}) \subseteq \operatorname{int}(L)$. First, suppose $\pi_m > 0$ (the case where $\pi_m < 0$ is symmetric). Consider the set $L \cap \{x \in \mathbb{R}^m : x_m = \frac{\delta^{m+1} + \delta^m - \delta}{\delta^m(\delta+1)}\}$, which contains the set $M := \{x \in \mathbb{R}^m : x_m =$

$\frac{\delta^{m+1}+\delta^m-\delta}{\delta^m(\delta+1)}$ and $x_j \in \{0,1\}$ for all $j = 1,\ldots,m-1\}$ in its relative interior. By Lemma 4.3, it follows that $\mathrm{int}(L)$ contains an integer point, which is a contradiction. Hence, we can assume that $\pi_m = 0$. Define the point $w \in \mathbb{Z}^m$ such that $w_m = 1$ and $w_i = \frac{1}{2}\,\mathrm{sgn}(\pi_i)(\mathrm{sgn}(\pi_i)+1)$ for all $i = 1,\ldots,m-1$. One can show that $w \in \mathrm{int}(L)\cap\mathbb{Z}^m$ (for the moment assume that this is true; we will show it at the end of the proof), which is a contradiction. It follows that $\psi^L(r) \geq \delta+1$ for at least one $r \in R$. Thus, the minimization problem

$$\begin{aligned} \min\ s_a + \textstyle\sum_{r\in R} s_r \quad \text{s.t.} \quad (\delta+1)s_r &\geq 1 \quad \forall r \in R,\\ s_a &\geq 0 \end{aligned} \tag{4.3}$$

is a strengthening of (4.2) which means that the optimal objective value of (4.2) is at most the optimal objective value of (4.3). An optimal solution for (4.3) is $\bar{s}_a = 0$ and $\bar{s}_r = \frac{1}{\delta+1}$ for all $r \in R$ with optimal objective value $\frac{2(m-1)}{\delta+1}$. For $\delta \in \mathbb{Z}_+$ large enough this implies that the optimal objective value of (4.2) can be arbitrarily close to zero.

We will now show that $w \in \mathrm{int}(L)\cap\mathbb{Z}^m$. For that, we show that $w = c+\lambda\pi$ for some $c \in C$ and $\lambda \in \mathbb{R}$. To simplify notation we define $\Delta := \delta^{m+1}+\delta^m-\delta$. Furthermore, for all $i = 1,\ldots,m-1$, let $F(\pi_i) := 1$ if $\pi_i > 0$, and $F(\pi_i) := -1$ otherwise. Consider the points $p_i := (\frac{1}{2},\ldots,\frac{1}{2},\frac{\Delta}{\delta^m(\delta+1)}) + \frac{m-1}{2}F(\pi_i)e_i$ for all $i = 1,\ldots,m-1$. Note that $\{f,p_1,\ldots,p_{m-1}\} \subseteq C$. Solving the system of equations

$$a_f f + \sum_{j=1}^{m-1} a_j p_j + \lambda\pi = w \qquad\qquad \text{and} \qquad\qquad a_f + \sum_{j=1}^{m-1} a_j = 1$$

yields

$$\begin{aligned} a_f &= \frac{\delta^2}{\Delta},\\ a_i &= \frac{\Delta\sum_{j=1}^{m-1}F(\pi_j)\pi_j - \delta^2(m-1)F(\pi_i)\pi_i}{\Delta(m-1)\sum_{j=1}^{m-1}F(\pi_j)\pi_j} \qquad \forall i = 1,\ldots,m-1,\\ \lambda &= \frac{\delta^2(m-1)}{2\Delta\sum_{j=1}^{m-1}F(\pi_j)\pi_j}. \end{aligned}$$

It is easy to check that $0 < a_f < 1$ and $0 < a_i < 1$ for all $i = 1,\ldots,m-1$. Therefore, $c := a_f f + \sum_{j=1}^{m-1} a_j p_j$ is a proper convex combination of points in C and $w = c+\lambda\pi \in C + \mathrm{lin}(\{\pi\}) \subseteq \mathrm{int}(L)$. Thus, $w \in \mathrm{int}(L)\cap\mathbb{Z}^m$. □

Theorem 4.4 implies that the strength of $\mathcal{S}^m(R^n_f)$ with respect to $\mathcal{S}^{m-1}(R^n_f)$ can become arbitrarily large and therefore the approximability of R_f^n by $\mathcal{S}^{m-1}(R_f^n)$ may be arbitrarily bad. Thus, in general, good approximations for R_f^n can be expected only by using inequalities for R_f^n of full split-dimension.

The result of Theorem 4.4 is quite unsatisfactory. It seems to run afoul of lots of experimentation in the literature which supports that in many cases even the usual split closure gives a good approximation of the mixed-integer hull (see, for instance, [AW10, BS08]). However, the key observation is that Theorem 4.4 describes a worst case scenario. In the proof of Theorem 4.4, a root vertex f and $2m-1$ rays are constructed whose precision grows with $\delta \in \mathbb{Z}_+$. Eventually, the growth of δ ensures that the approximation becomes worse and worse. Thus, it is natural to ask whether there is a positive result on the strength when the data size is bounded. In the next section, we shed light on this question. We show that under certain conditions usual split cuts give an approximation for R_f^n to within a constant factor. More precisely, we show that under specific assumptions on the data, $t(R_f^n, \mathcal{S}^1(R_f^n))$ cannot be arbitrarily large.

4.2 A positive result on the strength

From Theorem 4.4, one could conclude that inequalities of low split-dimension are of little value in approximating R_f^n. This is correct in general. However, by restricting the size of the data and under certain assumptions on the rays r^j, $j \in N$, inequalities of split-dimension equal to m can still be approximated to within a constant factor by using ordinary splits. This is the main result in this section. Our second result in this section deals with a special type of polyhedra.

Definition 4.5. *Let $b_1, \ldots, b_m \in \mathbb{Z}^m$ be vectors forming a basis of $\mathbb{Z}^m$ and let $v \in \mathbb{Z}^m$ be any integer point. We call the set $v + \operatorname{conv}(\{o, mb_1, \ldots, mb_m\})$ a regular lattice-free simplex (RLS).*

RLS's are maximal lattice-free integral polyhedra of split-dimension equal to m and whose max-facet-width is equal to m as well.

Let B_ω be a maximal lattice-free rational polyhedron of split-dimension equal to m and with max-facet-width at most ω. Let $\sum_{j \in N} \psi^{B_\omega}(r^j) s_j \geq 1$ be the cut associated with B_ω. We define $\mathcal{S}^1_\omega(R_f^n)$ to be the set obtained by adding to $\mathcal{S}^1(R_f^n)$ all such inequalities $\sum_{j \in N} \psi^{B_\omega}(r^j) s_j \geq 1$. Furthermore, let R be an RLS and let $\sum_{j \in N} \psi^R(r^j) s_j \geq 1$ be the cut associated with R. We define $\mathcal{S}^1_{RLS}(R_f^n)$ to be the set obtained by adding to $\mathcal{S}^1(R_f^n)$ all such inequalities $\sum_{j \in N} \psi^R(r^j) s_j \geq 1$ for which the corner rays of R are among the rays $r^1, \ldots, r^n$. Our results in this section may be summarized as follows:

- $t(\mathcal{S}^1_\omega(R^n_f), \mathcal{S}^1(R^n_f))$ is bounded from above by a constant which involves only the dimension m, the precision of the root vertex f, and the max-facet-width ω (Theorem 4.7).
- $t(\mathcal{S}^1_{RLS}(R^n_f), \mathcal{S}^1(R^n_f))$ is bounded from above by a constant which involves only the dimension m (Theorem 4.8).

We start with a general property of lattices that we need in our proofs.

Lemma 4.6. *Let* $\mathcal{L}(B) := \{y \in \mathbb{Z}^m : y = Bx$ for some $x \in \mathbb{Z}^m\}$ *be a proper sublattice of* $\mathbb{Z}^m$*, where* $B \in \mathbb{Z}^{m\times m}$ *is an invertible matrix. Then, for any* $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$ *such that* $B^\top f \in \mathbb{Z}^m$*, there exists some* $v \in \{B\lambda : -\frac{1}{2} \leq \lambda_i \leq \frac{1}{2}$ for all $i = 1, \ldots, m\} \cap \mathbb{Z}^m$ *such that* $f^\top v \notin \mathbb{Z}$.

Proof. By assumption, there exists $z \in \mathbb{Z}^m$ such that $B^\top f = z$. Since f is the unique solution to the system $\{B^\top x = z,\ x \in \mathbb{R}^m\}$ and $f \notin \mathbb{Z}^m$, the system $\{B^\top x = z,\ x \in \mathbb{Z}^m\}$ is infeasible. Applying the integral Farkas lemma (see, for instance, [BW05, Theorem 6.5]) there exists $y \in \mathbb{Q}^m$ such that $By \in \mathbb{Z}^m$ and $z^\top y \notin \mathbb{Z}$. Thus, there exists some $v \in \mathbb{Z}^m$ such that $y = B^{-1}v$. We obtain $f^\top v = z^\top y \notin \mathbb{Z}$. Now define $\bar{v} := B(y - \lfloor y \rceil)$, where $\lfloor y \rceil := (\lfloor y_1 \rceil, \ldots, \lfloor y_m \rceil)$. Then $\bar{v}$ has all required properties. $\square$

Theorem 4.7. *Let q be the precision of the root vertex f and let $\mathcal{S}^1_\omega(R^n_f)$ be the set obtained by adding to $\mathcal{S}^1(R^n_f)$ all cuts associated with a maximal lattice-free rational polyhedron of split-dimension m and with max-facet-width at most ω. Then $S^1(R^n_f) \subseteq \frac{mq\omega}{2}\mathcal{S}^1_\omega(R^n_f)$.*

Proof. Let $L = \{x \in \mathbb{R}^m : \Pi^\top x \leq \pi\}$ be a maximal lattice-free rational polyhedron of split-dimension equal to m and with max-facet-width at most ω, where $\Pi \in \mathbb{Z}^{m\times t}$, $\pi \in \mathbb{Z}^t$, and t denotes the number of facets of L. We denote by π^i the i-th column of Π and assume $\gcd(\pi^i_1, \ldots, \pi^i_m) = 1$ for all $i = 1, \ldots, t$. Since L is full-dimensional, the representation of L under these assumptions is unique. Let $\sum_{j\in N} \psi^L(r^j)s_j \geq 1$ be the cut associated with L. We prove the theorem by showing that

$$\min\left\{\sum_{j\in N} \psi^L(r^j)s_j : s \in S^1(R^n_f)\right\} \geq \frac{2}{mq\omega}. \tag{4.4}$$

The statement then follows from applying Lemma 4.1 and Observation 4.2. By scaling, we can assume that $\psi^L(r^j) = 1$ for all $j \in N$. We distinguish two cases.

Case 1: First, suppose that $(\pi^i)^\top f \notin \mathbb{Z}$ for some index $i \in \{1, \ldots, t\}$. Consider the split $S_i := \{x \in \mathbb{R}^m : \lfloor(\pi^i)^\top f\rfloor \leq (\pi^i)^\top x \leq \lceil(\pi^i)^\top f\rceil\}$. Let

$\sum_{j\in N}\psi^i(r^j)s_j \geq 1$ be the cut associated with S_i. Then for all $j \in N$ the coefficient $\psi^i(r^j)$ is given by

$$\psi^i(r^j) = \begin{cases} \frac{(\pi^i)^\top r^j}{\lceil(\pi^i)^\top f\rceil - (\pi^i)^\top f} & \text{if } (\pi^i)^\top r^j > 0, \\ 0 & \text{if } (\pi^i)^\top r^j = 0, \\ \frac{(\pi^i)^\top r^j}{\lfloor(\pi^i)^\top f\rfloor - (\pi^i)^\top f} & \text{if } (\pi^i)^\top r^j < 0. \end{cases} \tag{4.5}$$

We relax the minimization problem in (4.4) by taking only the split S_i. Then setting up the dual we obtain

$$\begin{aligned} \max\ y \quad \text{s.t.} \quad \psi^i(r^j)y &\leq 1 \quad \forall j \in N, \\ y &\geq 0. \end{aligned} \tag{4.6}$$

Observe, that $|(\pi^i)^\top r^j| = |(\pi^i)^\top(f + r^j) - (\pi^i)^\top f| < \omega$ for all $j \in N$. Since the precision of f is q, it follows that

$$\max\left\{\frac{1}{\lceil(\pi^i)^\top f\rceil - (\pi^i)^\top f}, \frac{1}{(\pi^i)^\top f - \lfloor(\pi^i)^\top f\rfloor}\right\} \leq q$$

and hence, $\psi^i(r^j) < q\omega$ for all $j \in N$. Thus, problem (4.6) can be strengthened to $\max\{y : y \leq \frac{1}{q\omega},\ y \geq 0\}$ with optimal objective value $\frac{1}{q\omega}$. This gives a lower bound for the minimization problem in (4.4). We have $\frac{1}{q\omega} \geq \frac{2}{mq\omega}$ since $m \geq 2$, by assumption.

Case 2: Now suppose that $(\pi^i)^\top f \in \mathbb{Z}$ for all $i = 1, \ldots, t$. Since L has split-dimension equal to m there exist $i_1, \ldots, i_m \in \{1, \ldots, t\}$ such that $\text{lin}(\{\pi^{i_1}, \ldots, \pi^{i_m}\}) = \mathbb{R}^m$. For simplicity, we assume $\text{lin}(\{\pi^1, \ldots, \pi^m\}) = \mathbb{R}^m$ and let $B := [\pi^1, \ldots, \pi^m]$ be the matrix with columns π^i, $i = 1, \ldots, m$. Since $B^\top f \in \mathbb{Z}^m$ but $f \notin \mathbb{Z}^m$, the lattice with basis matrix B is a proper sublattice of $\mathbb{Z}^m$. From Lemma 4.6, it follows that there exists some $v \in \mathbb{Z}^m$ such that $v^\top f \notin \mathbb{Z}$ and $v = \sum_{i=1}^m \lambda_i \pi^i$ with $-\frac{1}{2} \leq \lambda_i \leq \frac{1}{2}$ for all $i = 1, \ldots, m$. Consider the split $S_v := \{x \in \mathbb{R}^m : \lfloor v^\top f\rfloor \leq v^\top x \leq \lceil v^\top f\rceil\}$ and let $\sum_{j\in N}\psi^v(r^j)s_j \geq 1$ be the cut associated with S_v. The coefficients $\psi^v(r^j)$ are given by (4.5) (with π^i replaced by v). In addition, we have $|v^\top r^j| = |\sum_{i=1}^m \lambda_i(\pi^i)^\top r^j| \leq \sum_{i=1}^m |\lambda_i||(\pi^i)^\top r^j| < \frac{m\omega}{2}$. Thus, $\psi^v(r^j) < \frac{mq\omega}{2}$ for all $j \in N$, and similar reasoning as in the first case finishes the proof. □

In Theorem 4.7, we derived an upper bound for $t(\mathcal{S}^1_\omega(R^n_f), \mathcal{S}^1(R^n_f))$ which is dependent on the dimension, the precision of f, and the max-facet-width of the polyhedra that define the inequalities for $\mathcal{S}^1_\omega(R^n_f)$. In some cases better bounds can be obtained, for instance when only a particular family of inequalities is considered and certain assumptions on the rays are made. We now show that RLS's can be approximated with usual split cuts by a constant factor which is independent of the precision of f.

Theorem 4.8. *Let $\mathcal{S}^1_{RLS}(R^n_f)$ be the set obtained by adding to $\mathcal{S}^1(R^n_f)$ all cuts associated with RLS's whose corner rays are among the rays r^j, $j \in N$. Then $\mathcal{S}^1(R^n_f) \subseteq m^2 \mathcal{S}^1_{RLS}(R^n_f)$.*

Proof. Let R be an RLS such that its corner rays are among the rays $r^1, \dots, r^n$ and let $\sum_{j \in N} \psi^R(r^j) s_j \geq 1$ be the cut associated with R. By a unimodular transformation, it suffices to consider the RLS $R := \mathrm{conv}(\{o, me_1, \dots, me_m\})$. We show that

$$\min \left\{ \sum_{j \in N} \psi^R(r^j) s_j : s \in S^1(R^n_f) \right\} \geq \frac{1}{m^2}. \tag{4.7}$$

For the remainder of the proof we can assume that the set of rays consists of the corner rays of R only (see the discussion after Observation 4.2 on p. 24). By setting $r^j := me_j - f$ for all $j = 1, \dots, m$ and $r^{m+1} := -f$ we obtain $\psi^R(r^j) = 1$ for all $j = 1, \dots, m+1$. In order to show (4.7), we have to prove that $\frac{1}{m^2}$ is a lower bound for the problem

$$\min \sum_{j=1}^{m+1} s_j \quad \text{s.t.} \quad s \in \mathcal{S}^1(R^n_f). \tag{4.8}$$

Henceforth, we write $\sum f_j$ instead of $\sum_{j=1}^m f_j$ for simplicity. We consider two cases.

Case 1: Assume that $\sum f_j \notin \mathbb{Z}$ and $f_i \notin \mathbb{Z}$ for all $i = 1, \dots, m$. We relax problem (4.8) by defining the $m+1$ splits

$$\begin{aligned} S_i &:= \{x \in \mathbb{R}^m : \lfloor f_i \rfloor \leq x_i \leq \lceil f_i \rceil\} \quad \forall i = 1, \dots, m, \\ S_{m+1} &:= \left\{ x \in \mathbb{R}^m : \left\lfloor \sum_{j=1}^m f_j \right\rfloor \leq \sum_{j=1}^m x_j \leq \left\lceil \sum_{j=1}^m f_j \right\rceil \right\}. \end{aligned}$$

For $i = 1, \dots, m+1$, let $\sum_{j=1}^{m+1} \psi^i(r^j) s_j \geq 1$ be the cut associated with S_i. The optimal objective value of (4.8) is greater than or equal to the minimal value of $\sum_{j=1}^{m+1} s_j$ such that $\sum_{j=1}^{m+1} \psi^i(r^j) s_j \geq 1$ for all $i = 1, \dots, m+1$ and $s_j \geq 0$ for all $j = 1, \dots, m+1$. Here, the coefficients $\psi^i(r^j)$ can be computed with the help of (4.5). We obtain the optimization problem

$$\begin{aligned}
&\min \sum_{j=1}^{m+1} s_j \qquad \text{s.t.} \\
&\tfrac{m-f_i}{\lceil f_i\rceil - f_i} s_i + \tfrac{f_i}{f_i - \lfloor f_i\rfloor} \sum_{j=1, j\neq i}^{m+1} s_j \geq 1 \qquad \forall i = 1,\dots,m, \\
&\tfrac{m-\sum f_i}{\lceil \sum f_i\rceil - \sum f_i} \sum_{j=1}^{m} s_j + \tfrac{\sum f_i}{\sum f_i - \lfloor \sum f_i\rfloor} s_{m+1} \geq 1, \\
&s_i \geq 0 \qquad \forall i = 1,\dots,m+1.
\end{aligned} \tag{4.9}$$

Now consider the corresponding dual linear program:

$$\begin{aligned}
&\max \sum_{j=1}^{m+1} y_j \qquad \text{s.t.} \\
&\tfrac{m-f_i}{\lceil f_i\rceil - f_i} y_i + \sum_{j=1, j\neq i}^{m} \tfrac{f_j}{f_j - \lfloor f_j\rfloor} y_j + \\
&\qquad \tfrac{m-\sum f_k}{\lceil \sum f_k\rceil - \sum f_k} y_{m+1} \leq 1 \qquad \forall i = 1,\dots,m, \\
&\sum_{j=1}^{m} \tfrac{f_j}{f_j - \lfloor f_j\rfloor} y_j + \tfrac{\sum f_k}{\sum f_k - \lfloor \sum f_k\rfloor} y_{m+1} \leq 1, \\
&y_i \geq 0 \qquad \forall i = 1,\dots,m+1.
\end{aligned} \tag{4.10}$$

The objective value of any dual feasible solution is a lower bound for the optimal objective value of (4.9), so we will construct such solutions in the following. In the remainder of the proof, when we write *i-th column of* (4.10), then we mean the i-th $(m+1)$-dimensional column of (4.10) without the non-negativity restrictions. Without loss of generality let $f_1 \geq \cdots \geq f_m$. If $f_m \leq \frac{1}{2}$, then $\bar{y}_m = \frac{1}{2m-1}$ and $\bar{y}_j = 0$ for all $j \neq m$ is dual feasible since the m-th column of (4.10) is

$$\left(1,\dots,1,\frac{m-f_m}{1-f_m},1\right)^\top \leq \left(1,\dots,1,\frac{m-\frac{1}{2}}{1-\frac{1}{2}},1\right)^\top = (1,\dots,1,2m-1,1)^\top.$$

Note that $\frac{1}{2m-1} \geq \frac{1}{m^2}$ as $m \geq 2$. Now let $f_m > \frac{1}{2}$. We have $f_m < 1$, otherwise $\sum f_i \geq m f_m \geq m$, which is a contradiction to $\sum f_i < m$. Therefore, $f_m = 1-\epsilon$ for some $\epsilon \in (0, \frac{1}{2})$. For $\epsilon \leq \frac{1}{m+1}$ we obtain $\sum f_i \geq m f_m = m(1-\epsilon) \geq m(1-\frac{1}{m+1}) = m^2(m+1)^{-1} > m-1$. Thus, $\lfloor \sum f_i \rfloor = m-1$ and $\lceil \sum f_i \rceil = m$. The $(m+1)$-th column of (4.10) is therefore $(1,\dots,1,\frac{\sum f_i}{\sum f_i - (m-1)})^\top$. Since $\sum f_i \geq \frac{m^2}{m+1}$ we obtain $(1,\dots,1,\frac{\sum f_i}{\sum f_i-(m-1)})^\top \leq (1,\dots,1,m^2)^\top$ and a dual feasible solution is given by $\bar{y}_{m+1} = \frac{1}{m^2}$ and $\bar{y}_j = 0$ for all $j \neq m+1$. For the opposite case where $\epsilon \geq \frac{1}{m+1}$ we obtain for the m-th column of (4.10) that $(1,\dots,1,\frac{m-(1-\epsilon)}{1-(1-\epsilon)},1)^\top \leq (1,\dots,1,m^2,1)^\top$ and therefore the dual feasible solution $\bar{y}_m = \frac{1}{m^2}$ and $\bar{y}_j = 0$ for all $j \neq m$ can be chosen. This shows that the optimal objective value of (4.9) is at least $\frac{1}{m^2}$ and therefore (4.7) holds true.

Case 2: Now assume that one of the f_i's or $\sum f_i$ is integer. If $f_i \in \mathbb{Z}$, then S_i is not a split, so we drop it. As in the proof of Case 1 we set up the dual and obtain a maximization problem similar to (4.10) where the i-th column is missing. Thus, the analysis remains the same, but now in dimension $m-1$. If $\sum f_i \in \mathbb{Z}$, then S_{m+1} is not a split. We drop the split and with it the $(m+1)$-th column of the dual. A similar reasoning as in the proof of Case 1 shows that for f_m (again the smallest non-integer of the f_i's) we must have $f_m \leq \frac{1}{2}$ or $f_m = 1 - \epsilon$, where $\epsilon \geq \frac{1}{m+1}$. If several of the f_i's or $\sum f_i$ are integer, then we erase all the corresponding columns of the dual (4.10) as explained above. Note that at least one of the f_i's is non-integer since f is in the interior of R. In every case, the bound $\frac{1}{m^2}$ is attained again. □

We point out that the bound derived in Theorem 4.8 is not tight, in general. For instance, for the case $m = 2$, Theorem 4.8 gives $S^1(R^n_f) \subseteq 4\mathcal{S}^1_{RLS}(R^n_f)$. However, in [BBCM11, Theorem 1.6] it is shown that even $S^1(R^n_f) \subseteq 2\mathcal{S}^1_{RLS}(R^n_f)$ holds true in this case.

Remark 4.9. *Based on experiments with the help of a computer algebra system, we conjecture that $S^1(R^n_f) \subseteq m\mathcal{S}^1_{RLS}(R^n_f)$ holds true. However, unfortunately we have no convincing arguments at hand to support our claim.*

The main message of this chapter is that, in general, non-trivial facet-defining inequalities for $\text{conv}(P_I)$ of full split-dimension are needed in order to approximate $\text{conv}(P_I)$ closely. On the other hand, we were able to show that split cuts approximate $\text{conv}(P_I)$ to within a constant factor when specific assumptions on the data size are made. In Chapter 6, we will continue to evaluate facet-defining inequalities for $\text{conv}(P_I)$ in the special case $m = 2$. For that, we need more insight into the structural properties of maximal lattice-free convex sets in $\mathbb{R}^2$. We will provide the necessary relations in the next chapter. These results are then used in Chapter 6 to obtain a more refined evaluation of facet-defining inequalities for $\text{conv}(P_I)$ in dimension two.

CHAPTER 5

AREA – LATTICE WIDTH RELATIONS IN THE PLANE

In this chapter, we study the relation between the area and the lattice width (resp. the area and the covering minima) of a two-dimensional lattice-free convex set. Parts of our results are used in Chapters 6 and 8. This was the original motivation for this chapter. In addition, we obtain results which go beyond an application to mixed-integer cutting plane theory.

We derive results for general and centrally symmetric planar lattice-free convex set. A correspondence between the lattice width on the one hand and the covering minima on the other, allows us to reformulate our results in terms of the covering minima introduced by Kannan and Lovász [KL88]. We obtain a tight upper bound for the area for any given value of the lattice width. With *tight bound* we mean that the bound is best possible in the sense that there exist lattice-free convex sets satisfying the bound with equality. The lattice-free convex sets satisfying the upper bound are characterized. Lower bounds are studied as well. We further rectify a result of [KL88] with a new proof.

In Section 5.1, we review basic notions and relevant results from the literature which we will need in the course of this chapter. Section 5.2 summarizes our main results. In Section 5.3, we present formulas for the lattice width and the area of triangles. Section 5.4 contains the proofs for general planar lattice-free convex sets. The proofs for the centrally symmetric planar lattice-free convex sets are given in Section 5.5.

5.1 Preliminaries

We recall that $\mathcal{K}^2$ is the class of closed convex sets in $\mathbb{R}^2$ with non-empty interior. Throughout this chapter, we only consider the lattice $\mathbb{Z}^2$ for notational convenience. For simplicity, we use $w(K)$ instead of $w_{\mathbb{Z}^2}(K)$ to denote the lattice width of $K \in \mathcal{K}^2$ with respect to the lattice $\mathbb{Z}^2$. Note that $w(K) > 0$ for every $K \in \mathcal{K}^2$. The area of K is denoted by $A(K)$. We point out, that our results could also be formulated in terms of an arbitrary lattice. Let us briefly explain why the inequalities that we obtain in this chapter are unaffected by the choice of the lattice. Let $K \in \mathcal{K}^2$ and let Λ be an arbitrary (full-dimensional) lattice in $\mathbb{R}^2$. Then $w_\Lambda(K) = w_{T(\Lambda)}(T(K))$ for every linear transformation T in $\mathbb{R}^2$. Choosing T such that it maps Λ onto $\mathbb{Z}^2$ we obtain $w(T(K)) = w_{\mathbb{Z}^2}(T(K)) = w_{T(\Lambda)}(T(K)) = w_\Lambda(K)$. Since $A(T(K)) \cdot \det(\Lambda) = A(K)$, every inequality involving the area and the lattice width with respect to $\mathbb{Z}^2$ can be transformed to an inequality involving the area, the lattice width with respect to Λ, and $\det(\Lambda)$, where $\det(\Lambda)$ denotes the determinant of the lattice Λ. For figures it is sometimes more convenient to use the lattice of regular triangles, i.e. the lattice generated by the vectors $(1, 0)$ and $\frac{1}{2}(1, \sqrt{3})$. The following fact is frequently used in our proofs.

Proposition 5.1. *Every lattice-free $K \in \mathcal{K}^2$ is contained in a maximal lattice-free $H \in \mathcal{K}^2$.*

Proof. For a closed convex set $U \subseteq \mathbb{R}^2$ and a point $x \in \mathbb{R}^2$ we denote by $c(U, x)$ the topological closure of the convex hull of $U \cup \{x\}$. Since the topological closure of a convex set is again convex (see, for instance, [Roc72, Theorem 6.2]), the set $c(U, x)$ is a closed convex set for every closed convex set $U \subseteq \mathbb{R}^2$ and every $x \in \mathbb{R}^2$.

Let $(z_n)_{n=1}^\infty$ be a sequence of all elements of $\mathbb{Q}^2$. We define $U_0 := K$ and for every $n \in \mathbb{N}$ we set $U_n := c(U_{n-1}, z_n)$ if $c(U_{n-1}, z_n)$ is lattice-free, and $U_n := U_{n-1}$ otherwise. Let H be the topological closure of $\bigcup_{n=0}^\infty U_n$. Then H is a closed convex set and since $U_{n-1} \subseteq U_n$ for every $n \in \mathbb{N}$ it holds $K \subseteq H$. This implies that H has a non-empty interior. By construction, H is a closed convex set with non-empty interior, i.e. $H \in \mathcal{K}^2$. In addition, H is lattice-free: assume y is an integer point in the interior of H. Then there exists some $j \in \mathbb{N}$ such that y is in the interior of U_j and thus, U_j is not lattice-free. This contradicts the construction of U_j. Let us show that H is maximal lattice-free. Assume the opposite and let $L \in \mathcal{K}^2$ be lattice-free such that $H \subsetneq L$. Then $L \setminus H$ contains rational points which occur in the sequence $(z_n)_{n=1}^\infty$. Let $z_k \in L \setminus H$ be such a rational point. If $c(U_{k-1}, z_k)$ were lattice-free, then $z_k \in c(U_{k-1}, z_k) = U_k \subseteq H$, a contradiction. On the other hand we have $z_k \in L$ and $U_{k-1} \subseteq H \subsetneq L$. Therefore, $c(U_{k-1}, z_k) \subseteq L$ by the convexity of L. This implies that L is not lattice-free, again a contradiction. $\square$

We point out that the above proof is not constructive. For a constructive, but lengthy proof we refer to [BCCZ10, Corollary 2.2].

Remark 5.2. *Proposition 5.1 states that, given a lattice-free closed convex set $K \subseteq \mathbb{R}^2$ with non-empty interior, then there exists a maximal lattice-free closed convex set $H \subseteq \mathbb{R}^2$ with non-empty interior such that $K \subseteq H$. It is interesting that the non-emptiness of the interior of K is crucial to infer that H has a non-empty interior. If K were only assumed to be a lattice-free closed convex set in $\mathbb{R}^2$, then the interior of H could be both, empty or non-empty. For instance, consider the line $K = \{x \in \mathbb{R}^2 : x_2 = \sqrt{2}x_1\}$ having an irrational slope and containing the origin as its only integer point. K is maximal lattice-free (and thus $K = H$), but its interior is empty. Using the notation of the proof of Proposition 5.1, it holds $U_n = K$ for every $n \in \mathbb{N}$. The reason is that the set $c(K, z)$, $z \in \mathbb{Q}^2$, is either K itself (if $z \in K$), or it contains an integer point in its interior (if $z \notin K$): let $z \notin K$ and let ϵ be the distance from z to K. It is well-known that for every line in $\mathbb{R}^2$ with an irrational slope there exist integer points arbitrarily close to it (see, for instance, [Kem17, Theorem 2]). Therefore, there exists a point $y \in \mathbb{Z}^2$ at (strictly positive) distance less than ϵ from K. Since $c(K, z) = \mathrm{conv}(K \cup (K + z))$, one of the points y or $-y$ is in the interior of $c(K, z)$. Thus, the set $c(K, z)$ contains an integer point in its interior whenever $z \in \mathbb{Q}^2 \setminus K$. Obviously, this argument can be generalized to arbitrary lines in $\mathbb{R}^2$ having an irrational slope.*

A classification of all maximal lattice-free sets in $\mathcal{K}^2$ was given by Lovász [Lov89, Section 3, p. 192]. The refined classification which is stated below can be found in [DW10, Proposition 1].

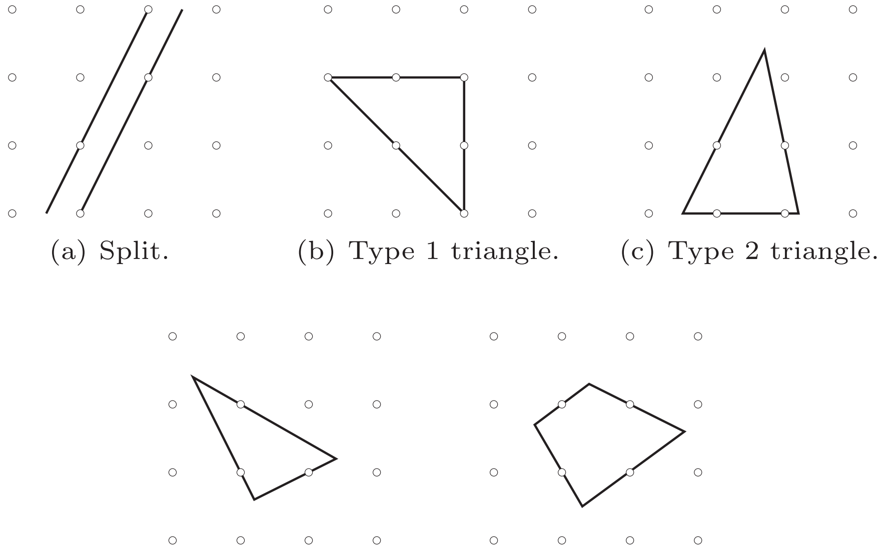
Proposition 5.3. ([DW10, Proposition 1] and [Lov89].) *Let $K \in \mathcal{K}^2$ be maximal lattice-free. Then K is either a split or a triangle or a quadrilateral. In particular, K is one of the following sets (see Fig. 5.1).*

I. A split $\{(x_1, x_2) \in \mathbb{R}^2 : \pi_0 \leq \pi_1 x_1 + \pi_2 x_2 \leq \pi_0 + 1\}$ with $\pi_0, \pi_1, \pi_2 \in \mathbb{Z}$ and $\gcd(\pi_1, \pi_2) = 1$.

II. A triangle which in turn is either a

- *(a) type 1 triangle, i.e. a triangle with integer vertices and exactly one integer point in the relative interior of each edge, or*
- *(b) type 2 triangle, i.e. a triangle with at least one fractional vertex v, exactly one integer point in the relative interior of the two edges incident to v and at least two integer points on the third edge, or*
- *(c) type 3 triangle, i.e. a triangle with exactly three integer points on the boundary, one in the relative interior of each edge. These three integer points form a triangle of area $\frac{1}{2}$.*

III. A quadrilateral containing exactly one integer point in the relative interior of each of its edges. These four integer points form a parallelogram of area 1.



(a) Split. (b) Type 1 triangle. (c) Type 2 triangle.

(d) Type 3 triangle. (e) Quadrilateral.

Figure 5.1: All types of maximal lattice-free sets in $\mathcal{K}^2$.

In the plane it is easy to construct examples of lattice-free convex sets with a lattice width of two. For instance, this is the case for every triangle of type 1. It may be surprising, that this value can be exceeded. This was noticed by Hurkens [Hur90], who also computed the best upper bound for the lattice width of a lattice-free convex set in the plane.

We point out that, throughout this chapter, sequences with n elements are indexed modulo n.

Theorem 5.4. [Hur90, p. 122] *Let $K \in \mathcal{K}^2$ be lattice-free. Then*

$$w(K) \leq 1 + \frac{2}{\sqrt{3}}, \tag{5.1}$$

with equality if and only if K is a triangle with vertices q_0, q_1, q_2 such that, for every i, the point $p_i := (1 - \frac{1}{\sqrt{3}})q_{i+1} + \frac{1}{\sqrt{3}}q_{i+2}$ belongs to $\mathbb{Z}^2$ (see Fig. 5.2).

The concept of covering minima is a useful tool in convex geometry. We will therefore give an alternative formulation of our results in terms of this concept. The notions lattice width and covering minima were introduced by Kannan and Lovász [KL86, KL88] for an arbitrary dimension. However, related results were obtained much earlier (see, for instance, Khinchin [Khi48]

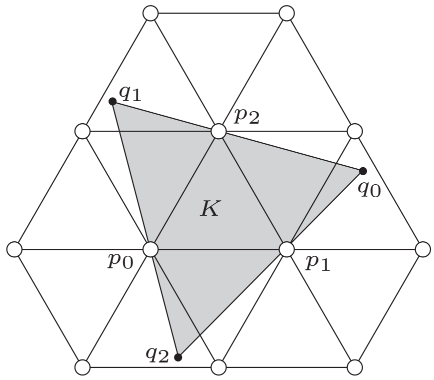


Figure 5.2: A lattice-free triangle with lattice width $1 + \frac{2}{\sqrt{3}}$.

and Fejes Tóth and Makai Jr. [FTM74]). In this chapter, we restrict our attention to dimension two.

Let $K \in \mathcal{K}^2$. For $j = 1, 2$ the j-th *covering minimum* is defined to be

$$\mu_j(K) := \inf \big\{ t \geq 0 : tK + \mathbb{Z}^2 \text{ intersects each } (2-j)\text{-dimensional affine subspace of } \mathbb{R}^2 \big\}.$$

This definition implies $0 < \mu_1(K) \leq \mu_2(K)$ for every $K \in \mathcal{K}^2$. If $K \in \mathcal{K}^2$ is a convex body, then the "inf" in the above definition becomes a "min". We note that the definition of $\mu_1(K)$ and $\mu_2(K)$ is invariant under translation of K since, for any $x \in \mathbb{R}^2$, $tK + \mathbb{Z}^2$ intersects each point (resp. line) in $\mathbb{R}^2$ if and only if $t(K-x)+x+\mathbb{Z}^2$ does (see [KL86, Observation 2.2] and [KL88, p. 582]). The second covering minimum $\mu_2(K)$ of K is the minimum value of $t \geq 0$ such that the sets $tK + z$ with $z \in \mathbb{Z}^2$ cover $\mathbb{R}^2$ and is also known under the name of *inhomogeneous minimum* (see [GL87, Section 13]). It turns out that some translate of K is lattice-free if and only if $\mu_2(K) \geq 1$ (see [KL88, p. 579][1]). Furthermore, for $t > 0$, an appropriate translate of tK is lattice-free if and only if $t \leq \mu_2(K)$. The first covering minimum $\mu_1(K)$ is the minimum value of $t \geq 0$ such that $tK + \mathbb{Z}^2$ intersects every line in $\mathbb{R}^2$. One can show that $\mu_1(K)w(K) = 1$ for every $K \in \mathcal{K}^2$ (see, for instance, [KL88, Lemma 2.3]). This leads to a correspondence between the lattice width on the one hand and the covering minima on the other, provided that K is lattice-free. For instance, Theorem 5.4 yields $\mu_2(K) \leq (1 + 2 \cdot (\sqrt{3})^{-1})\mu_1(K)$. Indeed, we know that $\mu_2(K) \cdot K$ has a lattice-free translate. From the linearity of the lattice width, we obtain that $1 + 2 \cdot (\sqrt{3})^{-1} \geq w(\mu_2(K) \cdot K) = \mu_2(K)w(K) = \mu_2(K) \cdot (\mu_1(K))^{-1}$. The results which we present in this chapter can therefore

[1] The authors in [KL88] deal with a different notion of lattice-freeness (which does not allow integer points on the boundary), but it is straightforward to transfer their results into our notion of lattice-freeness.

be expressed as a relation between the area and the covering minima of K, as well (see Corollaries 5.11 and 5.12).

Let $K \in \mathcal{K}^2$ be a convex body. We recall (see Chapter 2) that the support function of K is $h(K,u) = \max\{u^\top x : x \in K\}$, where $u \in \mathbb{R}^2$. Furthermore, the width of K along the vector $u \in \mathbb{R}^2$ is $w(K,u) = h(K,u) + h(K,-u)$, and the lattice width of K with respect to the lattice $\mathbb{Z}^2$ is $w(K) = \min\{w(K,u) : u \in \mathbb{Z}^2 \setminus \{o\}\}$. It is straightforward to show that $h(DK,u) = w(K,u)$ for every $u \in \mathbb{R}^2$, where DK is the difference body of K. Moreover, one has $h(K,u) = \|u\|_{K^*}$ for every $u \in \mathbb{R}^2$, where K^* is the polar body of K and $\|\cdot\|_{K^*}$ is the Minkowski functional of K^*.

A subset $\mathcal{X} \subseteq \mathcal{K}^2$ is said to *tile* $\mathbb{R}^2$ (or is a *tiling* of $\mathbb{R}^2$) if the union of the elements of $\mathcal{X}$ is $\mathbb{R}^2$ and their interiors are pairwise disjoint. We refer to [Sch93b, Section 4.1] for information on lattice tilings.

A set $K \in \mathcal{K}^2$ is called *symmetric in the origin* if $x \in K$ implies that $-x \in K$, and K is called *centrally symmetric* if there exists some $c \in \mathbb{R}^2$ such that the set $K - c = \{x - c : x \in K\}$ is symmetric in the origin. In this case, c is called the *center of symmetry* of K. If $K \in \mathcal{K}^2$ is centrally symmetric, then $\frac{1}{2}DK$ is a translate of K, and if K is symmetric in the origin, then $\frac{1}{2}DK = K$. The two-dimensional version of *Minkowski's first fundamental theorem* (see, for instance, [Gru07, Theorem 22.1]) states that if $K \in \mathcal{K}^2$ is symmetric in the origin and $\mathrm{int}(K) \cap \mathbb{Z}^2 = \{o\}$, then $A(K) \leq 4$. Furthermore, if $A(K) = 4$, then the sets $\frac{1}{2}K + z$ with $z \in \mathbb{Z}^2$ tile $\mathbb{R}^2$ (see, for instance, [GL87, p. 42, Theorem 2] and the preceding paragraph therein[2]). *Mahler's inequality* (see [Mah39a] and [Mah39b][3]) states that $A(K)A(K^*) \geq 8$ for every convex body $K \in \mathcal{K}^2$ which is symmetric in the origin. Moreover, Mahler's inequality is satisfied with equality if and only if K is a parallelogram. The following proposition is easy to show. It is a special case of a result of Hajós.

Proposition 5.5. [Haj41, Sections 1 and 2] *Let $P \subseteq \mathbb{R}^2$ be a parallelogram which is symmetric in the origin and such that its translates $P + z$ with $z \in \mathbb{Z}^2$ tile $\mathbb{R}^2$. Then, up to a unimodular transformation, $P = \frac{1}{2}\mathrm{conv}(\{\pm(1-\alpha, 1), \pm(1+\alpha, -1)\})$ for some $0 \leq \alpha < 1$.*

[2] This statement is not explicitly mentioned in [GL87], but it is a straightforward consequence of (i) $A(\frac{1}{2}K) = 1$, (ii) $\mathrm{int}(\frac{1}{2}K + u) \cap \mathrm{int}(\frac{1}{2}K + v) = \emptyset$ for all $u, v \in \mathbb{Z}^2$ with $u \neq v$, and (iii) Blichfeldt's theorem.

[3] In [Mah39b, p. 96], Mahler conjectured that for every $d \geq 2$ and every convex body $K \in \mathcal{K}^d$ which is symmetric in the origin the inequality $\mathrm{vol}(K)\,\mathrm{vol}(K^*) \geq 4^d \cdot (d!)^{-1}$ holds true. He verified his conjecture in [Mah39a, Section 3] for $d = 2$, where he also showed that equality is attained if and only if K is a parallelogram (see [Mah39a, p. 10]). However, for arbitrary d, no proof is known so far. In fact, the inequality has been analyzed by many mathematicians and is known under the name of *Mahler's conjecture*.

5.2 Main results

In this section, we present the main results of Chapter 5. Let us first give a brief summary. For a lattice-free set $K \in \mathcal{K}^2$ with given lattice width $w(K)$ we present a list of inequalities which relate its area $A(K)$ to its lattice width (Theorems 5.6 and 5.9). By Theorem 5.4, we know that $0 < w(K) \leq 1 + 2 \cdot (\sqrt{3})^{-1}$. Assuming that $w(K)$ is given, we derive bounds for the maximum and the minimum possible area of K. Broadly speaking, our bounds have the form $L(w(K)) \leq A(K) \leq U(w(K))$, where $L(w(K))$ (resp. $U(w(K))$) is the lower (resp. upper) bound for the area and depends only on the lattice width of K. The sets $K \in \mathcal{K}^2$ for which the upper bound is attained are characterized for every value of $w(K)$ between 0 and $1 + 2 \cdot (\sqrt{3})^{-1}$. The sets $K \in \mathcal{K}^2$ for which the lower bound is attained are characterized only for $w(K) \leq 2$. For the case of centrally symmetric sets $K \in \mathcal{K}^2$ we even give the complete list of inequalities, i.e. the lower and upper bound for $A(K)$ for a given $w(K)$ and a characterization of all sets $K \in \mathcal{K}^2$ for which equality is attained. Theorem 5.6 states the relation between the area and the lattice width of arbitrary lattice-free sets $K \in \mathcal{K}^2$.

Theorem 5.6. *Let $K \in \mathcal{K}^2$ be lattice-free with $w := w(K)$ and $A := A(K)$. Then*

$$A \leq \infty \qquad \text{for} \qquad 0 < w \leq 1, \tag{5.2}$$

$$A \leq \frac{w^2}{2(w-1)} \qquad \text{for} \qquad 1 < w \leq 2, \tag{5.3}$$

$$A \leq \frac{3w^2}{3w + 1 - \sqrt{1 + 6w - 3w^2}} \qquad \text{for} \qquad 2 < w \leq 1 + \frac{2}{\sqrt{3}}, \tag{5.4}$$

$$A \geq \frac{3}{8}w^2 \qquad \text{for} \qquad 0 < w \leq 1 + \frac{2}{\sqrt{3}} \tag{5.5}$$

(see Fig. 5.4(a)). Furthermore, the following statements hold.

I. *Equality in* (5.2) *is attained if and only if K is unbounded and contained in a split.*

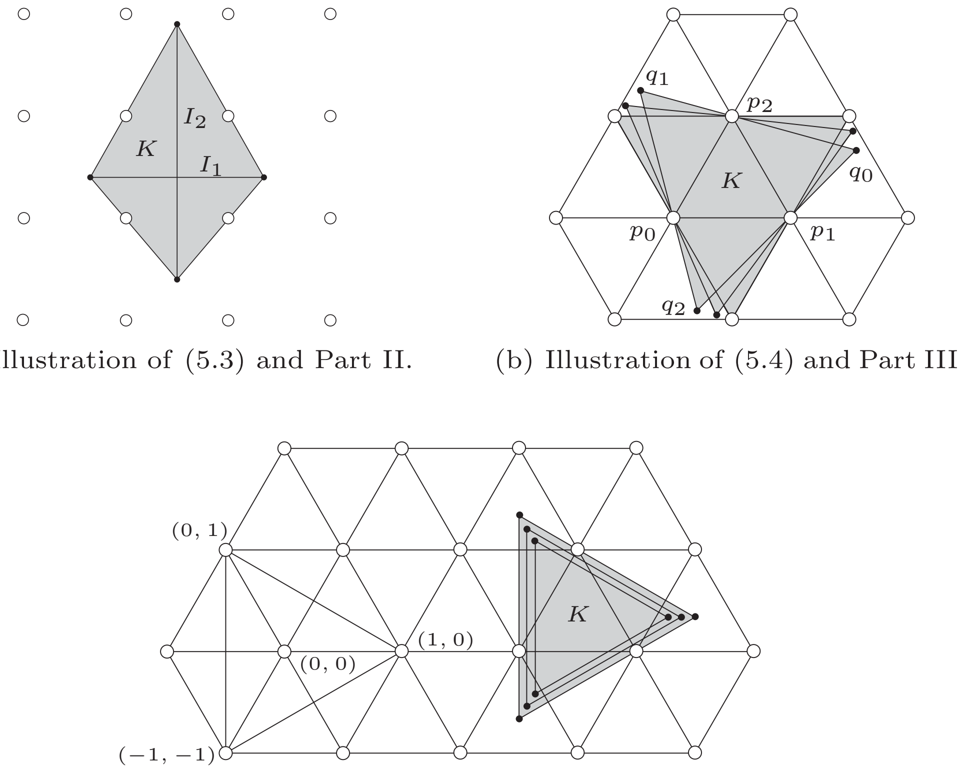
II. *Equality in* (5.3) *is attained if and only if, up to a unimodular transformation, $K = \operatorname{conv}(I_1 \cup I_2)$, where I_1 is a translate of $\operatorname{conv}(\{o, we_1\})$, I_2 is a translate of $\operatorname{conv}(\{o, \frac{w}{w-1}e_2\})$, and $I_1 \cap I_2 \neq \emptyset$ (see Fig. 5.3(a)).*

III. *Equality in* (5.4) *is attained if and only if K is a triangle with vertices q_0, q_1, q_2 such that, for every i, the point $p_i := (1-\lambda)q_{i+1} + \lambda q_{i+2}$ belongs to $\mathbb{Z}^2$ for*

$$\lambda := \frac{3w + 1 - \sqrt{1 + 6w - 3w^2}}{6w}$$

(see Fig. 5.3(b)).

IV. If $0 < w \leq 2$*, then equality in* (5.5) *is attained if and only if, up to a unimodular transformation,* K *is a translate of* $\frac{w}{2}\,\mathrm{conv}(\{(1,0),(0,1),(-1,-1)\})$ *(see Fig. 5.3(c)).*



(a) Illustration of (5.3) and Part II.

(b) Illustration of (5.4) and Part III.

(c) Illustration of (5.5) and Part IV.

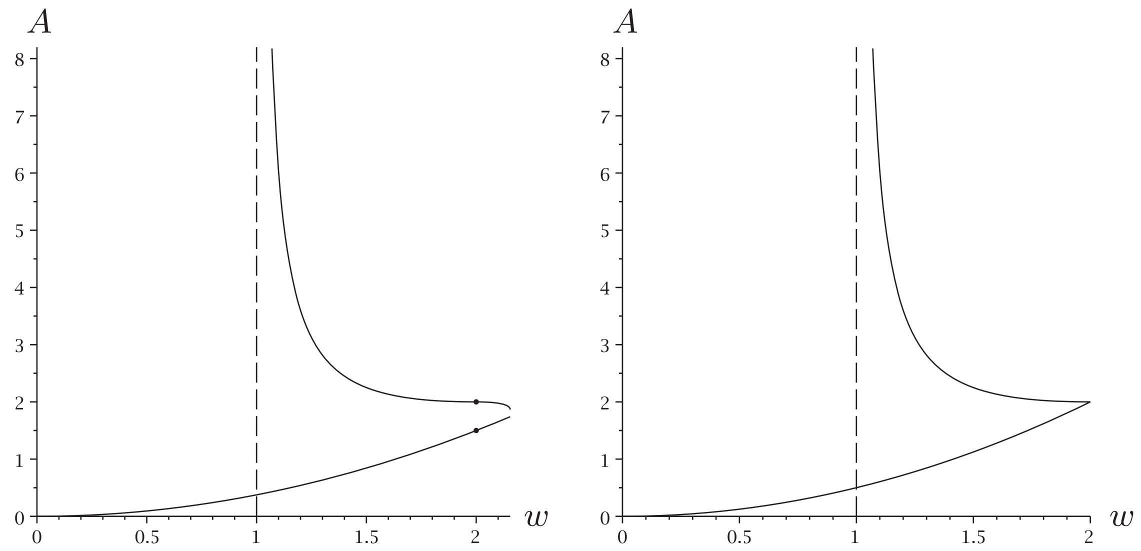
Figure 5.3: Examples of sets yielding equality in (5.3)–(5.5).

Remark 5.7. *It is straightforward to show that the upper bound in* (5.4) *is monotonically non-increasing for* $2 < w(K) \leq 1 + 2\cdot(\sqrt{3})^{-1}$*. In particular, it holds* $A(K) \leq 2$*.*

The bound (5.5) is not tight for $2 < w(K) \leq 1 + 2\cdot(\sqrt{3})^{-1}$. To see why, we need a result of Fejes Tóth and Makai Jr. [FTM74].

Theorem 5.8. [FTM74, Theorem 2] *Let* $K \in \mathcal{K}^2$ *with* $w := w(K)$ *and* $A := A(K)$*. Then it holds* $A \geq \frac{3}{8}w^2$*. Equality is attained if and only if, up to a unimodular transformation,* K *is a translate of* $\frac{w}{2}\,\mathrm{conv}(\{(1,0),(0,1),(-1,-1)\})$*.*

It is easy to see that $\frac{1}{2}w(K)\cdot\mathrm{conv}(\{(1,0),(0,1),(-1,-1)\})$ does not have a lattice-free translate for $w(K) > 2$. Thus, in view of Theorem 5.8, (5.5) is not tight when $2 < w(K) \leq 1 + 2\cdot(\sqrt{3})^{-1}$. The problem to determine the tight lower bound in this case is still open. We did not succeed to find it.



(a) Pairs (w, A) satisfying the inequalities in Theorem 5.6.

(b) Pairs (w, A) satisfying the inequalities in Theorem 5.9.

Figure 5.4: Bounds for the area in the general and centrally symmetric case.

A statement analogous to Theorem 5.6 can also be proved for centrally symmetric sets $K \in \mathcal{K}^2$. We show the following theorem.

Theorem 5.9. *Let $K \in \mathcal{K}^2$ be lattice-free and centrally symmetric with $w := w(K)$ and $A := A(K)$. Then*

$$0 < w \leq 2, \tag{5.6}$$

$$A \leq \infty \qquad \textit{for} \qquad 0 < w \leq 1, \tag{5.7}$$

$$A \leq \frac{w^2}{2(w-1)} \qquad \textit{for} \qquad 1 < w \leq 2, \tag{5.8}$$

$$A \geq \frac{1}{2}w^2 \qquad \textit{for} \qquad 0 < w \leq 2 \tag{5.9}$$

(see Fig. 5.4(b)). Furthermore, the following statements hold.

I. *The upper bound in* (5.6) *is attained if and only if, up to a unimodular transformation,*

$$K = \operatorname{conv}(\{\pm(1,0), \pm(0,1)\}) + \left(\frac{1}{2}, \frac{1}{2}\right).$$

(see Fig. 5.5(a)).

II. *Equality in* (5.7) *is attained if and only if K is unbounded and contained in a split.*

III. Equality in (5.8) *is attained if and only if, up to a unimodular transformation,*

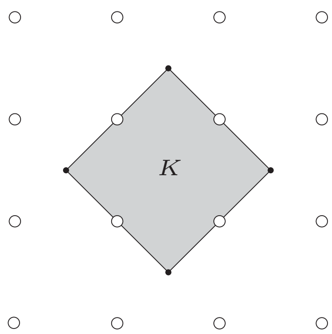
$$K = \mathrm{conv}\left(\left\{\pm\left(\frac{w}{2},0\right), \pm\left(0, \frac{w}{2(w-1)}\right)\right\}\right) + \left(\frac{1}{2}, \frac{1}{2}\right)$$

(see Fig. 5.5(b)).

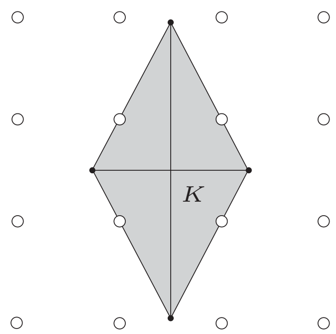
IV. Equality in (5.9) *is attained if and only if, up to a unimodular transformation, K is a translate of*

$$\frac{w}{2}\,\mathrm{conv}(\{\pm(1,\alpha), \pm(0,1)\})$$

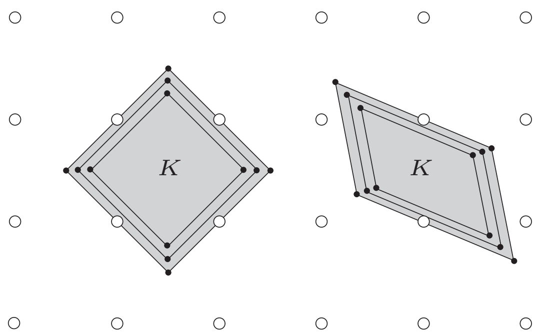
for some $0 \le \alpha < 1$ *that satisfies* $\max\{1+\alpha, 2-\alpha\} \ge w$ *(see Fig. 5.5(c)).*



(a) Illustration of (5.6) and Part I.



(b) Illustration of (5.8) and Part III.



(c) Illustration of (5.9) and Part IV.

Figure 5.5: Examples of sets yielding equality in (5.6), (5.8), and (5.9).

For centrally symmetric sets $K \in \mathcal{K}^2$, the lower bound in (5.9) was shown by Makai Jr. [Mak78, Theorem 3] who noticed that equality in (5.9) is attained only if K is a parallelogram, but the precise shape of all the parallelograms satisfying (5.9) was not stated. We also refer to Fejes Tóth [FT73] who characterized the extremal sets in a special case. However, to the best of our knowledge, an exact characterization of all the sets $K \in \mathcal{K}^2$ for which equality in (5.9) is attained is not known so far. This gap is now filled by Part IV of Theorem 5.9.

The upper bound in (5.6) is a consequence of a more general result due to Kannan and Lovász [KL88, Theorem 2.13]. Unfortunately, the proof of Theorem 2.13 in [KL88] does not seem to be correct[4], but implies the weaker result $0 < w(K) \leq 3$. Therefore, we prove the subsequent theorem.

Theorem 5.10. *Let $K \in \mathcal{K}^2$ be centrally symmetric. Then one has $\mu_2(K) \leq 2\mu_1(K)$.*

If $K \in \mathcal{K}^2$ is lattice-free, and thus $\mu_2(K) \geq 1$, then together with the relation $\mu_1(K)w(K) = 1$, Theorem 5.10 implies $w(K) \leq 2$. The results stated in Theorems 5.6 and 5.9 can also be expressed in terms of covering minima. In Corollary 5.11, the lower bound for $A(K)$ goes back to [FTM74, Theorem 2]. All the bounds for $A(K)$ in Corollaries 5.11 and 5.12 can be deduced from Theorems 5.6 and 5.9 in a straightforward way. We refer to [BHW93] and [Sch95] for further inequalities involving $\mu_1(K)$, $\mu_2(K)$, and $A(K)$.

Corollary 5.11. *Let $K \in \mathcal{K}^2$ with $A := A(K)$, $\mu_1 := \mu_1(K)$ and $\mu_2 := \mu_2(K)$. Then*

$$\mu_1 \leq \mu_2 \leq \left(1 + \frac{2}{\sqrt{3}}\right)\mu_1,$$

$$\begin{aligned}
A &\leq \infty && \text{for} \quad \mu_1 = \mu_2,\\
A &\leq \frac{1}{2\mu_1(\mu_2 - \mu_1)} && \text{for} \quad \mu_1 < \mu_2 \leq 2\mu_1,\\
A &\leq \frac{3}{\mu_1\left(3\mu_2 + \mu_1 - \sqrt{\mu_1^2 + 6\mu_1\mu_2 - 3\mu_2^2}\right)} && \text{for} \ 2\mu_1 < \mu_2 \leq \left(1 + \frac{2}{\sqrt{3}}\right)\mu_1,\\
A &\geq \frac{3}{8\mu_1^2} && \text{for} \quad \mu_1 \leq \mu_2 \leq \left(1 + \frac{2}{\sqrt{3}}\right)\mu_1.
\end{aligned}$$

The upper bounds for A are tight, whereas the lower bound for A is tight only for $\mu_1 \leq \mu_2 \leq 2\mu_1$.

[4] In the proof of Theorem 2.13 in [KL88] one claims that the covering minima of centrally symmetric convex bodies satisfy the inequalities $\mu_{k+1} \leq 2\mu_k$ for all $k = 1, \ldots, d-1$, where d is the dimension of the underlying space (see [KL88] for the explanation of the notation). In the proof of Theorem 2.13, p. 588, l. 11, it is inferred that $2(\alpha+\beta+\lambda_1) \leq 4(\alpha+\beta)$. However, one line before it is just shown that $\lambda_1 \leq 2(\alpha+\beta)$ and therefore the correct conclusion is $2(\alpha+\beta+\lambda_1) \leq 6(\alpha+\beta)$. Using the factor 6 instead of 4 in the rest of the proof leads to the weaker result $\mu_{k+1} \leq 3\mu_k$. To the best of our knowledge there is no revision or corrected version of this proof which would yield the assertion $\mu_{k+1} \leq 2\mu_k$. We have also tried to contact the authors by e-mail, but received no reply.

Corollary 5.12. *Let $K \in \mathcal{K}^2$ be centrally symmetric with $A := A(K)$, $\mu_1 := \mu_1(K)$ and $\mu_2 := \mu_2(K)$. Then*

$$\begin{aligned} &\mu_1 \leq \mu_2 \leq 2\mu_1, && && \\ &A \leq \infty && \text{for} && \mu_1 = \mu_2, \\ &A \leq \frac{1}{2\mu_1(\mu_2 - \mu_1)} && \text{for} && \mu_1 < \mu_2 \leq 2\mu_1, \\ &A \geq \frac{1}{2\mu_1^2} && \text{for} && \mu_1 \leq \mu_2 \leq 2\mu_1. \end{aligned}$$

The above bounds are tight.

Theorems 5.6 and 5.9 will be proved for maximal lattice-free convex sets first. Once the proof is established we show that it implies the validity of the inequalities for lattice-free convex sets which are not maximal. Before we present the proofs, we need some auxiliary results on triangles which we discuss in the next section.

5.3 Auxiliary results on triangles

An essential part of the proofs of Theorems 5.6 and 5.9 is concerned with analytical representations of the lattice width and the area of polytopes. In particular, we will need formulas for the lattice width of triangles in many places. We recall that for a triangle $T = \operatorname{conv}(\{t_0, t_1, t_2\}) \subseteq \mathbb{R}^2$ and a point $p \in \mathbb{R}^2$, p can be uniquely represented by $p = \lambda_0 t_0 + \lambda_1 t_1 + \lambda_2 t_2$ where $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_0 + \lambda_1 + \lambda_2 = 1$. The multipliers λ_0, λ_1, and λ_2 are called the barycentric coordinates of p with respect to the triangle T. For more information on barycentric coordinates we refer to [Cox69, Section 13.7]. The following lemma summarizes well-known facts about barycentric coordinates of triangles.

Lemma 5.13. *Let $Q := \operatorname{conv}(\{q_0, q_1, q_2\}) \subseteq \mathbb{R}^2$ be a triangle and let $p \in \mathbb{R}^2$ be represented in the form $p = x_0 q_0 + x_1 q_1 + x_2 q_2$ with $x_0 + x_1 + x_2 = 1$. We define $H_j := \operatorname{aff}(\{q_0, q_1, q_2\} \setminus \{q_j\})$ for $j = 0, 1, 2$. Then the following statements hold.*

I. The points p and q_j lie in the same open half-plane defined by H_j if and only if $x_j > 0$.

H_j separates the points p and q_j if and only if $x_j < 0$.

The point p lies on H_j if and only if $x_j = 0$.

II. The value $|x_j|$ is the ratio of the distance from p to H_j and the distance from q_j to H_j.

III. Let $P := \operatorname{conv}(\{p_0, p_1, p_2\}) \subseteq \mathbb{R}^2$ *be a triangle with* $p_i = \sum_{j=0}^{2} x_{i,j} q_j$ *and* $\sum_{j=0}^{2} x_{i,j} = 1$ *for all* $i = 0, 1, 2$*. Then the areas of* P *and* Q *are related by* $A(P) = |\det(x_{i,j})_{i,j=0,\dots,2}| A(Q)$*.*

The next lemma provides formulas for the area and the lattice width of triangles whose vertices are expressed in terms of the vertices of another triangle which coincides, up to a unimodular transformation, with the triangle $\operatorname{conv}(\{o, e_1, e_2\})$.

Lemma 5.14. *Let* $P := \operatorname{conv}(\{p_0, p_1, p_2\})$ *be a triangle such that* $p_0, p_1, p_2 \in \mathbb{Z}^2$ *are the only integer points in* P*. Let* $Q := \operatorname{conv}(\{q_0, q_1, q_2\})$ *be a triangle whose vertices are given by the barycentric coordinates with respect to* P*, that is, by a matrix* $B \in \mathbb{R}^{3\times 3}$ *such that*

$$\begin{pmatrix} q_0^\top & 1 \\ q_1^\top & 1 \\ q_2^\top & 1 \end{pmatrix} = B \begin{pmatrix} p_0^\top & 1 \\ p_1^\top & 1 \\ p_2^\top & 1 \end{pmatrix}.$$

Then

$$w(Q) = \min\{\|DBz\|_\infty : z \in \mathbb{Z}^3 \text{ and the coordinates of } z \text{ are not all equal}\}, \tag{5.10}$$

where

$$D := \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Furthermore, if $p_i = (1 - x_i) q_{i+1} + x_i q_{i+2}$ *with* $0 < x_i < 1$ *for all* $i = 0, 1, 2$ *(that is,* P *is contained in* Q*, see Fig. 5.6), then*

$$w(Q) = \frac{\min\left\{\max_{i=0,1,2} |x_i y_i + (1 - x_{i+1}) y_{i+1}| : y \in Y\right\}}{x_0 x_1 x_2 + (1 - x_0)(1 - x_1)(1 - x_2)} \tag{5.11}$$

and

$$A(Q) = \frac{1}{2\big(x_0 x_1 x_2 + (1 - x_0)(1 - x_1)(1 - x_2)\big)}, \tag{5.12}$$

where $Y := \{y \in \mathbb{Z}^3 \setminus \{o\},\ y_0 + y_1 + y_2 = 0\}$*.*

Proof. For $u \in \mathbb{Z}^2$ we have

$$w(Q, u) = \max\left\{|q_i^\top u - q_j^\top u| : 0 \le i < j \le 2\right\} = \left\| D \begin{pmatrix} q_0^\top \\ q_1^\top \\ q_2^\top \end{pmatrix} u \right\|_\infty$$

$$= \left\| D \begin{pmatrix} q_0^\top & 1 \\ q_1^\top & 1 \\ q_2^\top & 1 \end{pmatrix} \begin{pmatrix} u \\ k \end{pmatrix} \right\|_\infty = \left\| DB \begin{pmatrix} p_0^\top & 1 \\ p_1^\top & 1 \\ p_2^\top & 1 \end{pmatrix} \begin{pmatrix} u \\ k \end{pmatrix} \right\|_\infty = \|DBz\|_\infty ,$$

where $k \in \mathbb{Z}$ is arbitrary and

$$z := \begin{pmatrix} p_0^\top & 1 \\ p_1^\top & 1 \\ p_2^\top & 1 \end{pmatrix} \begin{pmatrix} u \\ k \end{pmatrix} = \begin{pmatrix} p_0^\top u + k \\ p_1^\top u + k \\ p_2^\top u + k \end{pmatrix} .$$

Clearly, $z \in \mathbb{Z}^3$. Since the vector z is the product of a unimodular matrix and an integer vector, it follows that $u = o$ if and only if the coordinates of z are all equal. This shows (5.10).

Let us now show (5.11). By assumption, we have

$$\begin{pmatrix} p_0^\top & 1 \\ p_1^\top & 1 \\ p_2^\top & 1 \end{pmatrix} = X \begin{pmatrix} q_0^\top & 1 \\ q_1^\top & 1 \\ q_2^\top & 1 \end{pmatrix} ,$$

where

$$X := B^{-1} = \begin{pmatrix} 0 & 1 - x_0 & x_0 \\ x_1 & 0 & 1 - x_1 \\ 1 - x_2 & x_2 & 0 \end{pmatrix} .$$

Since X is the matrix of barycentric coordinates of the vertices of P with respect to Q, B is the matrix of barycentric coordinates of the vertices of Q with respect to P. Direct computations yield

$$\det(X) = x_0 x_1 x_2 + (1 - x_0)(1 - x_1)(1 - x_2) > 0,$$

$$B = \frac{1}{\det(X)} \begin{pmatrix} -(1 - x_1)x_2 & x_0 x_2 & (1 - x_0)(1 - x_1) \\ (1 - x_1)(1 - x_2) & -(1 - x_2)x_0 & x_0 x_1 \\ x_1 x_2 & (1 - x_0)(1 - x_2) & -(1 - x_0)x_1 \end{pmatrix} ,$$

$$DB = \frac{1}{\det(X)} \begin{pmatrix} 1 - x_1 & -x_0 & x_0 + x_1 - 1 \\ x_1 + x_2 - 1 & 1 - x_2 & -x_1 \\ -x_2 & x_0 + x_2 - 1 & 1 - x_0 \end{pmatrix} .$$

Using the latter matrix relation we obtain for every $z = (z_0, z_1, z_2)^\top \in \mathbb{Z}^3$ that

$$DBz = \frac{1}{\det(X)} \begin{pmatrix} z_2 - z_1 & z_2 - z_0 & 0 & z_0 - z_2 \\ 0 & z_0 - z_2 & z_0 - z_1 & z_1 - z_0 \\ z_1 - z_2 & 0 & z_1 - z_0 & z_2 - z_1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ 1 \end{pmatrix} .$$

By the change of variables

$$y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} := \begin{pmatrix} z_2 - z_1 \\ z_0 - z_2 \\ z_1 - z_0 \end{pmatrix} \tag{5.13}$$

the latter amounts to

$$\begin{aligned} DBz &= \frac{1}{\det(X)} \begin{pmatrix} y_0 & -y_1 & 0 & y_1 \\ 0 & y_1 & -y_2 & y_2 \\ -y_0 & 0 & y_2 & y_0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ 1 \end{pmatrix} \\ &= \frac{1}{\det(X)} \begin{pmatrix} x_0 y_0 + (1 - x_1) y_1 \\ x_1 y_1 + (1 - x_2) y_2 \\ x_2 y_2 + (1 - x_0) y_0 \end{pmatrix}. \end{aligned}$$

Clearly, $y_0 + y_1 + y_2 = 0$ and, if the coordinates of z are not all equal, we have $y \neq o$. Conversely, for an arbitrary $y \in \mathbb{Z}^3 \setminus \{o\}$ with $y_0 + y_1 + y_2 = 0$, we can easily find an appropriate $z \in \mathbb{Z}^3$ satisfying (5.13) and such that the coordinates of z are not all equal: without loss of generality assume that $y_2 \neq 0$. Then the vector $z \in \mathbb{Z}^3$ defined by $z_0 := y_0$, $z_1 := y_0 + y_2$, and $z_2 := 2y_0 + y_2$ has all required properties. Thus, employing (5.10) and the previous derivation we arrive at

$$w(Q) = \frac{1}{\det(X)} \min \{\max \{|x_i y_i + (1 - x_{i+1}) y_{i+1}| : i = 0, 1, 2\} : y \in Y\}.$$

Equation (5.12) follows from Lemma 5.13 III and the fact that $\frac{1}{2} = A(P) = \det(X) A(Q)$. □

Lemma 5.14 allows us to compute the lattice width of an arbitrary triangle Q with the help of formula (5.10). This formula is only dependent on the barycentric coordinates of the vertices of Q with respect to a triangle P that is some unimodular transformation of $\operatorname{conv}(\{o, e_1, e_2\})$. In the special case where every vertex of P is a proper convex combination of two vertices of Q, formula (5.10) can be simplified to formula (5.11). Admittedly, both formulas are too technical to be of use for a given triangle Q. However, we will see in the next section that (5.11) simplifies nicely when Q is assumed to be a maximal lattice-free triangle of type 3 containing P.

5.4 Proofs for arbitrary sets

In this section, we prove Theorem 5.6. Our strategy is to show the statements in Theorem 5.6 only for maximal lattice-free sets in $\mathcal{K}^2$, and then to argue that they hold true for arbitrary lattice-free sets in $\mathcal{K}^2$ as well. This is possible since every lattice-free set in $\mathcal{K}^2$ is contained in some maximal lattice-free set in $\mathcal{K}^2$ by Proposition 5.1. Details of the proof will be given at the end of this section. Let us now focus on maximal lattice-free sets in $\mathcal{K}^2$. In Lemma 5.15, we characterize maximal lattice-free triangles of type 3 (see Fig. 5.1(d)) and their lattice width.

Lemma 5.15. *Let $P := \operatorname{conv}(\{p_0, p_1, p_2\})$ be a triangle such that $p_0, p_1, p_2 \in \mathbb{Z}^2$ are the only integer points in P. Let $Q := \operatorname{conv}(\{q_0, q_1, q_2\})$ be a triangle containing P such that $p_i = (1 - x_i)q_{i+1} + x_i q_{i+2}$ and $0 < x_i < 1$ for all $i = 0, 1, 2$ (see Fig. 5.6). Then the following statements hold.*

I. Q is a maximal lattice-free triangle of type 3 if and only if

(a) $x_i + x_j > 1$ for all $0 \le i < j \le 2$ or

(b) $x_i + x_j < 1$ for all $0 \le i < j \le 2$.

II. If (a) holds, then the lattice width of Q is given by

$$w(Q) = \frac{\min\{x_0, x_1, x_2\}}{x_0x_1x_2 + (1 - x_0)(1 - x_1)(1 - x_2)}.$$

III. If (a) holds, then $w(Q) \le 1 + \frac{2}{\sqrt{3}}$ with equality if and only if $x_0 = x_1 = x_2 = \frac{1}{\sqrt{3}}$.

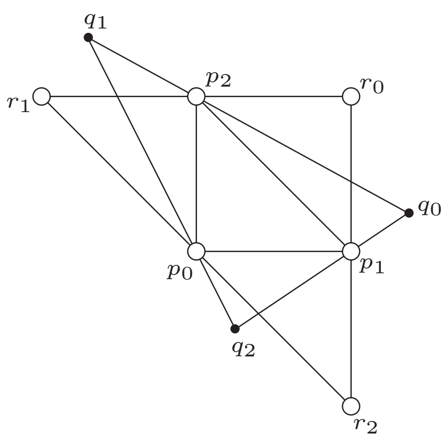


Figure 5.6: Points p_i, q_i, r_i, $i \in \{0, 1, 2\}$, as in the proof of Lemma 5.15.

Proof. Part I. Assume that Q is a maximal lattice-free triangle of type 3. For $i = 0, 1, 2$ we denote by H_i the closed half-plane with $q_{i+1}, q_{i+2} \in \operatorname{bd}(H_i)$ and $q_i \in H_i$. We also introduce for $i = 0, 1, 2$ the points $r_i := -p_i + p_{i+1} + p_{i+2} \in \mathbb{Z}^2$. By construction, p_i is the midpoint of $[r_{i+1}, r_{i+2}]$ for all $i = 0, 1, 2$.

Because of the latter property, and since $p_i \in \mathrm{bd}(H_i)$, we have $r_{i+1} \in H_i$ or $r_{i+2} \in H_i$ for all $i = 0, 1, 2$. For $i = 0, 1, 2$ we denote by $\tau(i)$ the set of all $k \in \{0, 1, 2\}$ such that $k \neq i$ and $r_k \in H_i$. By the above observations, it follows that $\tau(i) \neq \emptyset$ for every i. If for some $0 \leq i < j \leq 2$ one has $\tau(i) \cap \tau(j) \neq \emptyset$ we choose $k \in \tau(i) \cap \tau(j)$. Then $r_k \in H_0 \cap H_1 \cap H_2 = Q$, and by this we infer that $\mathrm{conv}(\{p_0, p_1, p_2, r_k\})$ is a subset of Q. Therefore, Q contains four distinct integer points. This contradicts the assumption that Q is a maximal lattice-free triangle of type 3. Thus, $\tau(i) \cap \tau(j) = \emptyset$ for $0 \leq i < j \leq 2$. Taking into account that $i \notin \tau(i)$ for $i = 0, 1, 2$, we see that $\tau(i)$ is a singleton for every i and is in fact one of the two possible cyclic shifts on $\{0, 1, 2\}$. In other words, either $\tau(i) = i + 1 \pmod 3$ for every i or $\tau(i) = i + 2 \pmod 3$ for every i. If $\tau(i) = i + 1 \pmod 3$ for every i, then $r_{i+2} \notin H_i$ for every i. This means, that the i-th barycentric coordinate of r_{i+2} with respect to Q is strictly negative. This barycentric coordinate is $x_{i+1} + x_{i+2} - 1$ since

$$\begin{aligned}
r_{i+2} &= -\, p_{i+2} + p_i + p_{i+1} \\
&= -\, ((1 - x_{i+2})q_i + x_{i+2}q_{i+1}) + \\
&\quad ((1 - x_i)q_{i+1} + x_i q_{i+2}) + ((1 - x_{i+1})q_{i+2} + x_{i+1}q_i) \\
&= (x_{i+1} + x_{i+2} - 1)q_i + (1 - x_i - x_{i+2})q_{i+1} + (1 - x_{i+1} + x_i)q_{i+2}.
\end{aligned}$$

Thus, we obtain (b). If $\tau(i) = i + 2 \pmod 3$ for every i, arguing in the same way we obtain (a).

For proving the converse, we assume that (a) or (b) is fulfilled and we show that Q is a maximal lattice-free triangle of type 3. Consider an arbitrary $p \in \mathbb{Z}^2 \setminus \{p_0, p_1, p_2\}$. We can represent p by $p = z_0p_0 + z_1p_1 + z_2p_2$ where $z_0, z_1, z_2 \in \mathbb{Z}$ and $z_0 + z_1 + z_2 = 1$. By symmetry, we may assume that $z_0 \leq z_1 \leq z_2$. Under these assumptions, we have $z_2 \geq 1$ and $z_0 \leq 0$. We compute the barycentric coordinates of p with respect to Q as follows:

$$\begin{aligned}
p &= z_0p_0 + z_1p_1 + z_2p_2 \\
&= z_0((1 - x_0)q_1 + x_0q_2) + z_1((1 - x_1)q_2 + x_1q_0) + z_2((1 - x_2)q_0 + x_2q_1) \\
&= (z_1x_1 + z_2(1 - x_2))q_0 + (z_0(1 - x_0) + z_2x_2)q_1 + (z_0x_0 + z_1(1 - x_1))q_2.
\end{aligned}$$

Case 1: If $z_1 \leq 0$, then the barycentric coordinate $z_0x_0 + z_1(1 - x_1)$ is less than or equal to zero. Now, if $z_0 = z_1 = 0$, then $z_2 = 1$ and $p = p_2$, a contradiction. Hence, we must have $z_0 < 0$ which implies that $z_0x_0 + z_1(1 - x_1) < 0$ and by this, in view of Lemma 5.13 I, $p \notin Q$.

Case 2: Assume that $z_1 \geq 1$. If (a) is fulfilled, then the barycentric coordinate $z_0x_0 + z_1(1 - x_1)$ is estimated as follows:

$$z_0x_0 + z_1(1 - x_1) < z_0x_0 + z_1x_0 = (z_0 + z_1)x_0 = (1 - z_2)x_0 \leq 0.$$

Consequently, $p \notin Q$. If (b) is fulfilled, the barycentric coordinate $z_0(1-x_0)+z_2x_2$ can be estimated analogously:

$$\begin{aligned} z_0(1-x_0)+z_2x_2 &< z_0(1-x_0)+z_2(1-x_0) \\ &= (z_0+z_2)(1-x_0) = (1-z_1)(1-x_0) \le 0. \end{aligned}$$

Thus, also in this case $p \notin Q$.

We have proved that Q is a lattice-free triangle containing only the points p_0, p_1, and p_2 on its boundary. Furthermore, each of the three edges of Q contains exactly one of the points p_0, p_1, p_2 in its relative interior. By Proposition 5.3 II (c), Q is a maximal lattice-free triangle of type 3. This shows the first part of the lemma.

Part II. In view of Lemma 5.14 it suffices to show that

$$\begin{aligned} g(x) &:= \min\left\{ \max_{i=0,1,2} |x_iy_i + (1-x_{i+1})y_{i+1}| : y \in \mathbb{Z}^3 \setminus \{o\},\ y_0+y_1+y_2=0 \right\} \\ &= \min\{x_0,x_1,x_2\} \end{aligned}$$

under the assumption that $x_i + x_j > 1$ for all $0 \le i < j \le 2$. Taking all six choices of $y \in \{-1,0,1\}^3 \setminus \{o\}$ with $y_0+y_1+y_2=0$ we easily verify that

$$g(x) \le \min_{i=0,1,2} \max\{x_{i+1}, 1-x_i\} = \min\{x_0,x_1,x_2\},$$

where the last equality is due to assumption (a). It remains to show the converse inequality. Consider $y \in \mathbb{Z}^3 \setminus \{o\}$ with $y_0+y_1+y_2=0$. Without loss of generality we assume that $y_j \ge 0$ and $y_{j+1} \ge 0$ for some $j \in \{0,1,2\}$ (one of the two vectors, y or $-y$, must have this property). If $y_j \ge 1$ and $y_{j+1} \ge 1$, then

$$\max_{i=0,1,2} |x_iy_i + (1-x_{i+1})y_{i+1}| \ge x_j + 1 - x_{j+1} \ge x_j \ge \min\{x_0,x_1,x_2\}.$$

Otherwise, one of the y_i's is equal to zero and the other two coordinates of y are equal to k and $-k$ for some $k \in \mathbb{N}$. Then we can replace y by $\frac{1}{k}y$ with the effect that $\max\{|x_iy_i + (1-x_{i+1})y_{i+1}| : i = 0,1,2\}$ decreases. It is therefore only necessary to check $y \in \{-1,0,1\}^3 \setminus \{o\}$ with $y_0+y_1+y_2=0$ in order to compute $g(x)$. Thus, $g(x) = \min\{x_0,x_1,x_2\}$.

Part III. Without loss of generality[5] let $x_0 \le x_1 \le x_2$.

[5] More precisely, let the smallest of the x_i's be denoted by u_0, the second smallest by u_1, and the largest by u_2. Then, $u_0 \le u_1 \le u_2$. The proof ought to continue by a case distinction on u_0. However, to avoid confusion, we omit the introduction of new variables and use x_i's instead of u_i's in the remainder of the proof.

Case 1: $x_0 \le \frac{1}{2}$. Then, since (a) holds, $x_1 > \frac{1}{2}$ and $x_2 > \frac{1}{2}$ and we have

$$\begin{aligned}\frac{1}{w(Q)} &= \frac{x_0x_1x_2 + (1-x_0)(1-x_1)(1-x_2)}{x_0}\\ &\ge \frac{x_0x_1x_2 + x_0(1-x_1)(1-x_2)}{x_0}\\ &= \frac{1}{2}(2x_1-1)(2x_2-1) + \frac{1}{2} > \frac{1}{2},\end{aligned} \tag{5.14}$$

which implies that $w(Q) < 2$.

Case 2: $x_0 > \frac{1}{2}$. We use the functions $\sigma_1(x) := x_0 + x_1 + x_2$ and $\sigma_2(x) := x_0x_1 + x_0x_2 + x_1x_2$. Clearly, $2\sigma_2(x) - \sigma_1(x) = (x_0 + x_1 - 1)x_2 + (x_0 + x_2 - 1)x_1 + (x_1 + x_2 - 1)x_0 \ge (2(x_0 + x_1 + x_2) - 3)x_0 = (2\sigma_1(x) - 3)x_0$ and by this

$$\begin{aligned}x_0x_1x_2 + (1-x_0)(1-x_1)(1-x_2) &= 1 - \sigma_1(x) + \sigma_2(x)\\ &\ge 1 - \frac{1}{2}\sigma_1(x) + \left(\sigma_1(x) - \frac{3}{2}\right)x_0\\ &= 1 - \frac{3}{2}x_0 + \left(x_0 - \frac{1}{2}\right)\sigma_1(x) \qquad (5.15)\\ &\ge 1 - \frac{3}{2}x_0 + \left(x_0 - \frac{1}{2}\right)3x_0\\ &= 1 - 3x_0 + 3x_0^2.\end{aligned}$$

Thus, we obtain

$$\frac{1}{w(Q)} = \frac{x_0x_1x_2 + (1-x_0)(1-x_1)(1-x_2)}{x_0} \ge 3x_0 - 3 + \frac{1}{x_0}.$$

Consequently, we can apply elementary calculus to compute the minimum of $3x_0 - 3 + \frac{1}{x_0}$ for x_0 satisfying $\frac{1}{2} < x_0 < 1$. The minimum is attained by $x_0 = \frac{1}{\sqrt{3}}$. Thus,

$$\frac{1}{w(Q)} \ge 3x_0 - 3 + \frac{1}{x_0} \ge 2\sqrt{3} - 3,$$

which implies $w(Q) \le 1 + \frac{2}{\sqrt{3}}$ and shows that the equality $w(Q) = 1 + \frac{2}{\sqrt{3}}$ is attained if and only if $x_0 = x_1 = x_2 = \frac{1}{\sqrt{3}}$. □

Remark 5.16. *In Parts II and III of Lemma 5.15 we assumed that (a) is satisfied. Of course, analogous statements could be derived if (b) were assumed. In this case, we have*

$$w(Q) = \frac{\min\{1-x_0, 1-x_1, 1-x_2\}}{x_0x_1x_2 + (1-x_0)(1-x_1)(1-x_2)}$$

and $w(Q) \le 1 + \frac{2}{\sqrt{3}}$ *with equality if and only if* $x_0 = x_1 = x_2 = 1 - \frac{1}{\sqrt{3}}$.

Remark 5.17. *From the proof of Part II of Lemma 5.15 it can be seen that the lattice width of a maximal lattice-free triangle of type 3 in* $\mathcal{K}^2$ *which contains the integer points* p_0, p_1, *and* p_2 *on its boundary is attained by (at least) one of the vectors* $(p_1 - p_0)^\perp$, $(p_2 - p_0)^\perp$, *or* $(p_2 - p_1)^\perp$, *where* $x^\perp := (-x_2, x_1)$ *denotes the orthogonal vector of* $x \in \mathbb{R}^2$. *Indeed, the vectors* $\pm(p_1 - p_0)^\perp$, $\pm(p_2 - p_0)^\perp$, *and* $\pm(p_2 - p_1)^\perp$ *correspond to the six choices of* $y \in \{-1, 0, 1\}^3 \setminus \{o\}$ *with* $y_0 + y_1 + y_2 = 0$. *In particular, if* $p_0 = (0, 0)$, $p_1 = (1, 0)$, *and* $p_2 = (0, 1)$, *then the lattice width is attained by (at least) one of the vectors* e_1, e_2, *or* $e_1 + e_2$.

Remark 5.18. *Lemma 5.15 I characterizes maximal lattice-free triangles of type 3 in terms of the barycentric coordinates of the three integer points on their boundary with respect to their vertices. Such an "if and only if" relation does not only exist for triangles of type 3, but for all maximal lattice-free triangles in* $\mathcal{K}^2$ *(see Proposition 5.3 II). Let us use the notation of Lemma 5.15 to explain this in more detail. Define* $K_{i,j} := \operatorname{sgn}(x_i + x_j - 1)$ *for* $0 \le i < j \le 2$ *and let Q be a triangle as in Lemma 5.15. From Lemma 5.15 I, it follows that Q is of type 3 if and only if* $K_{0,1} = K_{1,2} = K_{0,2} = 1$ *or* $K_{0,1} = K_{1,2} = K_{0,2} = -1$. *Moreover, it is straightforward to show that Q is of type 1 if and only if* $K_{0,1} = K_{1,2} = K_{0,2} = 0$ *(which implies* $x_0 = x_1 = x_2 = \frac{1}{2}$*), and that Q is not lattice-free if and only if there exist two of the* $K_{i,j}$*'s such that one of them is* 1 *and the other is* -1 *(and the third being arbitrary). Then the implication is that Q is of type 2 if and only if (i) two of the* $K_{i,j}$*'s are* 1 *(or* -1*), and the third is* 0 *(type 2a), or (ii) one of the* $K_{i,j}$*'s is* 1 *(or* -1*), and the other two are* 0 *(type 2b).*

This implies a classification of all the maximal lattice-free triangles in $\mathcal{K}^2$ *in terms of the* $K_{i,j}$*'s as described above. The advantage of the proposed classification is that it can be extended in a straightforward way to a classification of simplices in higher dimensions. Furthermore, it provides an alternative way to encode maximal lattice-free triangles in* $\mathcal{K}^2$ *based on barycentric coordinates (usually the vertex description and/or the facet description is used). This may help to find new structural properties of maximal lattice-free triangles in* $\mathcal{K}^2$ *and the cuts associated with them.*

The following two lemmas prepare the proof of Theorem 5.6. Parts of the proof of Lemma 5.19 are borrowed from [Hur90]. Nevertheless we need these parts for subsequent arguments.

Lemma 5.19. *Let* $K \in \mathcal{K}^2$ *be maximal lattice-free with* $[0, 1]^2 \subseteq K$ *and let* $w := w(K)$ *and* $A := A(K)$. *Then* $w \le 2$ *and either it holds* $w = 1$ *and* $A = \infty$ *(i.e. K is a split), or* $w > 1$ *and* $A \le \frac{w^2}{2(w-1)}$ *with equality* $A = \frac{w^2}{2(w-1)}$ *characterized by Part II of Theorem 5.6.*

Proof. From $[0,1]^2 \subseteq K$, it follows that $w \geq 1$. If $w = 1$, then, by the maximality of K, K is a split and $A = \infty$. Thus, we assume $w > 1$ and therefore K is a triangle of type 1, a triangle of type 2, or a quadrilateral. We first consider the case that K is a quadrilateral. The case where K is a triangle can be viewed as a degenerate version of a quadrilateral where one vertex becomes a convex combination of its two neighbor vertices. Then the arguments of the proof for the quadrilateral presented below just need to be adapted slightly. We explain this in further detail after we have shown the assertion for quadrilaterals.

Assume that K is a quadrilateral. By a_1, a_2, a_3, a_4 we denote the consecutive vertices of $[0,1]^2$. Let q_1, q_2, q_3, q_4 be consecutive vertices of K such that the point q_i' of $[0,1]^2$ closest to q_i lies in $[a_i, a_{i+1}]$. The distance from q_i to q_i' will be denoted by h_i and the distance from a_i to q_i' by t_i (see Fig. 5.7).

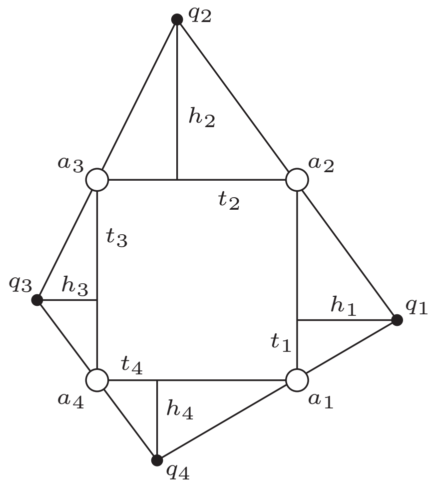


Figure 5.7: Maximal lattice-free quadrilateral in the proof of Lemma 5.19.

In particular, we have $t_i > 0$ for all $i = 1,2,3,4$. Taking into account the relations $h_i h_{i-1} = t_i(1 - t_{i-1})$ for $i = 1,2,3,4$, we obtain

$$\begin{aligned}
&1 - (h_1 + h_3)(h_2 + h_4) \\
&= 1 - h_2h_1 - h_1h_4 - h_4h_3 - h_3h_2 \\
&= 1 - t_2(1 - t_1) - t_1(1 - t_4) - t_4(1 - t_3) - t_3(1 - t_2) \\
&= (1 - t_1 - t_3)(1 - t_2 - t_4) \qquad (5.16) \\
&= \frac{\big(t_2(1 - t_1)t_4(1 - t_3) - t_1t_2t_3t_4\big)\big(t_1(1 - t_4)t_3(1 - t_2) - t_1t_2t_3t_4\big)}{t_1t_2t_3t_4} \\
&= \frac{(h_2h_1h_4h_3 - t_1t_2t_3t_4)(h_1h_4h_3h_2 - t_1t_2t_3t_4)}{t_1t_2t_3t_4}
\end{aligned}$$

$$= \frac{(h_1h_2h_3h_4 - t_1t_2t_3t_4)^2}{t_1t_2t_3t_4} \geq 0.$$

Without loss of generality we assume that $h_1 + h_3 \leq h_2 + h_4$. From the relations (5.16), it follows that $1 - (h_1 + h_3)(h_2 + h_4) \geq 0$ and therefore $h_1 + h_3 \leq 1$. Thus, the width of K along the vector e_1 is $h_1 + h_3 + 1 \leq 2$. For all vectors $u \in \mathbb{Z}^2 \setminus \{o, \pm e_1, \pm e_2\}$ we easily get $w(K,u) \geq w([0,1]^2, u) \geq 2$. Hence $w = h_1 + h_3 + 1 \leq 2$. Furthermore,

$$\begin{aligned} A &= 1 + \frac{1}{2}(h_1 + h_2 + h_3 + h_4) = 1 + \frac{1}{2}(w - 1 + h_2 + h_4) \\ &\leq 1 + \frac{1}{2}\left(w - 1 + \frac{1}{h_1 + h_3}\right) = 1 + \frac{1}{2}\left(w - 1 + \frac{1}{w-1}\right) = \frac{w^2}{2(w-1)}. \end{aligned}$$

If the equality $A = \frac{w^2}{2(w-1)}$ is attained, then $(h_1 + h_3)(h_2 + h_4) = 1 \Leftrightarrow h_2 + h_4 + 1 = \frac{w}{w-1}$. By (5.16), it follows $(1 - t_1 - t_3)(1 - t_2 - t_4) = 0$, which implies that $1 - t_1 - t_3 = 0$ or $1 - t_2 - t_4 = 0$. Using (5.16), we see that the equalities $1 - t_1 - t_3 = 0$ and $1 - t_2 - t_4 = 0$ imply one another. Indeed, $1 - t_1 - t_3 = \frac{1}{t_2t_4}(h_1h_2h_3h_4 - t_1t_2t_3t_4)$ and $1 - t_2 - t_4 = \frac{1}{t_1t_3}(h_1h_2h_3h_4 - t_1t_2t_3t_4)$. Hence, $1 - t_1 - t_3 = 0 \Leftrightarrow h_1h_2h_3h_4 - t_1t_2t_3t_4 = 0 \Leftrightarrow 1 - t_2 - t_4 = 0$. So we have $1 - t_1 - t_3 = 1 - t_2 - t_4 = 0$. This implies that $I_1 := [q_1, q_3]$ is a translate of $[o, we_1]$, and $I_2 := [q_2, q_4]$ is a translate of $[o, \frac{w}{w-1}e_2]$ such that $K = \operatorname{conv}(I_1 \cup I_2)$ with $I_1 \cap I_2 \neq \emptyset$, proving Part II of Theorem 5.6.

Now assume that K is a triangle of type 1 or type 2. Without loss of generality we assume that q_4 becomes a convex combination of q_1 and q_3, i.e. q_4 coincides with q_4'. This implies that $h_4 = 0$, $t_3 = 1$, and $t_1 = 0$ (see Fig. 5.8).

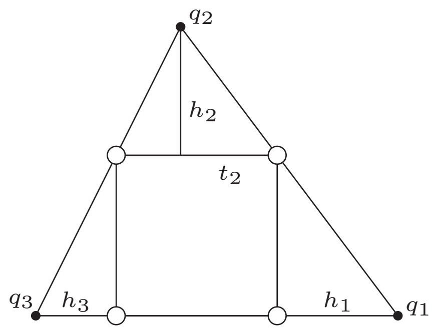


Figure 5.8: Maximal lattice-free triangle in the proof of Lemma 5.19.

Furthermore, $h_2 > 0$ and $h_1 + h_3 > 0$. In particular, we have $h_3h_2 = 1 - t_2$ and $h_2h_1 = t_2$. Adding these equations, we obtain $h_2(h_1 + h_3) = 1$. Without loss of generality we assume that $h_1 + h_3 \leq h_2$. From $h_2(h_1 + h_3) = 1$, it follows that $h_1 + h_3 \leq 1$. Thus, the width of K along the vector e_1 is $h_1 + h_3 + 1 \leq 2$. As above, for all vectors $u \in \mathbb{Z}^2 \setminus \{o, \pm e_1, \pm e_2\}$ it holds $w(K, u) \geq w([0,1]^2, u) \geq 2$. Hence $w = h_1 + h_3 + 1 \leq 2$. Moreover,

$$A = 1 + \frac{1}{2}(h_1 + h_2 + h_3) = 1 + \frac{1}{2}\left(h_1 + h_3 + \frac{1}{h_1 + h_3}\right)$$
$$= 1 + \frac{1}{2}\left(w - 1 + \frac{1}{w-1}\right) = \frac{w^2}{2(w-1)}.$$

Equality $A = \frac{w^2}{2(w-1)}$ is attained by all triangles of types 1 and 2. From $h_1 + h_3 + 1 = w$ and $h_2 + 1 = \frac{w}{w-1}$, Part II of Theorem 5.6 follows immediately. □

Remark 5.20. *Lemma 5.19 implies, in particular, that the lattice width of quadrilaterals and triangles of type 2 is at most two. On the other hand it is obvious that the lattice width of a split is equal to one and that the lattice width of a type 1 triangle is equal to two. Thus, triangles of type 3 are the only maximal lattice-free sets in $\mathcal{K}^2$ which admit a lattice width larger than two.*

Remark 5.21. *From the proof of Lemma 5.19 it can be seen that the lattice width of a maximal lattice-free quadrilateral (resp. triangle of type 2) in $\mathcal{K}^2$ which contains $[0,1]^2$ is attained by (at least) one of the vectors e_1 or e_2. More generally, let $v \in \mathbb{Z}^2$ and let $b_1, b_2 \in \mathbb{Z}^2$ form a basis of $\mathbb{Z}^2$. Then the lattice width of any maximal lattice-free quadrilateral (resp. triangle of type 2) in $\mathcal{K}^2$ which contains $v + \mathrm{conv}(\{o, b_1, b_2, b_1 + b_2\})$ is attained by (at least) one of the vectors $b_1^\perp$ or $b_2^\perp$, where $x^\perp := (-x_2, x_1)$ denotes the orthogonal vector of $x \in \mathbb{R}^2$.*

Lemma 5.22. *Let $K \in \mathcal{K}^2$ be a maximal lattice-free triangle with $w := w(K)$ and $A := A(K)$. Then $w > 1$ and (5.3) resp. (5.4) holds true. The equality case in both inequalities is characterized by Part II resp. Part III of Theorem 5.6.*

Proof. If K is a triangle of type 1 or type 2, then clearly $w > 1$. Consider a triangle K of type 3 and let $P = \mathrm{conv}(\{p_0, p_1, p_2\})$ as in Lemma 5.15, such that the relative interior of each edge of K contains a point from $\{p_0, p_1, p_2\}$. Then, for every $u \in \mathbb{Z}^2 \setminus \{o\}$, the width of K along u is strictly larger than the width of P along u, and furthermore, the lattice width of P is equal to 1. It follows that $w > 1$.

If K contains more than three integer points, then K is of type 1 or type 2. Thus, there is a unimodular transformation that maps K to a maximal lattice-free triangle $K' \in \mathcal{K}^2$ such that K' contains the square $[0,1]^2$. K' satisfies the assumptions of Lemma 5.19 and therefore our assertion follows directly from Lemma 5.19.

Now assume that every edge of K contains precisely one integer point, i.e. K is a triangle of type 3. We further define $K := Q = \mathrm{conv}(\{q_0, q_1, q_2\})$ with q_0, q_1, q_2 and Q given as in Lemma 5.15, and we also borrow the other notations of Lemma 5.15. Without loss of generality we assume that $x_0 \le x_1 \le x_2$ and $x_i + x_j > 1$ for all $0 \le i < j \le 2$. Let $f(x) := x_0x_1x_2 + (1-x_0)(1-x_1)(1-x_2)$. We recall that $w = w(Q) \le 1 + \frac{2}{\sqrt{3}}$ by Lemma 5.15 III.

Case 1: $x_0 \ge \frac{1}{2}$. If $1 < w < 2$, then in view of Lemmas 5.14 and 5.15 II we obtain

$$A = \frac{1}{2f(x)} = \frac{x_0}{f(x)} \cdot \frac{1}{2x_0} = \frac{w}{2x_0} \le w < \frac{w^2}{2(w-1)}.$$

Now assume $w \ge 2$. Taking into account Lemmas 5.14 and 5.15 II we infer

$$\begin{aligned} A &\le \max\left\{\frac{1}{2f(x)} : \frac{1}{2} \le x_0 \le x_1 \le x_2 < 1,\ x_0 = wf(x)\right\} \\ &= \frac{1}{2}\left(\min\left\{f(x) : \frac{1}{2} \le x_0 \le x_1 \le x_2 < 1,\ x_0 = wf(x)\right\}\right)^{-1}. \end{aligned}$$

Furthermore, using (5.15), we obtain

$$\begin{aligned} &\min\left\{f(x) : \frac{1}{2} \le x_0 \le x_1 \le x_2 < 1,\ x_0 = wf(x)\right\} \\ &= \frac{1}{w}\min\left\{x_0 : \frac{1}{2} \le x_0 \le x_1 \le x_2 < 1,\ x_0 = wf(x)\right\} \\ &\ge \frac{1}{w}\min\left\{x_0 : \frac{1}{2} \le x_0 < 1,\ x_0 \ge w(1 - 3x_0 + 3x_0^2)\right\}. \end{aligned} \tag{5.17}$$

We remark that (5.15) were derived for the case that $x_0 > \frac{1}{2}$, but it is easy to see that it holds true for $x_0 \ge \frac{1}{2}$ as well. The minimum in (5.17) is attained by x_0 equal to the smaller root of the equation $t = w(1 - 3t + 3t^2)$ since this root, which is equal to

$$\frac{3w + 1 - \sqrt{1 + 6w - 3w^2}}{6w},$$

lies in the interval $[\frac{1}{2}, 1)$. Thus, we obtain

$$\begin{aligned} A &\le \frac{1}{2}\left(\min\left\{f(x) : \frac{1}{2} \le x_0 \le x_1 \le x_2 < 1,\ x_0 = wf(x)\right\}\right)^{-1} \\ &\le \frac{w}{2} \cdot \frac{6w}{3w + 1 - \sqrt{1 + 6w - 3w^2}} = \frac{3w^2}{3w + 1 - \sqrt{1 + 6w - 3w^2}} \end{aligned}$$

which is the bound stated in (5.4). The characterization of the equality case follows directly by analyzing the equality case in the above estimates: equality holds if and only if $f(x) = 1 - 3x_0 + 3x_0^2$ and from (5.15) we easily see that this is true if and only if $x_0 = x_1 = x_2$. Hence, equality in (5.4) is attained if and only if

$$x_0 = x_1 = x_2 = \frac{3w + 1 - \sqrt{1 + 6w - 3w^2}}{6w},$$

showing Part III of Theorem 5.6.

Case 2: $x_0 < \frac{1}{2}$. This implies $x_1 > \frac{1}{2}$ and $x_2 > \frac{1}{2}$. Moreover, using the same arguments as in the proof of Case 1 of Part III of Lemma 5.15 (see p. 53), it follows that $w < 2$. In addition,

$$\begin{aligned}
I := \inf \Big\{ f(x) : 0 < x_0 < \frac{1}{2}, \frac{1}{2} < x_1 \le x_2 < 1, \\
x_0 + x_1 > 1,\, x_0 + x_2 > 1,\, x_0 = wf(x) \Big\} \\
= \frac{1}{w} \inf \Big\{ x_0 : 0 < x_0 < \frac{1}{2}, \frac{1}{2} < x_1 \le x_2 < 1, \\
x_0 + x_1 > 1,\, x_0 + x_2 > 1,\, x_0 = wf(x) \Big\} \\
\ge \frac{1}{w} \inf \Big\{ x_0 : 0 < x_0 < \frac{1}{2}, \frac{1}{2} < x_1 \le x_2 < 1, \\
x_0 + x_1 > 1,\, x_0 + x_2 > 1,\, x_0 \ge wf(x) \Big\}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
f(x) &= (x_0 + x_1 - 1)x_2 + (1 - x_0)(1 - x_1) \\
&\ge (x_0 + x_1 - 1)x_1 + (1 - x_0)(1 - x_1) \\
&= (1 - x_0)x_0 + (x_0 + x_1 - 1)^2 \\
&> (1 - x_0)x_0. \qquad (5.18)
\end{aligned}$$

Hence,

$$\begin{aligned}
I &\ge \frac{1}{w} \inf \left\{ x_0 : 0 < x_0 < \frac{1}{2},\, x_0 > wx_0(1 - x_0) \right\} \\
&= \frac{1}{w} \inf \left\{ x_0 : 1 - \frac{1}{w} < x_0 < \frac{1}{2} \right\} \qquad (5.19) \\
&= \frac{1}{w} \left(1 - \frac{1}{w} \right).
\end{aligned}$$

As in the previous case, it follows that

$$A = \frac{1}{2f(x)} \le \frac{1}{2I} \le \frac{w^2}{2(w-1)}. \tag{5.20}$$

We note that (5.20) can never hold with equality, for if equality would hold, then $f(x) = I$ and $I = \frac{w-1}{w^2} = \frac{1}{w}(1 - \frac{1}{w})$. By (5.19), the latter implies that $x_0 = 1 - \frac{1}{w}$ which, in turn, implies that $f(x) = I = (1 - x_0)x_0$. However, this is a contradiction to (5.18). $\square$

We notice that Theorem 5.4 is a consequence of Propositions 5.1 and 5.3, and Lemmas 5.19 and 5.22. In fact, the main steps of the proof in [Hur90] were incorporated in Lemmas 5.15, 5.19 and 5.22. Let us now prove Theorem 5.6 and Corollary 5.11.

Proof of Theorem 5.6. Let us first show (5.2)–(5.4). In view of Lemmas 5.19 and 5.22 the upper bounds for the area in (5.2)–(5.4) hold for maximal lattice-free sets in $\mathcal{K}^2$. Let $K \in \mathcal{K}^2$ be an arbitrary lattice-free set. By Proposition 5.1, there exists a maximal lattice-free set $H \in \mathcal{K}^2$ such that $K \subseteq H$. Obviously, we have $w(K) \le w(H)$ and $A(K) \le A(H)$. For $0 < w \le 1+2\cdot(\sqrt{3})^{-1}$ we define $F(w)$ to be the upper bound in (5.2)–(5.4), i.e.

$$F(w) := \begin{cases} \infty & \text{if } 0 < w \le 1, \\ \frac{w^2}{2(w-1)} & \text{if } 1 < w \le 2, \\ \frac{3w^2}{3w+1-\sqrt{1+6w-3w^2}} & \text{if } 2 < w \le 1 + \frac{2}{\sqrt{3}}. \end{cases}$$

By definition, F is monotonically non-increasing. Thus, it follows $A(K) \le A(H) \le F(w(H)) \le F(w(K))$. Therefore, the upper bounds for the area in (5.2)–(5.4) hold for arbitrary lattice-free sets in $\mathcal{K}^2$. The equality $A(K) = F(w(K))$ implies $K = H$. Hence, the characterizations of the equality cases for (5.2)–(5.4) follow from the characterizations of the equality cases in Lemmas 5.19 and 5.22. This shows Parts I–III.

The lower bound for the area in (5.5) and Part IV follow directly from Theorem 5.8. $\square$

Proof of Corollary 5.11. We use the relation $\mu_1(K)w(K) = 1$ and the fact that an appropriate translate of $\mu_2(K) \cdot K$ is lattice-free. Then we apply the bounds in Theorem 5.6 to $\mu_2(K) \cdot K$ and express the lattice width of $\mu_2(K) \cdot K$ as $\mu_2(K) \cdot (\mu_1(K))^{-1}$.

Let us illustrate this for the bound in (5.3). Since $\mu_2(K) \cdot K$ has a lattice-free translate, we obtain

$$\begin{aligned}(\mu_2(K))^2 A(K) &= A(\mu_2(K)\cdot K)\\ &\le \frac{(w(\mu_2(K)\cdot K))^2}{2(w(\mu_2(K)\cdot K)-1)} = \frac{(\mu_2(K))^2(w(K))^2}{2(\mu_2(K)w(K)-1)}.\end{aligned}$$

Since $\mu_2(K) > 0$ we can divide both sides by $(\mu_2(K))^2$ and substitute $w(K)$ by $\frac{1}{\mu_1(K)}$. Finally, we end up with

$$A(K) \le \frac{1}{2\mu_1(K)(\mu_2(K)-\mu_1(K))}.$$

The bound in (5.3) is valid for $1 < w(\mu_2(K)\cdot K) \le 2 \Leftrightarrow 1 < \mu_2(K)w(K) \le 2 \Leftrightarrow 1 < \mu_2(K)\frac{1}{\mu_1(K)} \le 2 \Leftrightarrow \mu_1(K) < \mu_2(K) \le 2\mu_1(K)$.

Analogously, the above procedure can be applied to the bounds in (5.2), (5.4) and (5.5). The tightness of the bounds in Corollary 5.11 follows from the characterizations of the equality cases in Theorem 5.6. □

5.5 Proofs for centrally symmetric sets

In this section, we prove Theorems 5.9 and 5.10 and Corollary 5.12. The upper bound in (5.6) states that $w(K) \le 2$ whenever $K \in \mathcal{K}^2$ is lattice-free and centrally symmetric. For maximal lattice-free sets from $\mathcal{K}^2$ this is a consequence of Remark 5.20 since only triangles of type 3 admit a lattice width larger than 2 (obviously, a triangle is not centrally symmetric). So it remains to verify it for centrally symmetric sets from $\mathcal{K}^2$ which are lattice-free, but not maximal. The fact that $w(K) \le 2$ has already been stated in [KL88, Theorem 2.13], but the proof of Kannan and Lovász does not seem to show this result (see p. 45). Therefore, we first show (5.6) by proving Theorem 5.10. Afterwards, we use (5.6) to show (5.7)–(5.9).

Proof of Theorem 5.10. Let $K \in \mathcal{K}^2$ be centrally symmetric. Our aim is to show that $\mu_2(K) \le 2\mu_1(K)$. Since both, $\mu_1(K)$ and $\mu_2(K)$, are homogeneous of degree -1 (i.e. $\mu_i(tK) = t^{-1}\mu_i(K)$ for $i = 1, 2$ and every $t > 0$), it suffices to consider the case $\mu_2(K) = 1$. Using $\mu_1(K)w(K) = 1$ it remains to prove that $w(K) \le 2$. For convenience we define $w := w(K)$.

Since $\mu_2(K) = 1$, K has a lattice-free translate. Without loss of generality we assume that K itself is lattice-free. By Proposition 5.1, there exists a maximal lattice-free $H \in \mathcal{K}^2$ with $K \subseteq H$. In particular, we have $w \le w(H)$ and $A(K) \le A(H)$. From Proposition 5.3, it follows that H is either a split or a triangle or a quadrilateral. If H is a split or a quadrilateral, then in view of Remark 5.20 one has $w \le w(H) \le 2$. Thus, let H be a triangle.

Assume, by contradiction, that $w > 2$. Then $w(H) > 2$ and by Remark 5.7 it follows $A(H) \le 2$. Let c be the center of symmetry of K. We consider the centrally symmetric set $L := H \cap (2c - H)$, where $2c - H$ is the reflection of H with respect to c. From $K \subseteq H$ and $K = 2c - K \subseteq 2c - H$, it follows that $K \subseteq H \cap (2c - H) = L$. Since H is a simplex and L is centrally symmetric with $L \subseteq H$, we have $A(L) \le \frac{2}{3} A(H)$ (see, for instance, [FR50, Satz 5]). Hence, $A(K) \le A(L) \le \frac{2}{3} A(H) \le \frac{2}{3} \cdot 2 = \frac{4}{3}$. On the other hand, by (5.5), we have $A(K) \ge \frac{3}{8} w^2 > \frac{3}{8} 2^2 = \frac{3}{2}$, which contradicts $A(K) \le \frac{4}{3}$. $\square$

The following lemma prepares the proof of Theorem 5.9. We recommend to look up the notation and statements presented in Section 5.1, in particular the part after Theorem 5.4, since they are frequently used in the proof of Lemma 5.23.

Lemma 5.23. *Let* $0 \le \alpha < 1$ *and let* $K_\alpha := \operatorname{conv}(\{\pm(1,\alpha), \pm(0,1)\})$. *Then*

$$\mu_2(K_\alpha) = \frac{1}{2} \max\{1 + \alpha, 2 - \alpha\}.$$

Proof. The unimodular transformation given by the matrix $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ maps K_α to $K_{1-\alpha}$. Thus, since the second covering minimum is invariant with respect to unimodular transformations, it suffices to consider the case $0 \le \alpha \le \frac{1}{2}$. For the sake of brevity we write $K := K_\alpha$. Direct computations show that $K^* = \operatorname{conv}(\{\pm(1-\alpha, 1), \pm(1+\alpha, -1)\}) = \operatorname{conv}(\{\pm(-\alpha, 1)\}) + \operatorname{conv}(\{\pm(1, 0)\})$. Hence, for every $u = (u_1, u_2) \in \mathbb{R}^2$, we have $h(K^*, u) = |u_2 - \alpha u_1| + |u_1|$. Furthermore,

$$\begin{aligned}
\mu_2(K) &= \min\left\{t \ge 0 : tK + \mathbb{Z}^2 = \mathbb{R}^2\right\} \\
&= \min\left\{t \ge 0 : \forall x \in \mathbb{R}^2\ \exists z \in \mathbb{Z}^2 \text{ such that } x - z \in tK\right\} \\
&= \min\left\{t \ge 0 : \forall x \in \mathbb{R}^2\ \exists z \in \mathbb{Z}^2 \text{ such that } \|x - z\|_K \le t\right\} \\
&= \max_{x \in \mathbb{R}^2} \min_{z \in \mathbb{Z}^2} \|x - z\|_K \\
&= \max_{x \in \mathbb{R}^2} \min_{z \in \mathbb{Z}^2} h(K^*, x - z). \qquad (5.21)
\end{aligned}$$

For $s, t \in \mathbb{R}$, we define by $d(s, 0) := |s - \lfloor s \rceil\,|$ the distance from s to its nearest integer, and by $d(s, t) := d(s - t, 0)$ the distance between s and t modulo 1.

Let us now analyze the minimization problem in (5.21) for a given $x = (x_1, x_2) \in \mathbb{R}^2$. We consider

$$h(K^*, x - z) = |x_2 - z_2 - \alpha(x_1 - z_1)| + |x_1 - z_1| \qquad (5.22)$$

with $z = (z_1, z_2)$ varying in $\mathbb{Z}^2$. Our aim is to find an integer vector z which minimizes (5.22). If we choose $z_1 = \lfloor x_1 \rceil$ and $z_2 = \lfloor x_2 - \alpha(x_1 - z_1) \rceil$, we see that $h(K^*, x - z) \le \frac{1}{2} + \frac{1}{2} = 1$. Hence, we found a $z \in \mathbb{Z}^2$ such that $h(K^*, x - z)$ is at most 1. Furthermore, for any $z \in \mathbb{Z}^2$ such that $z_1 \notin \{\lfloor x_1 \rfloor, \lceil x_1 \rceil\}$, we obtain $h(K^*, x - z) \ge |x_1 - z_1| \ge 1$. Thus, since we want to minimize (5.22), we can assume that $z_1 \in \{\lfloor x_1 \rfloor, \lceil x_1 \rceil\}$.

If $x_1 \in \mathbb{Z}$ we choose $z_1 = x_1$ and $z_2 = \lfloor x_2 \rceil$ and obtain $h(K^*, x - z) \le \frac{1}{2}$. Obviously, this is the best choice for $z \in \mathbb{Z}^2$ when $x_1 \in \mathbb{Z}$ is assumed.

Let us now assume that $x_1 \in \mathbb{R} \setminus \mathbb{Z}$. We introduce $\beta := x_1 - \lfloor x_1 \rfloor$ satisfying $0 < \beta < 1$. Since $z_1 \in \{\lfloor x_1 \rfloor, \lceil x_1 \rceil\}$, by assumption, we have

$$
\begin{aligned}
&\min_{z \in \mathbb{Z}^2} h(K^*, x - z) \\
&= \min \Big\{ \min_{z_2 \in \mathbb{Z}} \{|x_2 - z_2 - \alpha(x_1 - \lfloor x_1 \rfloor)| + |x_1 - \lfloor x_1 \rfloor|\}, \qquad (5.23) \\
&\qquad \min_{z_2 \in \mathbb{Z}} \{|x_2 - z_2 - \alpha(x_1 - \lceil x_1 \rceil)| + |x_1 - \lceil x_1 \rceil|\} \Big\} \\
&= \min \Big\{ \min_{z_2 \in \mathbb{Z}} \{|x_2 - z_2 - \alpha\beta| + \beta\}, \\
&\qquad \min_{z_2 \in \mathbb{Z}} \{|x_2 - z_2 + \alpha(1 - \beta)| + 1 - \beta\} \Big\} \\
&= \min \{ |x_2 - \alpha\beta - \lfloor x_2 - \alpha\beta \rceil| + \beta, \\
&\qquad |x_2 + \alpha(1 - \beta) - \lfloor x_2 + \alpha(1 - \beta) \rceil| + 1 - \beta \} \\
&= \min \{ d(x_2 - \alpha\beta, 0) + \beta,\ d(x_2 + \alpha(1 - \beta), 0) + 1 - \beta \}.
\end{aligned}
$$

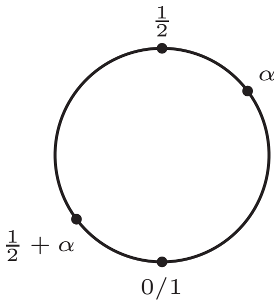
Thus, in both cases, $x_1 \in \mathbb{Z}$ and $x_1 \in \mathbb{R} \setminus \mathbb{Z}$, we computed the minimum of $h(K^*, x - z)$ for z varying in $\mathbb{Z}^2$ and given $x \in \mathbb{R}^2$. In order to compute the value in (5.21) we now have to maximize the minima for x varying in $\mathbb{R}^2$. Hence, the value in (5.21) is the maximum of $\frac{1}{2}$ and the value

$$
\begin{aligned}
&\max_{\substack{x \in \mathbb{R}^2 \\ x_1 \notin \mathbb{Z}}} \min_{z \in \mathbb{Z}^2} h(K^*, x - z) \\
&= \max_{\substack{0 < \beta < 1 \\ x_2 \in \mathbb{R}}} \min \{ d(x_2 - \alpha\beta, 0) + \beta,\ d(x_2 + \alpha(1 - \beta), 0) + 1 - \beta \} \\
&= \max_{\substack{0 < \beta < 1 \\ x_2 \in \mathbb{R}}} \min \{ d(x_2 + \alpha(1 - \beta), \alpha) + \beta,\ d(x_2 + \alpha(1 - \beta), 0) + 1 - \beta \} \\
&= \max_{\substack{0 < \beta < 1 \\ y \in \mathbb{R}}} \min \{ d(y, \alpha) + \beta,\ d(y, 0) + 1 - \beta \} \\
&\le \max_{\beta, y \in \mathbb{R}} \min \{ d(y, \alpha) + \beta,\ d(y, 0) + 1 - \beta \}.
\end{aligned}
$$

If $d(y, \alpha) + \beta$ and $d(y, 0) + 1 - \beta$ differ, then slightly perturbing β makes the minimum of these two values become larger. Hence, the latter maximum is attained by the β for which $d(y, \alpha) + \beta$ and $d(y, 0) + 1 - \beta$ coincide. In this case we have $\beta = \frac{1}{2}(d(y, 0) - d(y, \alpha) + 1)$. Thus

$$
\begin{aligned}
&\max_{\beta, y\in\mathbb{R}} \min\left\{d(y,\alpha)+\beta,\ d(y,0)+1-\beta\right\}\\
&= \max_{y\in\mathbb{R}} \frac{1}{2}\big(d(y,0)+d(y,\alpha)+1\big)\\
&= \frac{1}{2}\Big(1+\max_{y\in\mathbb{R}}\left\{d(y,0)+d(y,\alpha)\right\}\Big)\\
&= \frac{1}{2}(2-\alpha),
\end{aligned}
$$

where the last equality follows from the fact that $d(y,0)+d(y,\alpha)$ is maximized, for instance, at $y=\frac{1}{2}$. To see this, one can visualize $d(y,0)+d(y,\alpha)$ with the help of a circle (see Fig. 5.9).



(a) Circle with the regions for $\bar{y}$.

$$
\begin{matrix} d(y,0)\\ +\\ d(y,\alpha)\end{matrix} = \begin{cases} \alpha & \text{if } 0 \le \bar{y} < \alpha,\\ 2\bar{y}-\alpha & \text{if } \alpha \le \bar{y} < \frac{1}{2},\\ 1-\alpha & \text{if } \frac{1}{2} \le \bar{y} < \frac{1}{2}+\alpha,\\ 2(1-\bar{y})+\alpha & \text{if } \frac{1}{2}+\alpha \le \bar{y} < 1.\end{cases}
$$

(b) Formula for $d(y,0)+d(y,\alpha)$ with $\bar{y} := y \pmod 1$.

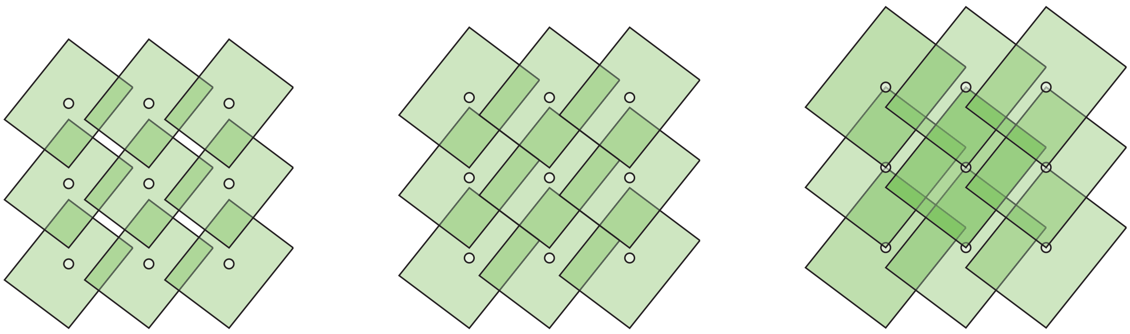
Figure 5.9: Computation of $d(y,0)+d(y,\alpha)$.

Since $\frac{1}{2}(2-\alpha)$ is at least $\frac{1}{2}$ we have shown that $\mu_2(K) \le \frac{1}{2}(2-\alpha)$. It remains to show that this value can be attained with equality. For that, consider the point $\bar{x} = (\frac{1}{2}, \frac{1}{2})$. By (5.21), we have

$$
\begin{aligned}
\mu_2(K) &\ge \min_{z\in\mathbb{Z}^2} h(K^*, \bar{x}-z)\\
&= \min\left\{ \min_{z_2\in\mathbb{Z}}\left\{\left|\frac{1-\alpha}{2}-z_2\right|+\frac{1}{2}\right\},\ \min_{z_2\in\mathbb{Z}}\left\{\left|\frac{1+\alpha}{2}-z_2\right|+\frac{1}{2}\right\}\right\}\\
&= \min\left\{\left|\frac{1-\alpha}{2}-0\right|+\frac{1}{2},\ \left|\frac{1+\alpha}{2}-1\right|+\frac{1}{2}\right\}\\
&= \min\left\{\frac{2-\alpha}{2},\ \frac{2-\alpha}{2}\right\} = \frac{1}{2}(2-\alpha),
\end{aligned}
$$

where the second equality follows from (5.23). Thus, $\mu_2(K) = \frac{1}{2}(2-\alpha)$. $\square$

Example 5.24. Let $\alpha = \frac{1}{4}$ and consider $K_{\frac{1}{4}} = \operatorname{conv}(\{\pm(1, \frac{1}{4}), \pm(0,1)\})$. By Lemma 5.23, we have $\mu_2(K_{\frac{1}{4}}) = \frac{7}{8}$. Thus, the sets $tK_{\frac{1}{4}} + z$ with $z \in \mathbb{Z}^2$ cover $\mathbb{R}^2$ for every $t \geq \frac{7}{8}$. On the other hand, for every positive $t < \frac{7}{8}$ there are points in the plane which are not covered by the sets $tK_{\frac{1}{4}} + z$ with $z \in \mathbb{Z}^2$. In particular, when we start with a small t close to zero and then incrementally increase t up to $\frac{7}{8}$, then it follows from the last part of the proof of Lemma 5.23 that the points $(\frac{1}{2}, \frac{1}{2}) + z$ with $z \in \mathbb{Z}^2$ are covered last (see Fig. 5.10). $\diamond$



(a) The sets $\frac{4}{5} \cdot K_{\frac{1}{4}} + \mathbb{Z}^2$. (b) The sets $\frac{7}{8} \cdot K_{\frac{1}{4}} + \mathbb{Z}^2$. (c) The sets $K_{\frac{1}{4}} + \mathbb{Z}^2$.

Figure 5.10: Illustration of Example 5.24.

We now apply Lemma 5.23 to prove Theorem 5.9.

Proof of Theorem 5.9. Since (5.6) has already been shown, the bounds (5.7) and (5.8) together with Parts II and III follow directly from Theorem 5.6. Part I is a consequence of (5.8), (5.9), and Part III. Indeed, if $w(K) = 2$, then the inequalities in (5.8) and (5.9) are satisfied with equality. Thus, we can apply Part III to infer Part I.

Let us now show (5.9) and Part IV. For the sake of brevity we write $w := w(K)$. For every $u \in \mathbb{Z}^2 \setminus \{o\}$ it holds $w \leq w(K, u) = h(DK, u) = \|u\|_{(DK)^*} = \min\{\lambda \geq 0 : u \in \lambda(DK)^*\}$ which implies $1 \leq \min\{\frac{1}{w}\lambda \geq 0 : u \in \frac{1}{w}\lambda \cdot w(DK)^*\} = \min\{t \geq 0 : u \in t \cdot w(DK)^*\} = \|u\|_{w(DK)^*}$. Therefore, we have $\|u\|_{w(DK)^*} \geq 1$ for every $u \in \mathbb{Z}^2 \setminus \{o\}$. It follows that o is the only interior integer point of $w(DK)^*$. Thus, by Minkowski's first fundamental theorem, $A(w(DK)^*) \leq 4$. Using this fact and Mahler's inequality we obtain

$$A(K) = \frac{A(DK)}{4} = \frac{A(DK)A((DK)^*)}{4A((DK)^*)} \geq \frac{2}{A((DK)^*)} \geq \frac{w^2}{2}.$$

This shows (5.9). It remains to characterize for which cases equality holds. So assume that $A(K) = \frac{1}{2}w^2$. Then, in view of Mahler's inequality, we must have $A(DK)A((DK)^*) = 8$, and in view of Minkowski's first fundamental theorem we must have $A(w(DK)^*) = 4$. Thus, $(DK)^*$ is a parallelogram and the sets $\frac{w}{2}(DK)^* + z$ with $z \in \mathbb{Z}^2$ tile $\mathbb{R}^2$ (see p. 40). By Proposition 5.5 we deduce that there exists a unimodular transformation T and a parameter $0 \leq \alpha < 1$ such that

$$T\left(\frac{w}{2}(DK)^*\right) = \frac{1}{2}\operatorname{conv}(\{\pm(1-\alpha, 1), \pm(1+\alpha, -1)\}). \tag{5.24}$$

Now we compute the duals of the left and the right hand side of (5.24). Direct computations yield

$$\left(\frac{1}{2}\operatorname{conv}(\{\pm(1-\alpha, 1), \pm(1+\alpha, -1)\})\right)^* = 2\operatorname{conv}(\{\pm(1,\alpha), \pm(0,1)\}).$$

Furthermore,

$$\begin{aligned}
&\left(T\left(\frac{w}{2}(DK)^*\right)\right)^* \\
&= \left\{u \in \mathbb{R}^2 : u^\top x \leq 1 \text{ for all } x \in T\left(\frac{w}{2}(DK)^*\right)\right\} \\
&= \left\{u \in \mathbb{R}^2 : u^\top T\frac{w}{2} \cdot \frac{2}{w}T^{-1}x \leq 1 \text{ for all } \frac{2}{w}T^{-1}x \in (DK)^*\right\} \\
&= \left\{u \in \mathbb{R}^2 : u^\top T\frac{w}{2}y \leq 1 \text{ for all } y \in (DK)^*\right\} \\
&= \left\{\left(T^\top\right)^{-1}\frac{2}{w} \cdot \frac{w}{2}T^\top u \in \mathbb{R}^2 : \left(\frac{w}{2}T^\top u\right)^\top y \leq 1 \text{ for all } y \in (DK)^*\right\} \\
&= \left\{\left(T^\top\right)^{-1}\frac{2}{w}v \in \mathbb{R}^2 : v^\top y \leq 1 \text{ for all } y \in (DK)^*\right\} \\
&= \frac{2}{w}\left(T^\top\right)^{-1} \cdot \left\{v \in \mathbb{R}^2 : v^\top y \leq 1 \text{ for all } y \in (DK)^*\right\} \\
&= \frac{2}{w}\left(T^\top\right)^{-1}(DK).
\end{aligned}$$

We infer

$$\frac{2}{w}\left(T^\top\right)^{-1}(DK) = 2\operatorname{conv}(\{\pm(1,\alpha), \pm(0,1)\}).$$

Clearly, the transformation $(T^\top)^{-1}$ is unimodular. Thus, up to a unimodular transformation, we have

$$\frac{1}{2}DK = \frac{w}{2}\operatorname{conv}(\{\pm(1,\alpha), \pm(0,1)\})$$

with $0 \leq \alpha < 1$. We note that $\frac{1}{2}DK$ is a translate of K. Furthermore, it holds $\mu_2(\frac{1}{2}DK) \geq 1$ if and only if $\frac{1}{2}DK$ has a lattice-free translate. Since K is lattice-free, by assumption, it follows that $\mu_2(\frac{1}{2}DK) \geq 1$. In view of Lemma 5.23 and the fact that μ_2 is homogeneous of degree -1 we get $1 \leq \mu_2(\frac{1}{2}DK) = \frac{2}{w} \cdot \frac{1}{2} \max\{1+\alpha, 2-\alpha\}$ which implies $w \leq \max\{1+\alpha, 2-\alpha\}$. This proves Part IV. $\square$

Corollary 5.12 is a straightforward consequence of Theorem 5.9.

Proof of Corollary 5.12. The proof is in an analogous manner to the proof of Corollary 5.11. $\square$

The two-dimensional results in this chapter are quite diverse. However, it is not immediately clear how to use them for the evaluation of facet-defining inequalities for $\operatorname{conv}(P_I)$. We will establish the connection to cutting plane theory in the next chapter where we resume the analysis on the evaluation of inequalities for $\operatorname{conv}(P_I)$ which we started in Chapter 4. Furthermore, in Chapter 8, we will discuss a particular class of three-dimensional polyhedra. Many proofs in Chapter 8 are based on a certain intersection of these polyhedra with a hyperplane. Hence, the two-dimensional results in this chapter are needed.

CHAPTER 6

A PROBABILISTIC MODEL FOR THE EVALUATION OF CUTTING PLANES

In the previous chapter, we investigated the relation between the area and the lattice width of lattice-free convex sets in the plane. Thereby, we gained valuable insights into the interplay of convexity, integrality, and lattice-freeness in dimension two. Nevertheless, we wandered off our actual subject – the evaluation of facet-defining inequalities for $\text{conv}(P_I)$. In this chapter, we return to cutting plane evaluation. Our aim is to apply a probabilistic model to study the two-dimensional case in more detail. For that, we will need parts of the results from Chapter 5.

The roadmap of this chapter is as follows. Our point of departure is the set P_I. We show that the non-trivial valid inequalities for $\text{conv}(P_I)$ can be classified into split, type 1, type 2, type 3, and quadrilateral inequalities. Using the strength measure of Goemans that we introduced in Chapter 4 (see p. 23), we analyze the benefit from adding a single non-split inequality on top of the split closure. Then, applying a probabilistic model, we show that the gain from adding a type 2 inequality decreases with decreasing lattice width of the triangle, on average. Our results suggest that this is also true for type 3 and quadrilateral inequalities.

In Section 6.1, we motivate the choice of our model. In particular, we reason why we restrict our attention to dimension two and why we focus on the addition of a single non-split inequality to the entire split closure. Furthermore, we explain why the model we consider is interesting. Section 6.2 introduces our probabilistic model and presents our main results. In Section 6.3, we study triangles of type 1. Our strategy for the analysis of triangles of types 2 and 3, and quadrilaterals is explained in Section 6.4. The analysis of triangles of type 2, quadrilaterals, and triangles of type 3 can be found in Sections 6.5, 6.6, and 6.7, respectively.

6.1 Motivation

In this chapter, we restrict our attention to the two-dimensional case by dealing with the mixed-integer set $P_I = \{(x, s) \in \mathbb{Z}^2 \times \mathbb{R}^n_+ : x = f + \sum_{j=1}^n r^j s_j\}$ with $f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $r^j \in \mathbb{Q}^2 \setminus \{o\}$ for all $j = 1, \ldots, n$. We recall that f is called the *root vertex* and that the vectors r^j are called *rays*. The reason for considering dimension two instead of an arbitrary dimension is that stronger statements can be made. This is, most of all, due to the fact that the complete characterization of all maximal lattice-free polyhedra in $\mathcal{K}^2$ is known from Proposition 5.3 (see p. 37). If such a classification were known for higher dimensions, then the tools which we will develop in this chapter could easily be extended. However, such a classification is not known yet.

As in Chapter 4, instead of $\operatorname{conv}(P_I)$, we consider the set $R_f(r^1, \ldots, r^n) = \operatorname{conv}(\{s \in \mathbb{R}^n_+ : f + \sum_{j=1}^n r^j s_j \in \mathbb{Z}^2\})$ which is the projection of $\operatorname{conv}(P_I)$ onto the space of the s-variables (see p. 24), and we write R^n_f instead of $R_f(r^1, \ldots, r^n)$ for simplicity. We recall that any lattice-free polyhedron $B \subseteq \mathbb{R}^2$ with f in its interior gives rise to a function $\psi^B : \mathbb{R}^2 \mapsto \mathbb{R}_+$ which is the Minkowski functional of $B - f$. The inequality $\sum_{j=1}^n \psi^B(r^j) s_j \geq 1$ is the *cut associated with* B and is valid for R^n_f (see p. 17).

Every lattice-free (rational) polyhedron $B \subseteq \mathbb{R}^2$ with $f \in \operatorname{int}(B)$ is contained[1] in a maximal lattice-free (rational) polyhedron $\bar{B} \subseteq \mathbb{R}^2$. It follows that $\psi^{\bar{B}}(r^j) \leq \psi^B(r^j)$ for all $j = 1, \ldots, n$ and that the cut associated with B is dominated by the cut associated with $\bar{B}$. This implies that the benefit from adding the latter cut on top of the split closure is at least as high as the benefit from adding the former. Thus, in the remainder of this chapter, we focus exclusively on cuts associated with maximal lattice-free rational polyhedra. Since the root vertex f is required to be in the interior of such polyhedra, all of them must have a non-empty interior. In other words, we are interested in cuts associated with two-dimensional maximal lattice-free rational polyhedra with non-empty interior. The complete classification of those polyhedra is known from Proposition 5.3. It follows that R^n_f has three

[1] Let B be represented by the system of inequalities $a_i^\top x \leq b_i$ for $i = 1, \ldots, p$. Then we consider each of the inequalities one by one: if an inequality can be removed such that the remaining system defines a lattice-free polyhedron, we remove it; if not, then we increase b_i until the facet defined by $a_i^\top x \leq b_i$ contains a relative interior integer point. By construction, applying this procedure to all the inequalities $a_i^\top x \leq b_i$ for $i = 1, \ldots, p$ yields a maximal lattice-free polyhedron.

types of non-trivial valid inequalities: split, triangle, and quadrilateral inequalities named after the corresponding two-dimensional object from which the inequality can be derived. Triangle inequalities are further subdivided into type 1, type 2, and type 3 inequalities (see Fig. 5.1 on p. 38). We note that a non-trivial valid inequality for R_f^n can correspond to more than one maximal lattice-free polyhedron.

We are interested in the quality of cuts associated with the maximal lattice-free polyhedra in Proposition 5.3. As in Chapter 4, we use the strength measure introduced by Goemans to evaluate the quality of a cut. Basu et al. [BBCM11] assess the strength of split, triangle, and quadrilateral inequalities in a non-probabilistic setting. They show that the closures of split and type 1 inequalities may produce an arbitrarily bad approximation of R_f^n. On the other hand, the closures of type 2 or type 3 or quadrilateral inequalities deliver good approximations of R_f^n in terms of the strength. This, however comes with a price. Up to unimodular transformations, there is only one split and only one triangle of type 1, but an infinite number of triangles of types 2 and 3, and quadrilaterals. Therefore, triangles of types 2 and 3, and quadrilaterals allow more degrees of freedom. Intuitively, this suggests that for real instances an approximation by adding all these cuts is hard to compute. Nevertheless, we have no formal argument at hand to support this claim. From a more practical point of view, one is interested in approximations of the mixed-integer hull that one can generate easily. Current state-of-the-art in computational integer programming is to experiment with split cuts and the split closure (see, for instance, [AW10] and [BS08]). This is the point of departure of our theoretical study.

The aim of this chapter is to shed light on the question of which average improvement a non-split inequality gives when added on top of the split closure. For that, we take any maximal lattice-free triangle or quadrilateral B, and we investigate all potential sets R_f^n such that we can generate a valid cut from B. For this, it is required that f is in the interior of B. We vary f uniformly at random in the interior of B. This defines our probability distribution. For each particular B and $f \in \operatorname{int}(B)$ we let n and $r^1, \dots, r^n$ attain arbitrary values. We compute a lower bound on the probability that the strength of adding the cut associated with B on top of the split closure is less than or equal to an arbitrary value. As a conclusion from our probabilistic analysis we obtain that the addition of a single type 2, type 3, or quadrilateral inequality to the split closure becomes less likely to be beneficial the closer the lattice-free set looks like a split.

We think that this complements nicely the analysis in [BBCM11]: there, the authors construct sequences of examples in which cuts from triangles of types 2 and 3, and quadrilaterals cannot be approximated to within a constant factor by the split closure. The approximation becomes worse as the triangles and quadrilaterals converge towards a split. From the results in this chapter, it follows that this geometrically counterintuitive situation occurs extremely rarely.

6.2 Probabilistic model and main results

Often, the quality of cuts is measured according to their worst case performance. An analysis based on this principle was carried out in Chapter 4. In this chapter, we apply a stochastic approach by introducing a probability distribution on all possible instances. We refer to [BCM10] and [HAN10] for other approaches on how to apply a probabilistic analysis to mixed-integer linear sets.

Basu et al. [BCM10] investigate the closures of split and triangle inequalities. These closures are compared in a worst case and an average case scenario. The root vertex f and the direction of the rays are assumed to be uniformly distributed. The measure of strength used by Basu et al. incorporates the objective function. In the worst case scenario a worst possible objective function vector is taken, whereas in the average case scenario the objective function vector is assumed to be uniformly distributed. The results show that, with quite high probability, the split closure produces a bad approximation of R_f^n when compared to the triangle closure in the worst case scenario. In the average case scenario, both closures provide rather good approximations of R_f^n with high probability.

He et al. [HAN10] compare split and type 1 inequalities with respect to dominance in terms of their coefficients and with respect to the volume which is cut off when adding a split or type 1 inequality to the linear programming relaxation. In their model, He et al. assume that both, f and the direction of the rays, are uniformly distributed. They provide guidelines on when a split inequality is likely to be more effective than a type 1 inequality and vice versa. The results suggest that split inequalities are more likely to perform well than type 1 inequalities. Unfortunately, they do not consider triangles of types 2 and 3, and quadrilaterals.

Our aim is to evaluate the benefit from adding a single cut associated with a maximal lattice-free rational triangle or quadrilateral on top of the split closure. For that, we introduce the following notation. Let Ω be the set of all maximal lattice-free rational polyhedra in $\mathbb{R}^2$ which contain f in the interior and let $\mathcal{S}$ (resp. $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$, $\mathcal{Q}$) be the subset of Ω containing all splits (resp. triangles of type 1, type 2, type 3, quadrilaterals). Observe that $R_f^n = \{s \in \mathbb{R}_+^n : \sum_{j=1}^n \psi^B(r^j)s_j \geq 1 \text{ for all } B \in \mathcal{S} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{Q}\}$. For a non-empty set $\mathcal{L} \subseteq \Omega$ we define $\mathcal{L}(R_f^n)$ to be the intersection of all cuts associated with the polyhedra in $\mathcal{L}$ and the trivial inequalities $s_j \geq 0$

for all $j = 1, \dots, n$. In the remainder of this chapter, $\mathcal{L}$ will always be $\mathcal{S}$ or $\mathcal{S} \cup \{B\}$ for some $B \in \Omega \setminus \mathcal{S}$. In this case, $\mathcal{L}(R_f^n)$ is a polyhedron (as the split closure of a rational polyhedron is again a polyhedron, see [CKS90, Theorem 3]). Moreover, $\mathcal{L}(R_f^n)$ is of covering type, since all cuts associated with the polyhedra in $\mathcal{L}$ are assumed to be in standard form.

In order to evaluate the gain from adding a triangle or quadrilateral inequality to the split closure we apply the following procedure. We fix a maximal lattice-free rational triangle or quadrilateral $B \in \Omega \setminus \mathcal{S}$, but we allow f to vary in its interior. We will show that by considering a specific set of rays that depends only on f, we obtain an upper bound on $t(\mathcal{F}(R_f^n), \mathcal{S}(R_f^n))$ where $\mathcal{F} = \mathcal{S} \cup \{B\}$. Depending on where f is located this bound may differ. By varying f over the entire area of B we compute the area for which the bound is below a certain value, say z, and compare it to the area of B. This gives a ratio that is a lower bound on the probability that the strength is less than or equal to z. In turn, 1 minus this probability is an upper bound on the chance that B improves upon the split closure by a value of more than z with respect to the strength.

Let $B \in \Omega \setminus \mathcal{S}$ and let $\mathcal{F} = \mathcal{S} \cup \{B\}$. The following observation is similar to Observation 4.2 (see p. 24) and follows easily from Lemma 4.1.

Observation 6.1.

$$t(\mathcal{F}(R_f^n), \mathcal{S}(R_f^n)) = \frac{1}{\min\{\sum_{j=1}^n \psi^B(r^j)s_j : s \in \mathcal{S}(R_f^n)\}}.$$

We recall that we can assume that the points $f + r^j$, $j = 1, \dots, n$, are on the boundary of B. Furthermore, a ray r^j is called a *corner ray* if the point $f + r^j$ is a vertex of B (see p. 25). In the course of this chapter we will deal with optimization problems of the following type:

$$\min \sum_{j=1}^{n} \psi^B(r^j)s_j \quad \text{s.t.} \quad s \in \mathcal{S}(R_f^n) \tag{6.1}$$

for some $B \in \Omega \setminus \mathcal{S}$. By the scaling of the rays, the objective function becomes $\sum_{j=1}^n s_j$.

For now, assume that all the corner rays of B (and only those) are present. We will explain later why we can make this assumption. This implies that the corner rays are fixed once f and B are chosen. We assume a continuous uniform distribution on f in the interior of B. Given $z \in \mathbb{R}$, $z > 1$, we define $P^B(z)$ to be the probability that $t(\mathcal{F}(R_f^n), \mathcal{S}(R_f^n))$ is less than or equal to z for f varying in the triangle or quadrilateral B, i.e.

$$P^B(z) := \frac{1}{A(B)} \int_{f \in \mathrm{int}(B)} \mathbf{1}\{t(\mathcal{F}(R_f^n), \mathcal{S}(R_f^n)) \le z\} df,$$

where $A(B)$ is the area of B and $\mathbf{1}$ is the *indicator function* defined as $\mathbf{1}\{x \leq z\} := 1$ if $x \leq z$, and $\mathbf{1}\{x \leq z\} := 0$ if $x > z$.

Let us now argue why we can assume that all corner rays of B are present in order to obtain the desired bound. Assume $k \geq 1$ corner rays $r^{n+1}, \ldots, r^{n+k}$ are missing and consider the set $R_f^{n+k} := \operatorname{conv}(\{s \in \mathbb{R}_+^{n+k} : f + \sum_{j=1}^{n+k} r^j s_j \in \mathbb{Z}^2\})$. We now apply Observation 6.1 to infer

$$t\big(\mathcal{F}(R_f^{n+k}), \mathcal{S}(R_f^{n+k})\big) \quad \geq \quad t\big(\mathcal{F}(R_f^{n}), \mathcal{S}(R_f^{n})\big) \tag{6.2}$$

$$\Longleftrightarrow$$

$$\min\left\{\sum_{j=1}^{n+k} s_j : s \in \mathcal{S}(R_f^{n+k})\right\} \quad \leq \quad \min\left\{\sum_{j=1}^{n} s_j : s \in \mathcal{S}(R_f^{n})\right\}.$$

The latter inequality follows from the fact that an optimal solution $\bar{s}$ for the latter minimization problem implies a feasible solution for the former minimization problem by setting $\bar{s}_j = 0$ for all $j = n+1, \ldots, n+k$. Indeed, if there would be a split which cuts off $(\bar{s}, 0)$ from $\mathcal{S}(R_f^{n+k})$, then the same split would cut off $\bar{s}$ from $\mathcal{S}(R_f^n)$, a contradiction. From inequality (6.2), it follows that $\mathbf{1}\{t(\mathcal{F}(R_f^{n+k}), \mathcal{S}(R_f^{n+k})) \leq z\} \leq \mathbf{1}\{t(\mathcal{F}(R_f^{n}), \mathcal{S}(R_f^{n})) \leq z\}$. Thus, by adding the corner rays we obtain a lower bound on $P^B(z)$.

We now explain why we can assume that only the corner rays of B are present in order to compute $t(\mathcal{F}(R_f^{n+k}), \mathcal{S}(R_f^{n+k}))$. We need two ingredients: first, the rays are scaled such that the points $f + r^j$, $j = 1, \ldots, n$, are on the boundary of B; second, the set of rays $\{r^1, \ldots, r^{n+k}\}$ contains all the corner rays of the polyhedron B and thus, every ray is a convex combination of the corner rays. Hence, by the explanation given on p. 25, (6.1) reduces to the problem where the objective function is $\sum_{j=1}^{k} s_j$ with $\{r^1, \ldots, r^k\}$, $k \in \{3, 4\}$, being exactly the set of corner rays for the given triangle or quadrilateral B. By Observation 6.1, this implies $t(\mathcal{F}(R_f^{n+k}), \mathcal{S}(R_f^{n+k})) = t(\mathcal{F}(R_f^{k}), \mathcal{S}(R_f^{k}))$. Thus, in the remainder of this chapter, in order to obtain the desired bounds, we assume that the set of rays consists of exactly the corner rays of B.

We recall that applying a unimodular transformation to a maximal lattice-free rational polyhedron $B \subseteq \mathbb{R}^2$ changes neither the lattice width nor does it affect the computation of the strength. In the remainder of this chapter, we informally call B *flat* whenever its lattice width is sufficiently close to 1. Our main results are summarized below.

Theorem 6.2. *Let T_i be a triangle of type $i \in \{1, 2\}$ and let $w := w(T_2)$. Then, for any $z > 1$, we have*

I.

$$P^{T_1}(z) \geq \begin{cases} 0 & \text{if } 1 < z \leq \frac{3}{2}, \\ \frac{3}{4}\left(\frac{2z-3}{z-1}\right)^2 & \text{if } \frac{3}{2} < z < 2, \\ 1 & \text{if } 2 \leq z < +\infty, \end{cases}$$

II.

$$P^{T_2}(z) \geq \begin{cases} 0 & \text{if } 1 < z \leq w, \\ g_1 & \text{if } w < z \leq \frac{w}{w-1}, \\ g_1 + g_2 & \text{if } \frac{w}{w-1} < z < +\infty, \end{cases}$$

with $g_1 = \frac{(z-w)(2wz-w-z)}{w^2(z-1)^2}$ *and* $g_2 = \frac{(w-1)^2(z-1)^2-1}{w^2(z-1)^2}$.

Theorem 6.2 II shows that for any given $z > 1$, $P^{T_2}(z)$ tends to 1 if $w(T_2)$ converges to 1, i.e. the probability that a flat type 2 triangle improves upon the split closure by a value of more than z goes to 0. This will be explained in further detail in Section 6.5.

The analysis of a type 3 triangle T_3 and a quadrilateral Q turns out to be more complex. We did not succeed in putting $P^{T_3}(z)$ (resp. $P^Q(z)$) into direct relation to $w(T_3)$ (resp. $w(Q)$) and z only. Instead we parametrize T_3 and Q in terms of the coordinates of their vertices. Using this more complicated parametrization we derive formulas for $P^{T_3}(z)$ and $P^Q(z)$. Then we discretize the coordinates of the vertices and evaluate the formulas with respect to our discretization. This qualitatively leads to the same conclusion as before: if T_3 and Q converge towards a split (meaning the lattice width converges to 1), then the probability that $t(\mathcal{F}(R_f^n), \mathcal{S}(R_f^n))$, with $\mathcal{F} = \mathcal{S} \cup \{T_3\}$ or $\mathcal{F} = \mathcal{S} \cup \{Q\}$, is less than or equal to z tends to 1. We refer to Sections 6.6 and 6.7 for the corresponding formulas.

6.3 Type 1 triangles

By a unimodular transformation, we assume that the type 1 triangle T_1 is given by $T_1 = \text{conv}(\{(0,0), (2,0), (0,2)\})$. Let

$$\begin{aligned} R_1 &:= \text{int}(\text{conv}(\{(1,0), (0,1), (1,1)\})), \\ R_2 &:= \text{int}(\text{conv}(\{(0,0), (1,0), (0,1)\})), \\ R_3 &:= \text{int}(\text{conv}(\{(0,1), (1,1), (0,2)\})), \\ R_4 &:= \text{int}(\text{conv}(\{(1,0), (1,1), (2,0)\})). \end{aligned}$$

Note that $\mathrm{int}(T_1) \setminus \cup_{j=1}^4 R_j$ is a set of area zero, so it can be neglected in the following probabilistic analysis. For given $f \in R_1 \cup R_2 \cup R_3 \cup R_4$ the three corner rays are $r^1 = (-f_1, -f_2)$, $r^2 = (2 - f_1, -f_2)$, and $r^3 = (-f_1, 2 - f_2)$. Let $\mathcal{F} = \mathcal{S} \cup \{T_1\}$. In [BBCM11, Theorem 6.1] it is shown that

$$t(\mathcal{F}(R_f^3), \mathcal{S}(R_f^3)) = \begin{cases} 2 & \text{if } f \in R_1, \\ \frac{3-f_1-f_2}{2-f_1-f_2} & \text{if } f \in R_2, \\ \frac{f_2+1}{f_2} & \text{if } f \in R_3, \\ \frac{f_1+1}{f_1} & \text{if } f \in R_4. \end{cases} \tag{6.3}$$

For the sake of brevity we write $\mathbf{1}\{t \leq z\}$ instead of $\mathbf{1}\{t(\mathcal{F}(R_f^3), \mathcal{S}(R_f^3)) \leq z\}$. Then

$$P^{T_1}(z) = \frac{1}{A(T_1)} \sum_{j=1}^{4} \int_{f \in R_j} \mathbf{1}\{t \leq z\} df.$$

We compute the four integrals separately. For that, we need to check when the corresponding functions in (6.3) attain a value which is less than or equal to z. Assume $f \in R_1$. Then $\int_{f \in R_1} \mathbf{1}\{t \leq z\} df$ is 0 if $z < 2$ and is $\frac{1}{2}$ if $z \geq 2$. Assume $f \in R_2$. We have $\frac{3-f_1-f_2}{2-f_1-f_2} \leq z \Leftrightarrow f_1 + f_2 \leq \frac{2z-3}{z-1}$ and thus $\int_{f \in R_2} \mathbf{1}\{t \leq z\} df$ is 0 if $z \leq \frac{3}{2}$, is $\frac{1}{2}$ if $z \geq 2$, and is $\frac{1}{2}(\frac{2z-3}{z-1})^2$ otherwise (here we used the fact that $0 < f_1 + f_2 < 1$ in R_2). Assume $f \in R_3$. Then $\frac{f_2+1}{f_2} \leq z \Leftrightarrow f_2 \geq \frac{1}{z-1}$. Hence, $\int_{f \in R_3} \mathbf{1}\{t \leq z\} df$ is 0 if $z \leq \frac{3}{2}$, is $\frac{1}{2}$ if $z \geq 2$, and is $\frac{1}{2}(\frac{2z-3}{z-1})^2$ otherwise (here we used $1 < f_2 < 2$). Finally, the case $f \in R_4$ is analogous to the previous case. Since $A(T_1) = 2$ it follows

$$P^{T_1}(z) = \begin{cases} 0 & \text{if } 1 < z \leq \frac{3}{2}, \\ \frac{3}{4}\left(\frac{2z-3}{z-1}\right)^2 & \text{if } \frac{3}{2} < z < 2, \\ 1 & \text{if } 2 \leq z < +\infty. \end{cases} \tag{6.4}$$

We note that the "=" in (6.4) becomes a "$\geq$" in Theorem 6.2 I since (6.4) were derived by assuming the presence of all the three corner rays of T_1. If at least one corner ray is missing, then our computations lead to a lower bound on $P^{T_1}(z)$ by (6.2) and the explanation in the paragraph after (6.2).

6.4 Strategy for triangles of types 2 and 3, and quadrilaterals

The analysis of type 1 triangles in Section 6.3 was quite easy. The reason is that – due to the assumption of having all the three corner rays present – the split closure is known: it is always defined by the trivial inequalities $s_j \geq 0$, $j = 1, 2, 3$, and a subset of the three split inequalities associated with the splits whose normal vectors are the normal vectors of the facets of the type 1 triangle (see [BBCM11, Lemmas 6.3 and 6.5] for a proof). We also note that $w(T_1) = 2$ and that this value is attained by precisely these vectors.

Using the split closure for triangles of types 2 and 3, and quadrilaterals would result in too complicated formulas. Thus, we choose another strategy. Instead of using the entire split closure we will take only one well-chosen split inequality (in addition to the trivial inequalities) and therefore obtain lower bounds on the desired probabilities. We emphasize that our proof technique which we describe below can only be used to show the weakness of an inequality, but not to show that an inequality is strong. Let $B \in \Omega \setminus \mathcal{S}$ be a triangle of type 2 or 3, or a quadrilateral. The split inequality which we choose will depend on the location of f in the interior of B. For that, we partition B into regions $R_1, \dots, R_p$ and select a single split for each region. The basic idea is to choose a split that contains f in its interior and such that the normal vector $u \in \mathbb{Z}^2 \setminus \{o\}$ of the split is a potential candidate for a vector for which $w(B)$ is attained, i.e. which satisfies $w(B) = w(B, u)$. We feel that choosing such a split is reasonable, even though we have no formal argument at hand to support our choice. The normal vectors of our candidate splits will always be among the vectors e_1, e_2, and $e_1 + e_2$ since we apply a unimodular transformation to B before we start our analysis. We do that to bring B into an appropriate form which is easy to handle (see Sections 6.5, 6.6, and 6.7 for details).

We now show that our simplification of using only one split inequality instead of the split closure leads to a lower bound on $P^B(z)$. Let $\mathcal{F} = \mathcal{S} \cup \{B\}$ for some $B \in \Omega \setminus \mathcal{S}$ and let $L \in \mathcal{S}$ be arbitrary. Then, by Observation 6.1,

$$
\begin{aligned}
&t\left(\mathcal{F}(R_f^k), \mathcal{S}(R_f^k)\right) \\
&= \left(\min\left\{\sum_{j=1}^{k} s_j : s \in \mathcal{S}(R_f^k)\right\}\right)^{-1} \\
&\leq \left(\min\left\{\sum_{j=1}^{k} s_j : s_j \geq 0 \text{ for } j = 1, \dots, k \text{ and } \sum_{j=1}^{k} \psi^L(r^j) s_j \geq 1\right\}\right)^{-1} \\
&= t\left(\left\{s \in \mathbb{R}_+^k : \sum_{j=1}^{k} \psi^L(r^j) s_j \geq 1 \text{ and } \sum_{j=1}^{k} \psi^B(r^j) s_j \geq 1\right\},\right. \\
&\qquad \left.\left\{s \in \mathbb{R}_+^k : \sum_{j=1}^{k} \psi^L(r^j) s_j \geq 1\right\}\right),
\end{aligned}
$$

since, by the scaling of the rays, we have $\psi^B(r^j) = 1$ for all $j = 1, \dots, k$.

For ease of notation we denote the latter by $\bar{t}(B, L)$, which is the strength of the polyhedron obtained by adding to $\mathbb{R}^k_+$ the cuts associated with B and L with respect to the polyhedron obtained by just adding to $\mathbb{R}^k_+$ the cut associated with L. It follows $\mathbf{1}\{t(\mathcal{F}(R^k_f), \mathcal{S}(R^k_f)) \le z\} \ge \mathbf{1}\{\bar{t}(B, L) \le z\}$ and therefore

$$P^B(z) \ge \frac{1}{A(B)} \sum_{j=1}^{p} \int_{f \in R_j} \mathbf{1}\{\bar{t}(B, L_{R_j}) \le z\} df, \tag{6.5}$$

where L_{R_j} is the single split which is used in region R_j to approximate the split closure, for $j = 1, \dots, p$. In the following, for simplicity, we write $\mathbf{1}\{\bar{t} \le z\}$ instead of $\mathbf{1}\{\bar{t}(B, L) \le z\}$ whenever B and L are clear from the context. In order to compute $\bar{t}(B, L)$ we need to solve an optimization problem of the type

$$\begin{aligned} \min\ & s_1 + \dots + s_k \\ \text{s.t. }\ & \psi^L(r^1)s_1 + \dots + \psi^L(r^k)s_k \ge 1, \\ & s_j \ge 0 \quad \text{for } j = 1, \dots, k. \end{aligned}$$

The values $\psi^L(r^1), \dots, \psi^L(r^k)$ are called the *coefficients of the cut associated with* L. We recall (see (4.5) on p. 31) that, in general, for a split $L = \{x \in \mathbb{R}^2 : \lfloor \pi^\top f \rfloor \le \pi^\top x \le \lceil \pi^\top f \rceil\}$, $\pi \in \mathbb{Z}^2 \setminus \{o\}$, which contains f in its interior, it is easy to verify that the coefficients of the cut associated with L, i.e. the coefficients of $\sum_{j=1}^k \psi^L(r^j)s_j \ge 1$, are

$$\psi^L(r^j) = \begin{cases} \frac{\pi^\top r^j}{\lceil \pi^\top f \rceil - \pi^\top f} & \text{if } \pi^\top r^j > 0, \\ 0 & \text{if } \pi^\top r^j = 0, \\ \frac{\pi^\top r^j}{\lfloor \pi^\top f \rfloor - \pi^\top f} & \text{if } \pi^\top r^j < 0, \end{cases} \tag{6.6}$$

for all $j = 1, \dots, k$. Thus, only the normal vector π of the split L is needed to compute the coefficients $\psi^L(r^1), \dots, \psi^L(r^k)$.

In Section 6.5, we will carefully explain how we compute the lower bound on $P^{T_2}(z)$ for triangles of type 2. Since the computations for quadrilaterals and triangles of type 3 in Sections 6.6 and 6.7 give no new insights we will only state intermediate results there.

6.5 Type 2 triangles

By a unimodular transformation, we assume that the type 2 triangle T_2 has one facet containing the points $(0, 0)$ and $(1, 0)$, one facet containing $(0, 1)$, and one facet containing $(1, 1)$. Furthermore, one vertex $a = (a_1, a_2)$ satisfies

$0 < a_1 < 1$ and $1 < a_2$. Thus, the other vertices are $b := (-\frac{a_1}{a_2-1}, 0)$ and $c := (\frac{a_2-a_1}{a_2-1}, 0)$. We assume that a is arbitrary but fixed and treat it as a parameter in the subsequent computations. We decompose T_2 into six regions (see Fig. 6.1):

$$
\begin{aligned}
R_1 &:= \operatorname{int}(\operatorname{conv}\{(0,0),(a_1,0),(a_1,1),(0,1)\}),\\
R_2 &:= \operatorname{int}(\operatorname{conv}\{(a_1,0),(1,0),(1,1),(a_1,1)\}),\\
R_3 &:= \operatorname{int}(\operatorname{conv}\{(0,0),(0,1),(b_1,b_2)\}),\\
R_4 &:= \operatorname{int}(\operatorname{conv}\{(1,0),(1,1),(c_1,c_2)\}),\\
R_5 &:= \operatorname{int}(\operatorname{conv}\{(0,1),(a_1,1),(a_1,a_2)\}),\\
R_6 &:= \operatorname{int}(\operatorname{conv}\{(a_1,1),(1,1),(a_1,a_2)\}).
\end{aligned}
$$

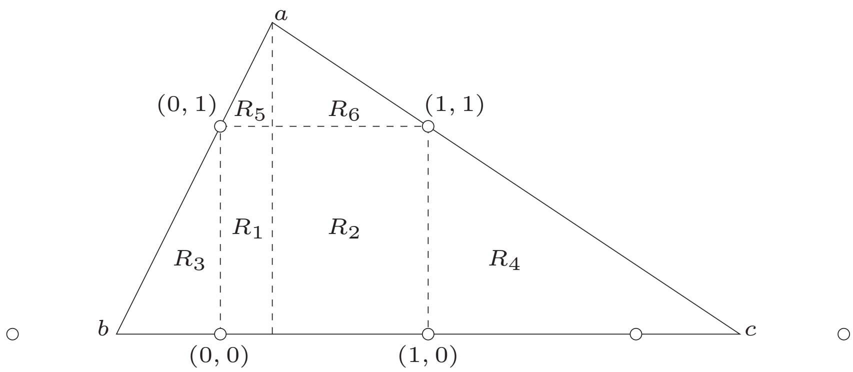


Figure 6.1: Decomposition of a type 2 triangle.

For given $f \in \cup_{j=1}^{6} R_j$ the three corner rays are $r^1 = (b_1 - f_1, -f_2)$, $r^2 = (c_1 - f_1, -f_2)$, and $r^3 = (a_1 - f_1, a_2 - f_2)$. For simplicity, let $w := w(T_2)$. From Remark 5.21 (see p. 57), it follows that w is attained by one of the vectors e_1 or e_2. Thus, $w = \min\{a_2, \frac{a_2}{a_2-1}\}$ which implies $w = a_2$ if $a_2 \leq 2$ and $w = \frac{a_2}{a_2-1}$ if $a_2 > 2$. In either case, direct computations show that

$$A(T_2) = \frac{(a_2)^2}{2(a_2-1)} = \frac{w^2}{2(w-1)}. \tag{6.7}$$

In regions R_3 and R_4 we use the split $S_1 := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_2 \leq 1\}$ and in regions R_5 and R_6 we use the split $S_2 := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1\}$. In regions R_1 and R_2 we choose either S_1 or S_2. We will use the one whose normal vector attains w, i.e. we choose S_1 if $a_2 \leq 2$ and S_2 if $a_2 > 2$. Such a split has another nice property: among all splits that contain f in their interior we select one which covers most of the area of the triangle.

6.5.1 Regions R_3 and R_4

Let $f \in R_3 \cup R_4$. In order to compute $\bar{t}(T_2, S_1)$ we have to solve the optimization problem

$$\begin{aligned} &\min\ s_1 + s_2 + s_3 \\ &\text{s.t. } \psi^{S_1}(r^1)s_1 + \psi^{S_1}(r^2)s_2 + \psi^{S_1}(r^3)s_3 \geq 1, \\ &\qquad s_j \geq 0 \quad \text{for } j = 1, 2, 3. \end{aligned} \tag{6.8}$$

Since we use S_1 we have $\pi = e_2$, and since $f \in R_3 \cup R_4$, it holds $0 < \pi^\top f < 1$. Thus, using (6.6), we obtain $\psi^{S_1}(r^1) = \psi^{S_1}(r^2) = 1$ and $\psi^{S_1}(r^3) = \frac{a_2 - f_2}{1 - f_2}$. Therefore, an optimal solution to (6.8) is $s^* = (0, 0, \frac{1-f_2}{a_2-f_2})$ with optimal objective value $\frac{1-f_2}{a_2-f_2}$. It follows $\bar{t}(T_2, S_1) = \frac{a_2-f_2}{1-f_2}$ for $f \in R_3 \cup R_4$.

6.5.2 Regions R_5 and R_6

Let $f \in R_5$. To compute $\bar{t}(T_2, S_2)$ we solve the following optimization problem: $\min \sum_{j=1}^{3} s_j$ s.t. $\sum_{j=1}^{3} \psi^{S_2}(r^j)s_j \geq 1,\ s_j \geq 0$ for $j = 1, 2, 3$. By applying (6.6) we obtain $\psi^{S_2}(r^1) = \frac{f_1 - b_1}{f_1}$, $\psi^{S_2}(r^2) = \frac{c_1 - f_1}{1 - f_1}$, and $\psi^{S_2}(r^3) = \frac{a_1 - f_1}{1 - f_1}$. Hence, we have to determine the minimum among the values in the set $\{(\psi^{S_2}(r^1))^{-1}, (\psi^{S_2}(r^2))^{-1}, (\psi^{S_2}(r^3))^{-1}\}$. Using our assumptions on the variables and parameters, it follows that $(\psi^{S_2}(r^3))^{-1} \geq 1$ and $(\psi^{S_2}(r^i))^{-1} \leq 1$ for $i \in \{1, 2\}$. One easily verifies that $(\psi^{S_2}(r^1))^{-1} \leq (\psi^{S_2}(r^2))^{-1} \Leftrightarrow a_1 \geq f_1$ which is satisfied in R_5 by assumption. Thus, $\bar{t}(T_2, S_2) = \frac{f_1 - b_1}{f_1}$ for $f \in R_5$.

Let $f \in R_6$. By symmetry, i.e. $a_1 \to 1 - a_1$ and $f_1 \to 1 - f_1$, we obtain $\bar{t}(T_2, S_2) = \frac{c_1 - f_1}{1 - f_1}$.

6.5.3 Regions R_1 and R_2

First assume $a_2 \leq 2$ and use the split S_1. Let $f \in R_1 \cup R_2$. The associated optimization problem is $\min \sum_{j=1}^{3} s_j$ s.t. $\sum_{j=1}^{3} \psi^{S_1}(r^j)s_j \geq 1,\ s_j \geq 0$ for $j = 1, 2, 3$ with $\psi^{S_1}(r^1) = \psi^{S_1}(r^2) = 1$ and $\psi^{S_1}(r^3) = \frac{a_2 - f_2}{1 - f_2}$. Hence, $\bar{t}(T_2, S_1) = \frac{a_2 - f_2}{1 - f_2}$ for $f \in R_1 \cup R_2$ and $a_2 \leq 2$.

Now assume that $a_2 > 2$. We use the split S_2. Let $f \in R_1$. The solution of the optimization problem $\min \sum_{j=1}^{3} s_j$ s.t. $\sum_{j=1}^{3} \psi^{S_2}(r^j)s_j \geq 1,\ s_j \geq 0$ for $j = 1, 2, 3$ with $\psi^{S_2}(r^1) = \frac{f_1 - b_1}{f_1}$, $\psi^{S_2}(r^2) = \frac{c_1 - f_1}{1 - f_1}$, and $\psi^{S_2}(r^3) = \frac{a_1 - f_1}{1 - f_1}$ is the minimum of $(\psi^{S_2}(r^1))^{-1}$ and $(\psi^{S_2}(r^2))^{-1}$ as $(\psi^{S_2}(r^3))^{-1} \geq 1$. It is easy to verify that $(\psi^{S_2}(r^1))^{-1} \leq (\psi^{S_2}(r^2))^{-1} \Leftrightarrow a_1 \geq f_1$. Therefore, we have $\bar{t}(T_2, S_2) = \frac{f_1 - b_1}{f_1}$ for $f \in R_1$ and $a_2 > 2$. Finally, assume $f \in R_2$. By symmetry, we obtain $\bar{t}(T_2, S_2) = \frac{c_1 - f_1}{1 - f_1}$ for $f \in R_2$ and $a_2 > 2$.

The following table summarizes the function $\bar{t}(T_2, S_i)$, for the corresponding $i \in \{1, 2\}$.

$\bar{t}(T_2, S_i)$ for $a_2 \leq 2$	$\bar{t}(T_2, S_i)$ for $a_2 > 2$	Location of f
$\frac{a_2 - f_2}{1 - f_2}$	$\frac{f_1 - b_1}{f_1}$	$f \in R_1$
$\frac{a_2 - f_2}{1 - f_2}$	$\frac{c_1 - f_1}{1 - f_1}$	$f \in R_2$
$\frac{a_2 - f_2}{1 - f_2}$	$\frac{a_2 - f_2}{1 - f_2}$	$f \in R_3$
$\frac{a_2 - f_2}{1 - f_2}$	$\frac{a_2 - f_2}{1 - f_2}$	$f \in R_4$
$\frac{f_1 - b_1}{f_1}$	$\frac{f_1 - b_1}{f_1}$	$f \in R_5$
$\frac{c_1 - f_1}{1 - f_1}$	$\frac{c_1 - f_1}{1 - f_1}$	$f \in R_6$

6.5.4 Approximation for $P^{T_2}(z)$

We now compute the integrals $\int_{f \in R_j} \mathbf{1}\{\bar{t}(T_2, S_i) \leq z\} df$ for $j = 1, \ldots, 6$ and the corresponding split S_1 or S_2 which we used above. For simplicity, let $\int_{f \in R_j} := \int_{f \in R_j} \mathbf{1}\{\bar{t} \leq z\} df$ for $j = 1, \ldots, 6$.

Let $f \in R_3$. Then $\mathbf{1}\{\bar{t} \leq z\} = 1 \Leftrightarrow \frac{a_2 - f_2}{1 - f_2} \leq z \Leftrightarrow f_2 \leq \frac{z - a_2}{z - 1}$. Observe that $\int_{f \in R_3} = 0$ if $z \leq a_2$. If $z > a_2$, then the area of the set $\{f \in R_3 : f_2 \leq \frac{z - a_2}{z - 1}\}$ is the difference of the area of two triangles (see Fig. 6.2(d) for a schematic representation of this area). Direct computations yield

$$\int_{f \in R_3} = \begin{cases} 0 & \text{if } z \leq a_2, \\ \frac{a_1(z - a_2)(z + a_2 - 2)}{2(a_2 - 1)(z - 1)^2} & \text{if } z > a_2. \end{cases}$$

Let $f \in R_5$. Then $\mathbf{1}\{\bar{t} \leq z\} = 1 \Leftrightarrow \frac{f_1 - b_1}{f_1} \leq z \Leftrightarrow f_1 \geq \frac{-b_1}{z - 1} = \frac{a_1}{(a_2 - 1)(z - 1)}$. We have $\int_{f \in R_5} = 0 \Leftrightarrow \frac{a_1}{(a_2 - 1)(z - 1)} \geq a_1 \Leftrightarrow z \leq \frac{a_2}{a_2 - 1}$. If $z > \frac{a_2}{a_2 - 1}$, then again we compute the difference of the area of two triangles (see Fig. 6.2(f)) and infer

$$\int_{f \in R_5} = \begin{cases} 0 & \text{if } z \leq \frac{a_2}{a_2 - 1}, \\ \frac{a_1(a_2 - 1)}{2} g_3 & \text{if } z > \frac{a_2}{a_2 - 1}, \end{cases}$$

where $g_3 = 1 - \frac{1}{(a_2 - 1)^2 (z - 1)^2}$.

By symmetry, the integrals for $f \in R_4$ and $f \in R_6$ are obtained by replacing a_1 with $1 - a_1$ in the formulas for $\int_{f \in R_3}$ and $\int_{f \in R_5}$ (see Fig.s 6.2(e) and 6.2(g)). Thus,

$$\int_{f \in R_4} = \begin{cases} 0 & \text{if } z \leq a_2, \\ \frac{(1 - a_1)(z - a_2)(z + a_2 - 2)}{2(a_2 - 1)(z - 1)^2} & \text{if } z > a_2, \end{cases}$$

$$\int_{f\in R_6} = \begin{cases} 0 & \text{if } z \le \frac{a_2}{a_2-1}, \\ \frac{(1-a_1)(a_2-1)}{2} g_3 & \text{if } z > \frac{a_2}{a_2-1}. \end{cases}$$

In order to compute $\int_{f\in R_1}$ and $\int_{f\in R_2}$ we distinguish the cases $a_2 \le 2$ and $a_2 > 2$. First, let us assume that $a_2 \le 2$. Computations in a similar manner as above (area of the shaded quadrilateral in Fig. 6.2(a)) lead to

$$\int_{f\in R_1} = \begin{cases} 0 & \text{if } z \le a_2, \\ \frac{a_1(z-a_2)}{z-1} & \text{if } z > a_2, \end{cases}$$

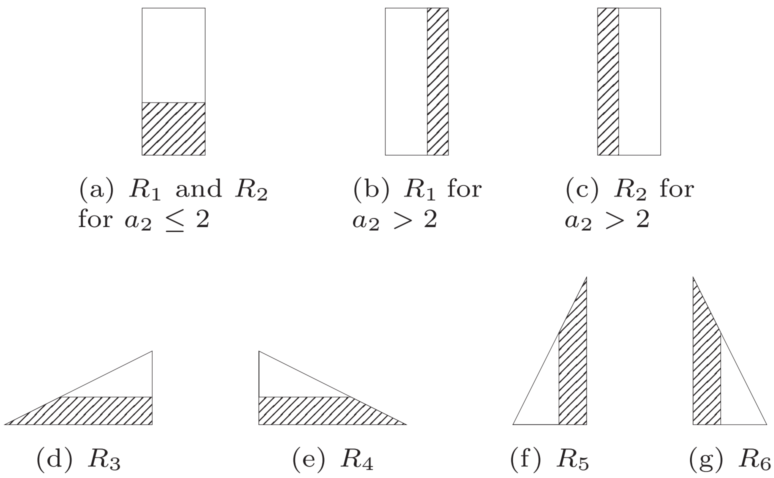
$$\int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le a_2, \\ \frac{(1-a_1)(z-a_2)}{z-1} & \text{if } z > a_2. \end{cases}$$

Now assume that $a_2 > 2$. Computing the corresponding area (see Fig.s 6.2(b) and 6.2(c)) yields

$$\int_{f\in R_1} = \begin{cases} 0 & \text{if } z \le \frac{a_2}{a_2-1}, \\ a_1 g_4 & \text{if } z > \frac{a_2}{a_2-1}, \end{cases}$$

$$\int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le \frac{a_2}{a_2-1}, \\ (1-a_1) g_4 & \text{if } z > \frac{a_2}{a_2-1}, \end{cases}$$

where $g_4 = 1 - \frac{1}{(a_2-1)(z-1)}$.



(a) R_1 and R_2 for $a_2 \le 2$ (b) R_1 for $a_2 > 2$ (c) R_2 for $a_2 > 2$

(d) R_3 (e) R_4 (f) R_5 (g) R_6

Figure 6.2: The shaded regions satisfy $\mathbf{1}\{\bar{t} \le z\} = 1$.

We can aggregate the formulas for regions R_1 and R_2, R_3 and R_4, and R_5 and R_6 in order to eliminate the parameter a_1. It follows

$$\int_{f\in R_1} + \int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le a_2 \text{ and } a_2 \le 2, \\ \frac{z-a_2}{z-1} & \text{if } z > a_2 \text{ and } a_2 \le 2, \end{cases}$$

$$\int_{f\in R_1} + \int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le \frac{a_2}{a_2-1} \text{ and } a_2 > 2, \\ 1 - \frac{1}{(a_2-1)(z-1)} & \text{if } z > \frac{a_2}{a_2-1} \text{ and } a_2 > 2, \end{cases}$$

$$\int_{f\in R_3} + \int_{f\in R_4} = \begin{cases} 0 & \text{if } z \le a_2, \\ \frac{(z-a_2)(z+a_2-2)}{2(a_2-1)(z-1)^2} & \text{if } z > a_2, \end{cases}$$

$$\int_{f\in R_5} + \int_{f\in R_6} = \begin{cases} 0 & \text{if } z \le \frac{a_2}{a_2-1}, \\ \frac{a_2-1}{2}\left(1 - \frac{1}{(a_2-1)^2(z-1)^2}\right) & \text{if } z > \frac{a_2}{a_2-1}. \end{cases}$$

We are now ready to state our results in terms of the lattice width. Reinterpreting the formulas above we obtain for $a_2 \le 2$ (i.e. $w = a_2$) the integrals

$$\int_{f\in R_1} + \int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le w, \\ \frac{z-w}{z-1} & \text{if } z > w, \end{cases}$$

$$\int_{f\in R_3} + \int_{f\in R_4} = \begin{cases} 0 & \text{if } z \le w, \\ \frac{(z-w)(z+w-2)}{2(w-1)(z-1)^2} & \text{if } z > w, \end{cases}$$

$$\int_{f\in R_5} + \int_{f\in R_6} = \begin{cases} 0 & \text{if } z \le \frac{w}{w-1}, \\ \frac{(w-1)^2(z-1)^2-1}{2(w-1)(z-1)^2} & \text{if } z > \frac{w}{w-1}. \end{cases}$$

For $a_2 > 2$ (i.e. $w = \frac{a_2}{a_2-1}$) we have

$$\int_{f\in R_1} + \int_{f\in R_2} = \begin{cases} 0 & \text{if } z \le w, \\ \frac{z-w}{z-1} & \text{if } z > w, \end{cases}$$

$$\int_{f\in R_3} + \int_{f\in R_4} = \begin{cases} 0 & \text{if } z \le \frac{w}{w-1}, \\ \frac{(w-1)^2(z-1)^2-1}{2(w-1)(z-1)^2} & \text{if } z > \frac{w}{w-1}, \end{cases}$$

$$\int_{f\in R_5} + \int_{f\in R_6} = \begin{cases} 0 & \text{if } z \le w, \\ \frac{(z-w)(z+w-2)}{2(w-1)(z-1)^2} & \text{if } z > w. \end{cases}$$

We note that any triangle of type 2 satisfies $1 < w \le 2$ (see Remark 5.20 on p. 57). Furthermore, it is easy to check that $w \le \frac{w}{w-1} \Leftrightarrow w \le 2$ for any $w > 1$. Hence, together with (6.5) and (6.7), we arrive at the following formula.

$$P^{T_2}(z) \geq \begin{cases} 0 & \text{if } 1 < z \leq w, \\ g_1 & \text{if } w < z \leq \frac{w}{w-1}, \\ g_1 + g_2 & \text{if } \frac{w}{w-1} < z < +\infty, \end{cases} \tag{6.9}$$

with $g_1 = \frac{(z-w)(2wz-w-z)}{w^2(z-1)^2}$ and $g_2 = \frac{(w-1)^2(z-1)^2-1}{w^2(z-1)^2}$.

Let $z > 1$ be given. We want to show that the probability to improve upon the split closure by a value of more than z when adding a type 2 triangle becomes the smaller the closer the type 2 triangle is to a split, i.e. the closer w is to 1. For w being sufficiently close to 1 we have $w < z \leq \frac{w}{w-1}$. Just substitute 1 for w in (6.9) to infer that $P^{T_2}(z)$ converges to 1 for any $z > 1$. Therefore, $1 - P^{T_2}(z)$ tends to 0. In other words, the chance that adding a flat type 2 triangle improves the split closure by a value of more than z with respect to our strength measure tends to 0. In terms of the vertex (a_1, a_2) this happens if a_2 converges either to 1 (i.e. T_2 converges to S_1) or infinity (i.e. T_2 converges to $S_2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < 0\}$).

Two special cases are of interest: $z = 2$ and $z = \frac{3}{2}$. Plugging in $z = 2$ in (6.9) yields

$$1 - P^{T_2}(2) \leq \frac{4(w-1)^2}{w^2}$$

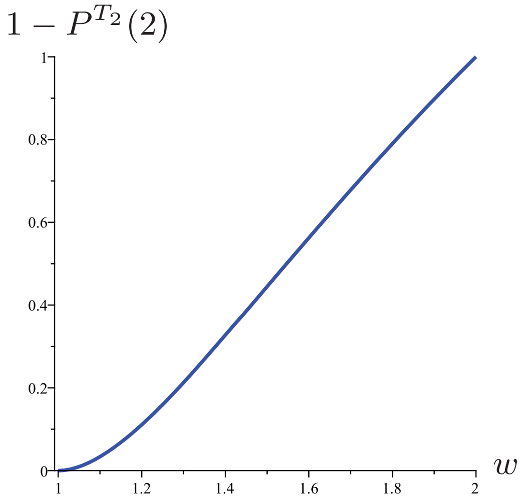
for any $1 < w \leq 2$ (see Fig. 6.3(a)) and plugging in $z = \frac{3}{2}$ in (6.9) leads to

$$P^{T_2}\left(\frac{3}{2}\right) \geq \begin{cases} 0 & \text{if } \frac{3}{2} \leq w \leq 2, \\ \frac{(3-2w)(4w-3)}{w^2} & \text{if } 1 < w < \frac{3}{2} \end{cases}$$

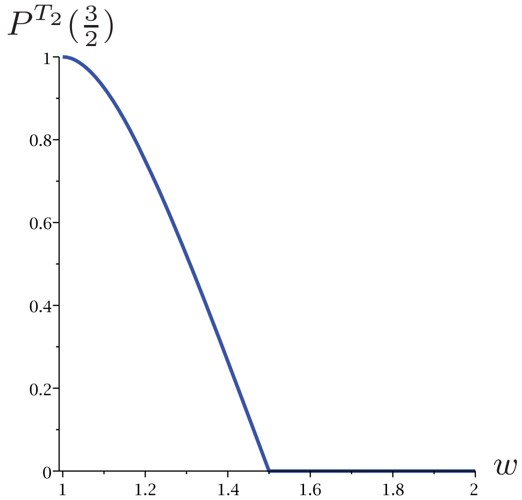
(see Fig. 6.3(b)).

$1 - P^{T_2}(2)$ is interpretable as the probability that the cut associated with T_2 closes more of the gap to R_f^3 than the cut associated with any type 1 triangle. On the other hand, $P^{T_2}(\frac{3}{2})$ can be seen as the probability that adding T_2 to the split closure is inferior to adding any type 1 triangle T_1 with rays going through the corners of T_1. The interpretation is based on the fact that any type 1 triangle (assuming corner rays) has a strength between $\frac{3}{2}$ and 2 (see (6.3) on p. 76).

In summary, we conclude that type 1 triangles are inferior to flat type 2 triangles by comparing the worst case strength (see [BBCM11, Theorems 6.1 and 8.6]). On the other hand, a type 1 triangle (assuming corner rays) guarantees a strength of $\frac{3}{2}$ and is therefore superior to a flat type 2 triangle on average. For instance, let T_2 be a type 2 triangle with $w(T_2) = 1.1$. Then $1-P^{T_2}(2) < 3.4\%$ and $P^{T_2}(\frac{3}{2}) > 92.5\%$. Thus, with a probability of less than



(a) An upper bound on $1 - P^{T_2}(2)$.



(b) A lower bound on $P^{T_2}(\frac{3}{2})$.

Figure 6.3: Bounds on $1 - P^{T_2}(2)$ and $P^{T_2}(\frac{3}{2})$.

3.4%, T_2 is better than any type 1 triangle T_1, and worse with a probability of more than 92.5% if corner rays are assumed for T_1. We point out that these probabilities are quite close to 0 and 1, respectively, even though we used just one split instead of the entire split closure. Furthermore, we emphasize that the above discussion is based on the fact that our proof technique is only appropriate to prove weakness of an inequality. Therefore, we only look at the contribution of flat type 2 triangles. For instance, the upper bound in Fig.6.3(a) is rather weak for values of w which are not sufficiently close to 1, but since our argumentation is against flat type 2 triangles, we obtain a quite strong bound.

6.6 Quadrilaterals

The vertices of the quadrilateral Q are denoted by $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$, and $d = (d_1, d_2)$. By a unimodular transformation, we assume that the point $(0, 0)$ (resp. $(1, 0)$, $(0, 1)$, $(1, 1)$) is in the relative interior of the facet with vertices b and c (resp. b and d, a and c, a and d). This implies $1 < a_2$ and $b_2 < 0$. We further assume $0 < a_1 \leq b_1 < 1$ and $-b_2 \leq a_2 - 1$ as otherwise Q can be flipped vertically or horizontally with respect to the line $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \frac{1}{2}\}$ or $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \frac{1}{2}\}$ (which is a unimodular transformation). The parameters a_1, a_2, b_1, and b_2 are assumed to be arbitrary but fixed. This implies

$$c_1 = -\frac{a_1 b_1}{(a_2-1)b_1 - a_1 b_2}, \qquad c_2 = \frac{c_1 b_2}{b_1},$$
$$d_1 = \frac{(a_2-a_1)(1-b_1)-(1-a_1)b_2}{(a_2-1)(1-b_1)-(1-a_1)b_2}, \qquad d_2 = \frac{(d_1-1)b_2}{b_1-1}.$$

One easily verifies $c_1 < 0$, $0 < c_2 < 1$, $1 < d_1$, and $0 < d_2 < 1$. Furthermore, direct computations show that $c_2 \le d_2$ and $A(Q) = \frac{1}{2}(a_2 - b_2 + d_1 - c_1)$. It follows from Remark 5.21 (see p. 57) that $w := w(Q) = \min\{a_2 - b_2, d_1 - c_1\}$. Thus, without loss of generality we assume $w = a_2 - b_2$. We decompose Q into four regions (see Fig. 6.4):

$$\begin{aligned}
R_1 &:= \operatorname{int}(Q \cap \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_2 \le \eta\}),\\
R_2 &:= \operatorname{int}(Q \cap \{(x_1,x_2) \in \mathbb{R}^2 : \eta \le x_2 \le 1\}),\\
R_3 &:= \operatorname{int}((Q \setminus \{R_1 \cup R_2\}) \cap \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le \theta\}),\\
R_4 &:= \operatorname{int}((Q \setminus \{R_1 \cup R_2\}) \cap \{(x_1,x_2) \in \mathbb{R}^2 : \theta \le x_1 \le 1\}),
\end{aligned}$$

where $\eta := \frac{-b_2}{w-1}$ and $\theta := \frac{a_1 b_1 (a_2-1)(1-b_1) - b_1 a_1 (1-a_1) b_2}{b_1 (a_2-1)(1-b_1) - a_1 (1-a_1) b_2}$. It is tedious but easy to verify that $c_2 \le \eta \le d_2$ and $a_1 \le \theta \le b_1$.

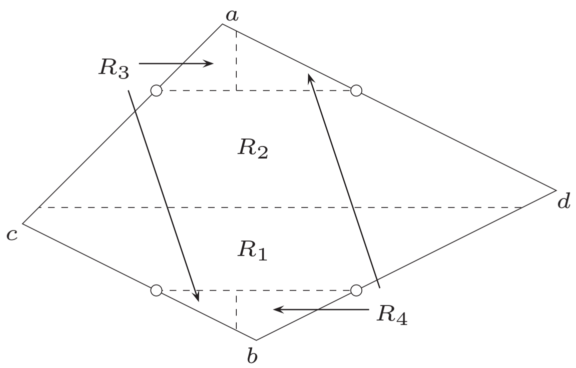


Figure 6.4: Decomposition of a quadrilateral.

As in the case of a type 2 triangle, instead of taking the entire split closure, we use a single well-chosen split inequality. In regions R_1 and R_2 we use the split $S_1 := \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_2 \le 1\}$ and in regions R_3 and R_4 we use the split $S_2 := \{(x_1,x_2) \in \mathbb{R}^2 : 0 \le x_1 \le 1\}$. Since the computations of $\bar{t}(Q,S_1)$ for $f \in R_1 \cup R_2$ and $\bar{t}(Q,S_2)$ for $f \in R_3 \cup R_4$ are straightforward (see Subsections 6.5.1 – 6.5.3 for an illustration on how to do that for a triangle of type 2) we only state the results:

$\bar{t}(Q, S_i)$	Location of f
$\frac{f_2-b_2}{f_2}$	$f \in R_1$
$\frac{a_2-f_2}{1-f_2}$	$f \in R_2$
$\frac{f_1-c_1}{f_1}$	$f \in R_3$
$\frac{d_1-f_1}{1-f_1}$	$f \in R_4$

As in Subsection 6.5.4, we now compute the integrals in order to obtain a lower bound on $P^Q(z)$. For simplicity, let $\int_{f \in R_j} := \int_{f \in R_j} \mathbf{1}\{\bar{t} \le z\} df$ for $j = 1, \ldots, 4$.

6.6.1 Regions R_1 and R_2

Let $f \in R_1$. Then $\mathbf{1}\{\bar{t} \le z\} = 1 \Leftrightarrow \frac{f_2-b_2}{f_2} \le z \Leftrightarrow f_2 \ge \frac{-b_2}{z-1}$. Hence, it follows that $\int_{f \in R_1} = 0$ if $\frac{-b_2}{w-1} = \eta \le \frac{-b_2}{z-1} \Leftrightarrow z \le w$. If $z > w$, then we distinguish into $\frac{-b_2}{z-1} \ge c_2$ and $\frac{-b_2}{z-1} < c_2$. If $\frac{-b_2}{z-1} \ge c_2$, then the area to compute is a trapezoid with area A_1. If $\frac{-b_2}{z-1} < c_2$, then the area to compute is composed of two trapezoids with aggregate area A_2. Since the computation of A_1 and A_2 is on an exercise level, we only state the results:

$$\begin{aligned} A_1 &= \frac{1}{2}\left(\frac{-b_2}{w-1} - \frac{-b_2}{z-1}\right)\left(\frac{w-(b_1-a_1)}{w-1} + \frac{z-b_1}{z-1} + \frac{a_1(z-1+b_2)}{(a_2-1)(z-1)}\right), \\ A_2 &= \frac{1}{2}\left(\frac{-b_2}{w-1} - c_2\right)\left(\frac{w-(b_1-a_1)}{w-1} + \frac{a_1(b_2-1)-(a_2-1)b_1}{a_1b_2-(a_2-1)b_1}\right) \\ &\quad + \frac{1}{2}\left(c_2 - \frac{-b_2}{z-1}\right)\left(\frac{z}{z-1} + \frac{a_1(b_2-1)-(a_2-1)b_1}{a_1b_2-(a_2-1)b_1}\right). \end{aligned}$$

We obtain

$$\int_{f \in R_1} = \begin{cases} 0 & \text{if } 1 < z \le w, \\ A_1 & \text{if } w < z \le \frac{c_2-b_2}{c_2}, \\ A_2 & \text{if } \frac{c_2-b_2}{c_2} < z < +\infty. \end{cases}$$

Let $f \in R_2$. Then $\mathbf{1}\{\bar{t} \le z\} = 1 \Leftrightarrow \frac{a_2-f_2}{1-f_2} \le z \Leftrightarrow f_2 \le \frac{z-a_2}{z-1}$. Thus, $\int_{f \in R_2} = 0$ if $\frac{z-a_2}{z-1} \le \eta = \frac{-b_2}{w-1} \Leftrightarrow z \le w$. Otherwise, we distinguish into $\frac{z-a_2}{z-1} \le d_2$ and $\frac{z-a_2}{z-1} > d_2$. In the first case the area to compute is a trapezoid with area A_3. In the second case the area to compute is composed of two trapezoids with aggregate area A_4, where

$$\begin{aligned} A_3 = \frac{1}{2}\left(\frac{z-a_2}{z-1} - \frac{-b_2}{w-1}\right) \cdot \\ \left(\frac{w-(b_1-a_1)}{w-1} + \frac{z-1+a_1}{z-1} + \frac{(z-a_2)(b_1-1)}{b_2(z-1)}\right), \end{aligned}$$

$$A_4 = \frac{1}{2}\left(\frac{z-a_2}{z-1} - d_2\right)\left(\frac{z}{z-1} + \frac{a_2(1-b_1)-(1-a_1)b_2}{(a_2-1)(1-b_1)-(1-a_1)b_2}\right)$$
$$+\frac{1}{2}\left(d_2 - \frac{-b_2}{w-1}\right)\cdot$$
$$\left(\frac{w-(b_1-a_1)}{w-1} + \frac{a_2(1-b_1)-(1-a_1)b_2}{(a_2-1)(1-b_1)-(1-a_1)b_2}\right).$$

We infer

$$\int_{f\in R_2} = \begin{cases} 0 & \text{if } 1 < z \le w, \\ A_3 & \text{if } w < z \le \frac{a_2-d_2}{1-d_2}, \\ A_4 & \text{if } \frac{a_2-d_2}{1-d_2} < z < +\infty. \end{cases}$$

6.6.2 Regions R_3 and R_4

Applying the same procedure as in the previous subsection leads to

$$A_5 = \frac{1}{2}\left(\frac{-c_1}{d_1-c_1-1} - \frac{-c_1}{z-1}\right)\left(\frac{(a_2-1)(d_1-1)}{(1-a_1)(d_1-c_1-1)} + \right.$$
$$\left.\frac{(a_2-1)(z-1+c_1)}{(1-a_1)(z-1)} + \frac{c_2}{d_1-c_1-1} + \frac{c_2}{z-1}\right),$$
$$A_6 = \frac{1}{2}\left(\frac{-c_1}{d_1-c_1-1} - a_1\right)\cdot$$
$$\left(\frac{(a_2-1)(2-a_1)}{1-a_1} - \frac{a_1b_2}{b_1} + \frac{c_1(a_2-1)+c_2(1-a_1)}{(1-a_1)(d_1-c_1-1)}\right)$$
$$+\frac{1}{2}\left(a_1 - \frac{-c_1}{z-1}\right)\left(a_2 - 1 - \frac{a_1b_2}{b_1} + \frac{a_1c_2 - c_1(a_2-1)}{a_1(z-1)}\right),$$
$$A_7 = \frac{1}{2}\cdot\frac{(d_1-1)(z-d_1+c_1)}{(z-1)(d_1-c_1-1)}\cdot$$
$$\left(\frac{c_2(1-a_1)+(a_2-1)(d_1-1)}{(1-a_1)(d_1-c_1-1)} + \frac{(a_2-1)(d_1-1)}{(1-a_1)(z-1)} - \frac{b_2(z-d_1)}{b_1(z-1)}\right),$$
$$A_8 = \frac{1}{2}\left(b_1 - \frac{-c_1}{d_1-c_1-1}\right)\left(\frac{c_2(1-a_1)+c_1(a_2-1)}{(1-a_1)(d_1-c_1-1)} + \right.$$
$$\left.\frac{(a_2-1)(2-b_1)-b_2(1-a_1)}{1-a_1}\right) + \frac{1}{2}\left(\frac{z-d_1}{z-1} - b_1\right)\cdot$$
$$\left(\frac{(a_2-1)(z-d_1)}{(a_1-1)(z-1)} - \frac{b_2(d_1-1)}{(1-b_1)(z-1)} + \frac{(a_2-1)(2-b_1)-b_2(1-a_1)}{1-a_1}\right),$$

and finally

$$\int\limits_{f\in R_3} = \begin{cases} 0 & \text{if } 1 < z \le d_1 - c_1, \\ A_5 & \text{if } d_1 - c_1 < z \le \frac{a_1 - c_1}{a_1}, \\ A_6 & \text{if } \frac{a_1 - c_1}{a_1} < z < +\infty, \end{cases}$$

$$\int\limits_{f\in R_4} = \begin{cases} 0 & \text{if } 1 < z \le d_1 - c_1, \\ A_7 & \text{if } d_1 - c_1 < z \le \frac{d_1 - b_1}{1 - b_1}, \\ A_8 & \text{if } \frac{d_1 - b_1}{1 - b_1} < z < +\infty. \end{cases}$$

6.6.3 Approximation for $P^Q(z)$

Adding the integrals and dividing by the area of the quadrilateral gives the lower bound on $P^Q(z)$ which we wanted. Thus,

$$P^Q(z) \ge \frac{1}{A(Q)} \sum_{j=1}^{4} \int\limits_{f\in R_j} \mathbf{1}\{\bar{t} \le z\} df. \tag{6.10}$$

We did not succeed in showing algebraically that this lower bound tends to 1 when w converges to 1. However, we performed simulations supporting our conjecture. In these simulations we let w converge to 1 for several values for a_1, a_2, b_1, and b_2. Concretely, we discretized the parameters within their ranges: $a_1 \in \{0.001, 0.002, \ldots, 0.999\}$, $b_1 \in \{a_1, a_1 + 0.001, \ldots, 0.999\}$, $a_2 \in \{1.999, 1.998, \ldots, 1.001\}$, and $b_2 \in \{-(a_2-1), -(a_2-1)+0.001, \ldots, -0.001\}$. In all cases we observed that the lower bound became close to one. This is, of course, not a proof, but it gives strong indication that the convergence property should hold in general.

We point out that under the additional assumption $a_1 = b_1$ (implying $a_1 = \theta = b_1$ and $c_2 = \eta = d_2$) the lower bound in (6.10) coincides with the lower bound of a type 2 triangle which is stated in (6.9). This can be seen by simply substituting a_1 for b_1 in the formulas for $A_1, \ldots, A_8$.

Corollary 6.3. *Let $Q^=$ be a quadrilateral meeting the assumptions stated at the beginning of this section (see p. 85) which, in addition, satisfies $a_1 = b_1$. Moreover, let $w := w(Q^=)$. Then*

$$P^{Q^=}(z) \ge \begin{cases} 0 & \text{if } 1 < z \le w, \\ \frac{(z-w)(2wz-w-z)}{w^2(z-1)^2} & \text{if } w < z \le \frac{w}{w-1}, \\ \frac{(z-w)(2wz-w-z)+(w-1)^2(z-1)^2-1}{w^2(z-1)^2} & \text{if } \frac{w}{w-1} < z < +\infty. \end{cases}$$

Moreover, for any $z > 1$, $P^{Q^=}(z)$ tends to 1 if w converges to 1.

6.7 Type 3 triangles

By a unimodular transformation, we assume that the type 3 triangle T_3 satisfies $T_3 \cap \mathbb{Z}^2 = \{(0,0),(1,0),(0,1)\}$ and that each facet of T_3 contains one of these points in its relative interior. The three vertices of T_3 are denoted by $a=(a_1,a_2)$, $b=(b_1,b_2)$, and $c=(c_1,c_2)$. We further assume $1<a_1$, $0<a_2<1$, $0<b_1<1$, and $b_1+b_2<0$. Let a_1, a_2, and b_1 be arbitrary but fixed. This implies

$$\begin{aligned} b_2 &= -\frac{a_2(1-b_1)}{a_1-1}, \\ c_1 &= \frac{a_1(a_1-1)b_1}{(a_1-1)(1-a_2)b_1 - a_1a_2(1-b_1)}, \\ c_2 &= -\frac{a_1a_2(1-b_1)}{(a_1-1)(1-a_2)b_1 - a_1a_2(1-b_1)}. \end{aligned}$$

One easily verifies $b_2<0$, $c_1<0$, $1<c_2$, and $0<c_1+c_2<1$. Furthermore, we have $A(T_3)=\frac{1}{2}(a_1+a_2-b_2-c_1)$. It follows from Remark 5.17 (see p. 54) that $w := w(T_3) = \min\{c_2-b_2, a_1-c_1, a_1+a_2-(b_1+b_2)\}$. Without loss of generality we assume $w = c_2-b_2 \le a_1-c_1 \le a_1+a_2-(b_1+b_2)$. During this section we consider the three splits $S_1 := \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_2\le 1\}$, $S_2 := \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1\le 1\}$, and $S_3 := \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1+x_2\le 1\}$. We decompose T_3 into six regions (see Fig. 6.5):

$$\begin{aligned} R_1 &:= \operatorname{int}(T_3 \cap \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_2\le\alpha\}), \\ R_2 &:= \operatorname{int}(T_3 \cap \{(x_1,x_2)\in\mathbb{R}^2 : \alpha\le x_2\le 1\}), \\ R_3 &:= \operatorname{int}((T_3 \setminus \{R_1\cup R_2\}) \cap \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1\le\beta\}), \\ R_4 &:= \operatorname{int}((T_3 \setminus \{R_1\cup R_2\}) \cap \{(x_1,x_2)\in\mathbb{R}^2 : \beta\le x_1\le 1\}), \\ R_5 &:= \operatorname{int}((T_3 \setminus \cup_{j=1}^4 R_j) \cap \{(x_1,x_2)\in\mathbb{R}^2 : 0\le x_1+x_2\le\gamma\}), \\ R_6 &:= \operatorname{int}((T_3 \setminus \cup_{j=1}^4 R_j) \cap \{(x_1,x_2)\in\mathbb{R}^2 : \gamma\le x_1+x_2\le 1\}), \end{aligned}$$

where $\alpha := \frac{-b_2}{c_2-1-b_2}$, $\beta := \frac{-c_1}{a_1-1-c_1}$, and $\gamma := \frac{-(b_1+b_2)}{a_1+a_2-1-(b_1+b_2)}$. We point out that R_5 could be empty. It is tedious but easy to verify that $a_2<\alpha<1$, $b_1<\beta<1$, and $0<\gamma<c_1+c_2$.

For each region, we use a single split inequality to approximate the split closure. In regions R_1 and R_2 we use the split S_1, in regions R_3 and R_4 we use the split S_2, and in regions R_5 and R_6 we use the split S_3. Thus, in each region R_j we choose a split which covers R_j and the convex hull of $(0,0)$, $(1,0)$, and $(0,1)$. The following table states the values for $\bar{t}(T_3,S_i)$ for the regions R_1 to R_6 and the corresponding $i\in\{1,2,3\}$.

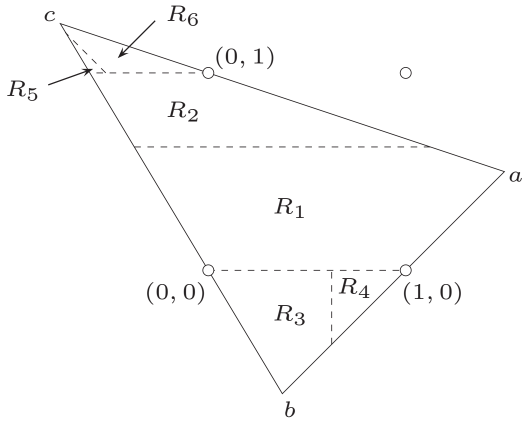


Figure 6.5: Decomposition of a type 3 triangle.

$\bar{t}(T_3, S_i)$	Location of f
$\frac{f_2-b_2}{f_2}$	$f \in R_1$
$\frac{c_2-f_2}{1-f_2}$	$f \in R_2$
$\frac{f_1-c_1}{f_1}$	$f \in R_3$
$\frac{a_1-f_1}{1-f_1}$	$f \in R_4$
$\frac{f_1+f_2-(b_1+b_2)}{f_1+f_2}$	$f \in R_5$
$\frac{a_1+a_2-(f_1+f_2)}{1-(f_1+f_2)}$	$f \in R_6$

The computation of the integrals $\int_{f \in R_j} \mathbf{1}\{\bar{t} \leq z\} df$ for $j = 1, \ldots, 6$ gives no new insights. Hence, we only state the results. For simplicity, let $\int_{f \in R_j} := \int_{f \in R_j} \mathbf{1}\{\bar{t} \leq z\} df$ for $j = 1, \ldots, 6$. Let

$$\begin{aligned}
A_1 &= \frac{1}{2}\left(\frac{-b_2}{w-1} - \frac{-b_2}{z-1}\right) \cdot \\
&\quad \left(\frac{b_1}{w-1} + \frac{b_1}{z-1} + \frac{a_1}{1-a_2}\left(\frac{c_2-1}{w-1} + \frac{z-1+b_2}{z-1}\right)\right), \\
A_2 &= \frac{1}{2}\left(\frac{-b_2}{w-1} - a_2\right)\left(\frac{(1-a_2)b_1 + a_1(c_2-1)}{(1-a_2)(w-1)} - \frac{a_2b_1 - a_1b_2)}{b_2}\right) \\
&\quad + \frac{1}{2}\left(a_2 - \frac{-b_2}{z-1}\right)\left(\frac{a_2b_1 - (a_1-1)b_2}{a_2(z-1)} - \frac{a_2b_1 - (a_1+1)b_2)}{b_2}\right), \\
A_3 &= \frac{1}{2}\left(\frac{z-c_2}{z-1} - \frac{-b_2}{w-1}\right) \cdot \\
&\quad \left(\frac{b_1}{w-1} - \frac{b_1(z-c_2)}{b_2(z-1)} + \frac{a_1}{1-a_2}\left(\frac{c_2-1}{w-1} + \frac{c_2-1}{z-1}\right)\right),
\end{aligned}$$

$$
\begin{aligned}
A_4 &= \frac{1}{2}\left(\frac{-c_1}{a_1-c_1-1}-\frac{-c_1}{z-1}\right)\left(\frac{a_2}{a_1-c_1-1}+\frac{a_2(z-1+c_1)}{(a_1-1)(z-1)}\right),\\
A_5 &= \frac{1}{2}\cdot\frac{a_2}{b_1(a_1-1)}\cdot\left(\frac{b_1(z-1)+c_1}{z-1}\right)^2,\\
A_6 &= \frac{1}{2}\left(\frac{z-a_1}{z-1}-\frac{-c_1}{a_1-c_1-1}\right)\left(\frac{a_2}{a_1-c_1-1}+\frac{a_2}{z-1}\right),\\
A_7 &= \frac{1}{2}\cdot\left(\frac{-b_2}{a_1+a_2-(b_1+b_2)-1}-1\right)^2,\\
A_8 &= \frac{1}{2}\left(\frac{b_1}{a_1+a_2-(b_1+b_2)-1}+\frac{b_1}{b_2}\right)\left(\frac{-b_2}{a_1+a_2-(b_1+b_2)-1}-1\right),\\
A_9 &= \frac{1}{2}\left(-\frac{b_1+b_2}{b_2}-\frac{b_1+b_2}{z-1}\right)\left(\frac{-b_2}{z-1}-1\right),\\
A_{10} &= \frac{1}{2}\cdot\left(\frac{b_2(z-(a_1+a_2))}{(b_1+b_2)(z-1)}-1\right)^2,\\
A_{11} &= \frac{1}{2}\left(\frac{-b_1(z-(a_1+a_2))}{(b_1+b_2)(z-1)}+\frac{b_1}{b_2}\right)\left(\frac{b_2(z-(a_1+a_2))}{(b_1+b_2)(z-1)}-1\right),\\
A_{12} &= \frac{1}{2}\left(-\frac{b_1}{b_2}-\frac{a_1+a_2-1}{a_1+a_2-(b_1+b_2)-1}\right)\left(\frac{-b_2}{a_1+a_2-(b_1+b_2)-1}-1\right),\\
A_{13} &= \frac{1}{2}\cdot(c_2-1)^2,\\
A_{14} &= \frac{1}{2}\cdot\left(\frac{b_1}{b_2}-c_1\right)(c_2-1),\\
A_{15} &= \frac{1}{2}\left(-\frac{a_1+a_2-1}{a_1+a_2-(b_1+b_2)-1}-\frac{b_1}{b_2}\right)\left(\frac{-b_2}{a_1+a_2-(b_1+b_2)-1}-1\right),\\
A_{16} &= \frac{1}{2}\left(1-(c_1+c_2)-\frac{a_1+a_2-1}{z-1}\right)\left(c_2-\frac{z-a_2}{z-1}\right),\\
A_{17} &= \frac{1-a_2}{z-1}\left(1-(c_1+c_2)-\frac{a_1+a_2-1}{z-1}\right).
\end{aligned}
$$

We obtain

$$
\int_{f\in R_1}+\int_{f\in R_2}=\begin{cases}0 & \text{if } 1<z\le w,\\ A_1+A_3 & \text{if } w<z\le \frac{a_2-b_2}{a_2},\\ A_2+A_3 & \text{if } \frac{a_2-b_2}{a_2}<z<+\infty,\end{cases}
$$

$$\int_{f\in R_3} + \int_{f\in R_4} = \begin{cases} 0 & \text{if } 1 < z \le a_1 - c_1, \\ A_4 + A_6 & \text{if } a_1 - c_1 < z \le \frac{b_1 - c_1}{b_1}, \\ A_4 - A_5 + A_6 & \text{if } \frac{b_1 - c_1}{b_1} < z < +\infty, \end{cases}$$

$$\int_{f\in R_5} = \begin{cases} 0 & \text{if } 1 < z < a_1 + a_2 - (b_1 + b_2) \\ & \quad \text{or } a_1 + a_2 - b_1 - 1 > 0, \\ A_7 - A_8 - A_9 & \text{if } a_1 + a_2 - (b_1 + b_2) \le z \le 1 - b_2 \\ & \quad \text{and } a_1 + a_2 - b_1 - 1 \le 0, \\ A_7 - A_8 & \text{if } 1 - b_2 < z < +\infty \\ & \quad \text{and } a_1 + a_2 - b_1 - 1 \le 0, \end{cases}$$

$$\int_{f\in R_6} = \begin{cases} 0 & \text{if } 1 < z < a_1 + a_2 - (b_1 + b_2), \\ A_{10} - A_{11} - A_{12} & \text{if } a_1 + a_2 - (b_1 + b_2) \le \\ & \quad z \le \frac{a_1 + a_2 - (c_1 + c_2)}{1 - (c_1 + c_2)} \text{ and} \\ & \quad a_1 + a_2 - b_1 - 1 \le 0, \\ A_{10} - A_{11} & \text{if } a_1 + a_2 - (b_1 + b_2) \le \\ & \quad z \le \frac{a_1 + a_2 - (c_1 + c_2)}{1 - (c_1 + c_2)} \text{ and} \\ & \quad a_1 + a_2 - b_1 - 1 > 0, \\ A_{13} - A_{14} - A_{15} + A_{16} + A_{17} & \text{if } \frac{a_1 + a_2 - (c_1 + c_2)}{1 - (c_1 + c_2)} < z < +\infty \\ & \quad \text{and } a_1 + a_2 - b_1 - 1 \le 0, \\ A_{13} - A_{14} + A_{16} + A_{17} & \text{if } \frac{a_1 + a_2 - (c_1 + c_2)}{1 - (c_1 + c_2)} < z < +\infty \\ & \quad \text{and } a_1 + a_2 - b_1 - 1 > 0. \end{cases}$$

This leads to the lower bound on $P^{T_3}(z)$ which we wanted, namely

$$P^{T_3}(z) \ge \frac{1}{A(T_3)} \sum_{j=1}^{6} \int_{f\in R_j} \mathbf{1}\{\bar{t} \le z\} df.$$

As for quadrilaterals, we did not succeed in showing algebraically the convergence of this lower bound to 1 when w converges to 1. However, simulations with discretized parameters $a_1 \in \{3, 4, \ldots, 1\,000\,000\}$, $a_2 \in \{0.001, 0.002, \ldots, 0.999\}$, and $b_1 \in \{0.001, 0.002, \ldots, 0.999\}$ (where $b_1 < \frac{a_2}{a_1 + a_2 - 1}$) such that w converges to 1 suggest that this is the case.

CHAPTER 7

ON FINITENESS OF LATTICE-FREE POLYHEDRA

In the previous three chapters, we focused on the evaluation of cutting planes with a particular emphasis on dimension two. Now, we return to the relation between cutting planes and lattice-free polyhedra.

Our point of departure is Lemma 3.3 (see p. 15) in which we assign to every valid inequality for $\operatorname{conv}(P_I)$ a lattice-free polyhedron. In particular, we show that every non-trivial facet-defining inequality for $\operatorname{conv}(P_I)$ can be derived from a lattice-free rational polyhedron having the root vertex f in its interior (see Assumption 3.5 on p. 17 and the preceding paragraph). In this chapter, we want to analyze those polyhedra which correspond to valid inequalities for $\operatorname{conv}(P_I)$ that are "important" in a cutting plane framework. We consider a class $\mathcal{Z}$ of polyhedra to be "important" if the cuts associated with the polyhedra in $\mathcal{Z}$ ensure that a cutting plane algorithm which is based on these cuts stops after a finite number of steps with an optimal mixed-integer point[1]. We recall that the precision of a rational polyhedron P is the smallest natural number s such that the set $sP = \{sx : x \in P\}$ is an integral polyhedron. If we fix a dimension d and a precision s, then our main result in this chapter is that, up to unimodular transformations, the number of lattice-free rational polyhedra of precision s which are not properly contained in another lattice-free rational polyhedron of precision s

[1] By assumption, the feasible region of our underlying MILP (1.1) (see p. 1) is non-empty and its linear programming relaxation is bounded. Thus, there is indeed a (bounded) mixed-integer solution.

is finite. The special case $s = 1$ (meaning lattice-free integral polyhedra) is important for cutting plane generation. To explain why, let us briefly review the main results from the literature concerning the relation between mixed-integer cutting plane theory and lattice-free polyhedra.

The study of lattice-free polyhedra was motivated by research in mixed-integer linear optimization in the 1970s. Balas [Bal71] ignited the application of lattice-free convex sets for cutting plane generation. Originally, he used a hypersphere S passing through the vertices of a translate of the unit hypercube which contains an optimal vertex x^* of the linear programming relaxation. In order to determine the coefficients of a cut, he computed the intersection of S with certain half-lines emanating from x^*. He obtained these half-lines from the simplex tableau associated with x^* (in our notation x^* corresponds to f and the half-lines are the rays r^j). Balas mentioned that, in principle, any lattice-free closed convex set with x^* in its interior could be employed. Thus, given the way the cuts are constructed, it suffices to consider lattice-free closed convex sets with non-empty interior which are not properly contained in another lattice-free closed convex set. In other words, one can restrict to maximal lattice-free sets in $\mathcal{K}^d$ for some dimension d. It was Lovász [Lov89] who characterized these sets first, in particular he showed that every such set is a polyhedron (see [Lov89, Proposition 3.2]). In view of the algorithmic applications that we have in mind, we further restrict our attention to rational polyhedra, i.e. we consider maximal lattice-free rational polyhedra in $\mathcal{K}^d$ for some dimension d. We refer to Proposition 2.1 on p. 12 for a summary of the most important properties of these polyhedra.

The fact that any lattice-free polyhedron with non-empty interior can be used to derive a cutting plane poses the question: which of these polyhedra should be used within a cutting plane framework to ensure finite convergence? Del Pia and Weismantel gave an answer to this question. They proved that it is enough to use only cuts associated with lattice-free integral polyhedra (see [DPW11, Theorem 4]) to find an optimal mixed-integer point in a finite number of applied rounds. Let $d \in \mathbb{N}$ and let $\mathcal{I}^d$ denote the set of all lattice-free integral polyhedra in $\mathbb{R}^d$ which are not properly contained in another lattice-free integral polyhedron[2]. From the results of Del Pia and Weismantel, it follows that only the elements in $\mathcal{I}^d$ are needed for finite convergence. Therefore, it is natural to ask for a characterization of the set $\mathcal{I}^d$.

In this chapter we shed light on a class of polyhedra which is a generalization of the class $\mathcal{I}^d$. As a corollary of the results of this chapter, we infer that $\mathcal{I}^d / \operatorname{Aff}(\mathbb{Z}^d)$, i.e. the set of distinct (meaning that no two elements coincide by

[2]We point out that this notion of inclusion-maximality is different from the notion of maximal lattice-freeness. The precise relation will be explained in more detail in the next section.

applying a unimodular transformation) elements in $\mathcal{I}^d$, is a finite set whose number of elements is bounded by a constant which is only dependent on d.

Theorem 7.1. *Let $d \in \mathbb{N}$. Then there exists a constant $C(d) \in \mathbb{N}$, depending only on d, and polyhedra $P_1, \dots, P_{C(d)} \in \mathcal{I}^d$ such that every $P \in \mathcal{I}^d$ satisfies $P = UP_j + v$ for some unimodular matrix $U \in \mathbb{Z}^{d \times d}$, integral vector $v \in \mathbb{Z}^d$, and $j \in \{1, \dots, C(d)\}$.*

In Section 7.1, we introduce the notation which we use in the course of this chapter. In particular, we define the two basic sets $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$. The finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ is shown in Section 7.2. Our finiteness proof relies on the existence of a volume bound for polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$. In Section 7.3 we investigate such a bound in more detail. Section 7.4 is devoted to a discussion of the relation between $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$.

7.1 Preliminaries and main results

We recall that $\mathcal{P}^d$ denotes the set of (not necessarily full-dimensional) polyhedra in $\mathbb{R}^d$. Furthermore, if Λ is a (full-dimensional) lattice in $\mathbb{R}^d$, then a polyhedron $P \in \mathcal{P}^d$ is called Λ-free if $\operatorname{int}(P) \cap \Lambda = \emptyset$. In this chapter, we restrict Λ to be $s\mathbb{Z}^d = \{sx : x \in \mathbb{Z}^d\}$ for some $s \in \mathbb{N}$. Let π denote the projection onto the first $d-1$ coordinates, i.e. the mapping $\pi(x) := (x_1, \dots, x_{d-1})$, where $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. This implies $\pi(\Lambda) = s\mathbb{Z}^{d-1}$. We are concerned with the interplay of the following three properties of polyhedra:

- integrality (abbreviated with "i"),
- Λ-freeness (abbreviated with "f" and an additional "s" in brackets to indicate the dependency on $\Lambda = s\mathbb{Z}^d$),
- inclusion-maximality in a given class (abbreviated with "m").

By $\mathcal{P}^d_{\mathrm{i}}$ we denote the set of integral polyhedra belonging to $\mathcal{P}^d$, by $\mathcal{P}^d_{\mathrm{if}}(s)$ the set of Λ-free polyhedra belonging to $\mathcal{P}^d_{\mathrm{i}}$, and by $\mathcal{P}^d_{\mathrm{ifm}}(s)$ the set of elements of $\mathcal{P}^d_{\mathrm{if}}(s)$ which are maximal within $\mathcal{P}^d_{\mathrm{if}}(s)$ with respect to inclusion. We are interested in polyhedra in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ which do not coincide modulo $\operatorname{Aff}(\Lambda)$, since this identification preserves affine properties relative to the lattice Λ. In particular, two polyhedra $P, Q \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ which coincide up to an affine transformation in $\operatorname{Aff}(\Lambda)$ contain the same number of lattice points in $\mathbb{Z}^d$ and Λ on corresponding faces. We are now ready to present our main result.

Theorem 7.2. *Let $d, s \in \mathbb{N}$. Then $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ is a finite set.*

Let us assume that $P \in \mathcal{P}^d$ is a maximal lattice-free rational polyhedron with non-empty interior and let the precision of P be s. Then sP is an integral polyhedron. Moreover, the maximality and lattice-freeness of P with respect to the standard lattice $\mathbb{Z}^d$ transfers one-to-one into a maximality and Λ-freeness of sP with respect to the lattice $\Lambda = s\mathbb{Z}^d$. Thus, instead of analyzing "maximal lattice-free rational polyhedra" (which correspond to valid inequalities for $\operatorname{conv}(P_I)$ when rational data is assumed) we can equivalently consider the more convenient set of "maximal Λ-free integral polyhedra". Indeed, from an analytical point of view, the latter set is easier to handle since results from the literature stated in terms of integral polyhedra can be used.

We now relate maximal Λ-free integral polyhedra to the set $\mathcal{P}^d_{\mathrm{ifm}}(s)$. Let $\mathcal{C}^d_{\mathrm{fm}}(s)$ be the class of Λ-free convex sets in $\mathbb{R}^d$ which are not properly contained in another Λ-free convex set. The elements of $\mathcal{C}^d_{\mathrm{fm}}(s)$ are polyhedra (see [Lov89, Proposition 3.2]). Thus, $\mathcal{C}^d_{\mathrm{fm}}(s)$ is the class of maximal Λ-free polyhedra in $\mathbb{R}^d$. Since $\mathcal{C}^d_{\mathrm{fm}}(s)$ is a set of polyhedra, we use a more intuitive notation and change the "$\mathcal{C}$" to "$\mathcal{P}$", i.e. we define $\mathcal{P}^d_{\mathrm{fm}}(s) := \mathcal{C}^d_{\mathrm{fm}}(s)$. Let $\mathcal{P}^d_{\mathrm{fmi}}(s) := \mathcal{P}^d_{\mathrm{i}} \cap \mathcal{P}^d_{\mathrm{fm}}(s)$ be the class of maximal Λ-free integral polyhedra in $\mathbb{R}^d$. By definition we have $\mathcal{P}^d_{\mathrm{fmi}}(s) \subseteq \mathcal{P}^d_{\mathrm{ifm}}(s)$. We point out that the order of the subscripted indices "i", "f", and "m" indicates how the classes $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$ are defined. Since the difference between the two classes $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$ is crucial for the remainder of this chapter, we emphasize that $\mathcal{P}^d_{\mathrm{fmi}}(s)$ contains the Λ-free integral polyhedra in $\mathbb{R}^d$ which are not properly contained in another Λ-free convex set, whereas $\mathcal{P}^d_{\mathrm{ifm}}(s)$ contains the Λ-free integral polyhedra in $\mathbb{R}^d$ which are not properly contained in another Λ-free integral polyhedron. In Section 7.4, we will investigate the relation between $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$ in more detail. Let us now summarize the known inclusions:

$$\begin{gathered}\mathcal{P}^d_{\mathrm{fmi}}(s) \subseteq \mathcal{P}^d_{\mathrm{ifm}}(s), \qquad (7.1)\\ \mathcal{P}^d_{\mathrm{ifm}}(s) \subseteq \mathcal{P}^d_{\mathrm{if}}(s) \subseteq \mathcal{P}^d_{\mathrm{i}} \subseteq \mathcal{P}^d,\\ \mathcal{P}^d_{\mathrm{fmi}}(s) = \mathcal{P}^d_{\mathrm{i}} \cap \mathcal{P}^d_{\mathrm{fm}}(s) \subseteq \mathcal{P}^d_{\mathrm{fm}}(s) = \mathcal{C}^d_{\mathrm{fm}}(s) \subseteq \mathcal{P}^d.\end{gathered}$$

The finiteness of the set $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$ follows directly from (7.1) and Theorem 7.2. In particular, if we choose $s = 1$, we obtain that for every dimension d, up to unimodular transformations, there is only a finite number of maximal lattice-free integral polyhedra. One more consequence of Theorem 7.2 is the following. If we fix the dimension d and choose some integer $s \geq 1$, then Theorem 7.2 implies that, up to unimodular transformations, there is only a finite number of lattice-free polytopes with vertices in $\frac{1}{s}\mathbb{Z}^d$ which are not properly contained in another lattice-free polytope with vertices in $\frac{1}{s}\mathbb{Z}^d$. In particular, Theorem 7.1 follows from Theorem 7.2 and $\mathcal{I}^d = \mathcal{P}^d_{\mathrm{ifm}}(1)$.

7.2 Proof of Theorem 7.2

In this section, we prove Theorem 7.2. If $d = 1$, then it is easy to see that $\mathcal{P}^1_{\text{fmi}}(s) = \mathcal{P}^1_{\text{ifm}}(s) = \{[k, k+s] : k \in s\mathbb{Z}\}$. Consequently, up to a Λ-preserving transformation, we have only one single polyhedron. Therefore, in the remainder of this section we assume that $d \geq 2$. We note that for every $d, s \in \mathbb{N}$ the set $\text{conv}(\{o, sde_1, \ldots, sde_d\})$ belongs to $\mathcal{P}^d_{\text{fmi}}(s)$ (and thus to $\mathcal{P}^d_{\text{ifm}}(s)$), i.e. the sets which we consider are indeed non-empty. Let us first highlight the main steps of the proof of Theorem 7.2.

1. **Reduction to polytopes.** Every unbounded $P \in \mathcal{P}^d_{\text{ifm}}(s)$ is the direct product of an affine space and a polytope in $\mathcal{P}^k_{\text{ifm}}(s)$ for some $1 \leq k \leq d$ (see Proposition 7.3). Thus, it suffices to verify finiteness only for polytopes within $\mathcal{P}^d_{\text{ifm}}(s)$.

2. **Bounding $|P \cap \Lambda|$.** Consider a polytope $P \in \mathcal{P}^d_{\text{ifm}}(s)$. We construct an upper bound on the number of points of Λ on the boundary of P. For that, we use the lattice diameter. The *lattice diameter* of P with respect to Λ is defined as the maximum of $|l \cap P \cap \Lambda| - 1$ over all lines l in $\mathbb{R}^d$. We show that the lattice diameter of P is bounded from above by a constant which is only dependent on d and s. This is done as follows.

   We assume by contradiction that, for some line l, $|l \cap P \cap \Lambda| - 1$ is a large number M. By a Λ-preserving transformation, we can choose $l = \text{lin}(\{e_d\})$. Let P' be the projection of P onto the first $d-1$ coordinates. Then $\pi(l) = o$ and from $P \in \mathcal{P}^d_{\text{ifm}}(s)$ it follows $\text{int}(P') \cap \pi(\Lambda) \neq \emptyset$ (see Lemma 7.9). Let $p \in \text{int}(P') \cap \pi(\Lambda)$ be arbitrary. We construct a k-dimensional simplex S with vertices $o = p_0, p_1, \ldots, p_k$ in $\mathbb{Z}^{d-1}$ such that p is the only point of $\pi(\Lambda)$ in the relative interior of S. This construction is the key ingredient in our proof (see Lemma 7.8). Let $\lambda_0, \ldots, \lambda_k$ be the barycentric coordinates of p with respect to S. By results of Hensley [Hen83] and Lagarias and Ziegler [LZ91] (see Theorem 7.6), the λ_i's are bounded from below by a constant which is only dependent on d and s. The length of $(p+l) \cap P$ is bounded from below in terms of λ_0 and M. More precisely, when choosing a larger M, then the lower bound on the length of $(p+l) \cap P$ also becomes larger. On the other hand, since P is Λ-free, the length of $(p+l) \cap P$ is at most s. So, if M is too large, this leads to a contradiction.

   The upper bound on the lattice diameter implies an upper bound on $|P \cap \Lambda|$ (see Lemma 7.10).

3. **Conclusion of finiteness.** The upper bound on $|P \cap \Lambda|$ together with results of Hensley [Hen83] and Lagarias and Ziegler [LZ91] (see Theo-

rem 7.7) implies an upper bound on the volume of P (see Theorem 7.5). All bounds only depend on d and s. This, in turn, yields finiteness of $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ (see Theorem 7.4).

The fact that we can restrict to the study of polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ is a consequence of the following proposition. We point out that a similar result is true for the set $\mathcal{P}^d_{\mathrm{fm}}(s)$ as well (see [Lov89, Proposition 3.1]).

Proposition 7.3. *Let $d, s \in \mathbb{N}$ and let $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$. Then there exists some $k \in \{1, \dots, d\}$ and a polytope $P' \in \mathcal{P}^k_{\mathrm{ifm}}(s)$ such that $P \equiv P' \times \mathbb{R}^{d-k}$ $(\mathrm{mod}\ \operatorname{Aff}(\Lambda))$.*

Proof. If P is bounded, the assertion is trivial as we let $k = d$ and $P' = P$. Let P be unbounded. By an inductive argument, it suffices to show the existence of a polyhedron $P' \in \mathcal{P}^{d-1}_{\mathrm{ifm}}(s)$ such that $P \equiv P' \times \mathbb{R}$ $(\mathrm{mod}\ \operatorname{Aff}(\Lambda))$.

Since P is unbounded, the recession cone of P contains non-zero vectors. Since P is integral, the recession cone of P is an integral polyhedron (see, for instance, [Sch86, Section 16.2]). Thus, the recession cone of P contains a non-zero integer vector u. By scaling, we can assume that $u \in \Lambda$.

Applying a Λ-preserving transformation, we assume that $u = t \cdot s \cdot e_d$ for some $t \in \mathbb{N}$. It follows that the polyhedron $P' := \pi(P) \subseteq \mathbb{R}^{d-1}$ is $\pi(\Lambda)$-free. In fact, assume there exists a point $p' \in \operatorname{int}(P') \cap \pi(\Lambda)$. Then $\operatorname{int}(P) \cap \pi^{-1}(p')$ is non-empty and contains infinitely many points of Λ, a contradiction to the choice of P.

Since P' is $\pi(\Lambda)$-free, $\pi^{-1}(P')$ is Λ-free. By construction, $P \subseteq \pi^{-1}(P')$, and since P is maximal in $\mathcal{P}^d_{\mathrm{if}}(s)$ we even have $P = \pi^{-1}(P')$. Furthermore, $P' \in \mathcal{P}^{d-1}_{\mathrm{ifm}}(s)$. In fact, if P' were not maximal in $\mathcal{P}^{d-1}_{\mathrm{if}}(s)$ we could find $P'' \in \mathcal{P}^{d-1}_{\mathrm{if}}(s)$ such that $P' \subsetneq P''$. Then P is properly contained in the Λ-free integral polyhedron $\pi^{-1}(P'')$, a contradiction to the assumptions on P. By construction, $P \equiv P' \times \mathbb{R}$ $(\mathrm{mod}\ \operatorname{Aff}(\Lambda))$. □

The following theorem is well-known (see, for instance, [LZ91, Theorem 2]).

Theorem 7.4. *Let $d, s \in \mathbb{N}$ and let $\mathcal{X} \subseteq \mathcal{P}^d_{\mathrm{i}}$ be a set of integral polytopes. Then the following statements are equivalent.*

(i) The set $\mathcal{X}/\operatorname{Aff}(s\mathbb{Z}^d)$ is finite.

(ii) There exists a constant $K(d, s)$, which depends only on d and s, such that for every $X \in \mathcal{X}$ it holds $\operatorname{vol}(X) \leq K(d, s)$.

Due to Theorem 7.4, it remains to show the existence of a constant $V(d, s)$, which is only dependent on d and s, and which satisfies $\operatorname{vol}(P) \leq V(d, s)$ for every polytope $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$. Thus, in the remainder of this section we prepare the proof of the following theorem.

Theorem 7.5. *Let $d, s \in \mathbb{N}$. Then there exists a constant $V(d,s)$, which is only dependent on d and s, such that for every polytope $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ it holds $\operatorname{vol}(P) \leq V(d,s)$.*

Once Theorem 7.5 is proven, Theorem 7.2 is a direct consequence of Proposition 7.3 and Theorems 7.4 and 7.5. Thus, let us now prove Theorem 7.5.

The proof of Theorem 7.5 relies on results of Hensley [Hen83] and Lagarias and Ziegler [LZ91]. Hensley showed that the volume and the number of integer points of a d-dimensional integral polyhedron with precisely $k > 0$ interior integer points is bounded from above by a constant which depends only on d and k. Lagarias and Ziegler improved these bounds and generalized parts of Hensley's results. For the proof of Theorem 7.5, we make use of two results from [Hen83] and [LZ91]. We recommend to look up the notions *simplex* and *barycentric coordinates* on p. 10 since they are frequently used in the following. The next theorem states that, for every d-dimensional integral simplex S in $\mathbb{R}^d$ with precisely one interior point p in $s\mathbb{Z}^d$, all barycentric coordinates of p with respect to S are bounded from below by a (strictly) positive constant which depends only on d and s.

Theorem 7.6. ([Hen83, Theorem 3.1] and [LZ91, Lemma 2.2].) *Let $d, s \in \mathbb{N}$. Then there exists a constant $\lambda^*(d,s) > 0$, which is only dependent on d and s, such that for every d-dimensional integral simplex $S \in \mathcal{P}^d$ with $\operatorname{int}(S) \cap s\mathbb{Z}^d = \{p\}$, all barycentric coordinates $\lambda_0, \dots, \lambda_d$ of p with respect to S satisfy $\lambda_i \geq \lambda^*(d,s)$.*

We point out that, in the formulation of Theorem 7.6, $\lambda^*(d,s)$ is not necessarily best possible. Once some $\lambda^*(d,s)$ is known, then any smaller positive constant works as well. Thus, it is always possible (and will be convenient later) to choose the values $\lambda^*(d,s)$ to be non-increasing in $d \in \mathbb{N}$. In fact, the currently best known concrete values for $\lambda^*(d,s)$, given in [LZ91, Lemma 2.2], are non-increasing in d. The following theorem will be used to finalize the proof of Theorem 7.5.

Theorem 7.7. ([Hen83, Theorem 3.6] and [LZ91, Theorem 1].) *Let $d, s, k \in \mathbb{N}$. Then there exists a constant $W(d,s,k)$, depending only on d, s, and k, such that for every polytope $Q \in \mathcal{P}^d_{\mathrm{i}}$ with $1 \leq |\operatorname{int}(Q) \cap s\mathbb{Z}^d| \leq k$ it holds $\operatorname{vol}(Q) \leq W(d,s,k)$.*

We have mentioned all results from the literature that are needed to prove Theorem 7.5. Let us now show our assertion. We point out that in the remainder of this section, for all statements and proofs, we always assume that $d \geq 2$ is the underlying dimension and that $\Lambda = s\mathbb{Z}^d$ for an integer $s \geq 1$.

Let us now introduce two classes of polyhedra which are needed for our proofs. Let $a \in \Lambda$ and let $\mathcal{X}^d(a)$ be the class of all polyhedra $P \in \mathcal{P}^d_{\mathrm{i}}$

such that $a \in \operatorname{relbd}(P)$ and $\operatorname{relint}(P) \cap \Lambda \neq \emptyset$. On $\mathcal{X}^d(a)$ we introduce the partial order $\preceq$ as follows: for $P, Q \in \mathcal{X}^d(a)$ we define $P \preceq Q$ if and only if $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q)$. Let us verify that the binary relation $\preceq$ is indeed a partial order. The property $P \preceq P$ is obvious. If $P \preceq Q$ and $Q \preceq P$, then $\operatorname{relint}(P) = \operatorname{relint}(Q)$. Since P and Q are closed (as they are polyhedra), it follows $P = Q$. If $P \preceq Q$ and $Q \preceq R$, then $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q) \subseteq \operatorname{relint}(R)$. Thus $P \preceq R$.

By $\mathcal{R}^d(a)$ we denote the set of the minimal elements of the poset $(\mathcal{X}^d(a), \preceq)$, i.e. the set of the elements $Q \in \mathcal{X}^d(a)$ such that there exists no $P \in \mathcal{X}^d(a)$ with $P \preceq Q$ and $P \neq Q$. We emphasize that elements of $\mathcal{X}^d(a)$ and $\mathcal{R}^d(a)$ do not have to be full-dimensional. Furthermore, for every $P \in \mathcal{X}^d(a)$ there exists some $Q \in \mathcal{R}^d(a)$ such that $Q \preceq P$. Indeed, if P is bounded, this follows from the fact that the set of all $Q \in \mathcal{X}^d(a)$ satisfying $Q \preceq P$ is finite as $\left|P \cap \mathbb{Z}^d\right| < +\infty$. If P is unbounded we replace P by $\bar{P} = \operatorname{conv}(P \cap B \cap \mathbb{Z}^d)$, where B is a sufficiently large box centered at a and such that $\operatorname{relint}(\bar{P}) \cap \Lambda \neq \emptyset$. Then we apply the argument for the bounded case to $\bar{P}$. In particular, it follows that all elements of $\mathcal{R}^d(a)$ are bounded.

We remark that for $P, Q \in \mathcal{X}^d(a)$ the condition $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q)$ holds if and only if one has $P \subseteq Q$ and $\operatorname{relint}(P) \cap \operatorname{relint}(Q) \neq \emptyset$. Indeed, first assume that $\operatorname{relint}(P) \subseteq \operatorname{relint}(Q)$. Since P and Q are closed, this implies $P \subseteq Q$. From $P \in \mathcal{X}^d(a)$, it follows that $\operatorname{relint}(P) \neq \emptyset$ and thus we obtain $\operatorname{relint}(P) \cap \operatorname{relint}(Q) = \operatorname{relint}(P) \neq \emptyset$. For the converse assume that $P \subseteq Q$ and $\operatorname{relint}(P) \cap \operatorname{relint}(Q) \neq \emptyset$. Then

$$\emptyset \neq \operatorname{relint}(P) = \operatorname{relint}(P \cap Q) = \operatorname{relint}(P) \cap \operatorname{relint}(Q) \subseteq \operatorname{relint}(Q),$$

where the non-emptiness of $\operatorname{relint}(P)$ follows from $P \in \mathcal{X}^d(a)$, the first equality from $P \subseteq Q$, and the second equality from [Roc72, Theorem 6.5]).

It turns out that the elements of $\mathcal{R}^d(a)$ have a very specific shape which is described below.

Lemma 7.8. *Let $a \in \Lambda$ and $P \in \mathcal{R}^d(a)$. Then P has the following properties.*

I. P is a simplex of dimension $k \in \{1, \dots, d\}$.

II. The point a is a vertex of P.

III. The set $\operatorname{relint}(P) \cap \Lambda$ consists of precisely one point.

IV. The facet F of P opposite to the vertex a satisfies $F \cap \mathbb{Z}^d = \operatorname{vert}(F)$.

Proof. Let $P \in \mathcal{R}^d(a)$ and let $q \in \operatorname{relint}(P) \cap \Lambda$ be arbitrary. Consider the point $2q - a$ which is the reflection of a with respect to q. First assume that $2q - a \in P$. Then $q \in \operatorname{relint}(P) \cap \operatorname{relint}([a, 2q - a])$ and $[a, 2q - a] \subseteq P$. Thus, since $P \in \mathcal{R}^d(a)$, we have $P = [a, 2q - a]$. Again, since $P \in \mathcal{R}^d(a)$, q is the only point of Λ in $\operatorname{relint}(P)$. For such a P, Parts I–IV follow immediately. In the remainder of the proof let $2q - a \notin P$.

Parts I and II. Let b be the intersection point of $[a, q\rangle$ and $\operatorname{relbd}(P)$ (such an intersection point exists since P is bounded as it is an element of $\mathcal{R}^d(a)$). Since $q \in \operatorname{relint}([a, b])$ we have $q = (1 - \lambda)a + \lambda b$ for some $0 < \lambda < 1$. Consider a facet F of P which contains b. Since P is integral, F is also integral, i.e. $F = \operatorname{conv}(F \cap \mathbb{Z}^d)$. By Carathéodory's theorem (see, for instance, [Sch93a, Theorem 1.1.4]), there exist affinely independent points $q_1, \ldots, q_k \in F \cap \mathbb{Z}^d$ such that $b = \lambda_1 q_1 + \cdots + \lambda_k q_k$ for some $\lambda_1, \ldots, \lambda_k > 0$ with $\lambda_1 + \cdots + \lambda_k = 1$. Thus, $q = (1 - \lambda)q_0 + \lambda\lambda_1 q_1 \cdots + \lambda\lambda_k q_k$, where $q_0 := a$. The point $a = q_0$ does not belong to $\operatorname{aff}(F)$. In fact, otherwise $a \in P \cap \operatorname{aff}(F) = F$ and since $b \in F$ we get $q \in F$, a contradiction to $q \in \operatorname{relint}(P)$. Hence $q_0, \ldots, q_k$ are affinely independent. Since $P \in \mathcal{R}^d(a)$, we have $P = \operatorname{conv}(\{q_0, \ldots, q_k\})$. Therefore P is a simplex of dimension k and a is a vertex of P.

In the remainder of the proof let $P = \operatorname{conv}(\{q_0, \ldots, q_k\})$ with $q_0 := a$ and $q_1, \ldots, q_k$ defined as above.

Part III. For $j = 0, \ldots, k$ let P_j be the simplex with vertices $\{q, q_0, \ldots, q_k\} \setminus \{q_j\}$. It can be verified with straightforward arguments that $P = P_0 \cup \cdots \cup P_k$ and that the relative interiors of the simplices P_j are pairwise disjoint. For proving Part III, we argue by contradiction. We assume that $\operatorname{relint}(P) \cap \Lambda$ contains a point q' with $q' \neq q$. Let us show that $q' \in P_0$. Assume the contrary. Then $q' \in P_j$ for some $j \in \{1, \ldots, k\}$. Let F be the face of P_j with $q' \in \operatorname{relint}(F)$. Since $q' \notin P_0$, a is a vertex of F. The existence of $F \subsetneq P$ with $q' \in \operatorname{relint}(F)$ and $a \in \operatorname{vert}(F)$ contradicts the fact that $P \in \mathcal{R}^d(a)$. Hence $q' \in P_0$. We define $Q := \operatorname{conv}((P_0 \cap \mathbb{Z}^d) \setminus \{q\})$. Since $q' \in \operatorname{relint}(P)$, and $q', q_1, \ldots, q_k \in Q$, the polytope Q has the same dimension as P. We have $[a, q\rangle \cap Q = [b', b]$, where $b' \in \operatorname{relint}(P)$ and $b \in \operatorname{relbd}(P)$. Since $q \in \operatorname{relint}([a, b'])$ one has $q = (1 - \lambda)a + \lambda b'$ for some $0 < \lambda < 1$. Let now G be the facet of Q containing b'. The point $a = q_0$ does not belong to $\operatorname{aff}(G)$. In fact, otherwise $\operatorname{aff}(G)$ would contain $[b', b]$, which implies $\operatorname{aff}(G) \cap \operatorname{relint}(Q) \neq \emptyset$, a contradiction. Using Carathéodory's theorem, let $p_1, \ldots, p_m$ be affinely independent vertices of G such that $b' = \lambda_1 p_1 + \cdots + \lambda_m p_m$ for some $\lambda_1, \ldots, \lambda_m > 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. Then $q = (1 - \lambda)p_0 + \lambda\lambda_1 p_1 + \cdots + \lambda\lambda_m p_m$ with $p_0 := a$. Since $p_0 \notin \operatorname{aff}(G)$ and since $p_1, \ldots, p_m \in G$ are affinely independent, we see that $p_0, \ldots, p_m$ are affinely independent. The simplex $S = \operatorname{conv}(\{p_0, \ldots, p_m\})$ is properly contained in P (as $b \notin S$), contains the point a on its relative boundary and satisfies $q \in \operatorname{relint}(S) \cap \operatorname{relint}(P)$, a contradiction to the fact that $P \in \mathcal{R}^d(a)$. This shows Part III.

Part IV. We argue by contradiction. Let F be the facet of P opposite to a and assume that $\mathrm{vert}(F) \subsetneq F \cap \mathbb{Z}^d$. Let $S_1, \ldots, S_m$ be a triangulation constructed on the points $F \cap \mathbb{Z}^d$. Then $S_1, \ldots, S_m$ are simplices with pairwise disjoint interiors having the same dimension as F and such that $F \cap \mathbb{Z}^d = \bigcup_{i=1}^m \mathrm{vert}(S_i)$, $F = \bigcup_{i=1}^m S_i$, and for every S_i, $\mathrm{vert}(S_i)$ are the only integer points in S_i. By assumption, we have $S_i \neq F$ for every i. Moreover, there exists a simplex S_j such that $[a, q\rangle \cap S_j$ is non-empty. Let b be the point $[a, q\rangle \cap S_j$. Further on, let G be the face of S_j with $b \in \mathrm{relint}(G)$. By construction, $q \in \mathrm{relint}(\bar{P})$ where $\bar{P} := \mathrm{conv}(\{a\} \cup G)$ and $\bar{P} \subsetneq P$. This contradicts the fact that $P \in \mathcal{R}^d(a)$. □

Lemma 7.8 and the following Lemma 7.9 are used to prove Lemma 7.10.

Lemma 7.9. *Let $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ be a polytope. Then $\mathrm{int}(\pi(P)) \cap \pi(\Lambda) \neq \emptyset$.*

Proof. If $P' := \pi(P)$ satisfies $\mathrm{int}(P') \cap \pi(\Lambda) = \emptyset$, then $\pi^{-1}(P')$ is Λ-free and integral, and then in view of the maximality of P, one has $\pi^{-1}(P') \subseteq P$ which contradicts the boundedness of P. □

In the following lemma we prove that the number of points of Λ on the boundary of a polytope $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ is bounded by a constant which is only dependent on d and s.

Lemma 7.10. *Let $d, s \in \mathbb{N}$. Then there exists a constant $N(d,s)$, which is only dependent on d and s, such that for every polytope $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ it holds $|P \cap \Lambda| \leq N(d,s)$.*

Proof. Let $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ be a polytope. We explicitly construct an upper bound $N(d,s)$ on the number of points in $P \cap \Lambda$. Assume, by contradiction, that $|P \cap \Lambda| \geq M^d + 1$, where

$$M := \left\lceil \frac{1}{\lambda^*(d-1,s)} + 1 \right\rceil$$

with $\lambda^*(d-1,s)$ defined as in Theorem 7.6. Thus, there exist two distinct points $v, w \in P \cap \Lambda$ such that $\frac{1}{s}v \equiv \frac{1}{s}w \pmod{M}$. Then we can choose $M+1$ pairwise distinct points $z_0, \ldots, z_M$ in $P \cap \Lambda \cap \mathrm{aff}(\{v,w\})$ such that $\mathrm{conv}(\{z_0, \ldots, z_M\}) \cap \Lambda = \{z_0, \ldots, z_M\}$. Performing a Λ-preserving transformation we assume that $z_j = j \cdot s e_d$ for $j = 0, \ldots, M$. One has $\pi(z_j) = o$ for every $j = 0, \ldots, M$. Since $M \geq 2$ (which follows from $\lambda^*(d-1,s) > 0$), o is a boundary point of $P' := \pi(P)$, otherwise P would not be Λ-free. By Lemma 7.9, $\mathrm{int}(P') \cap \pi(\Lambda) \neq \emptyset$.

By construction, P' is integral and belongs to $\mathcal{X}^{d-1}(o)$. Thus, there exists a polytope $Q \in \mathcal{R}^{d-1}(o)$ with $\mathrm{relint}(Q) \subseteq \mathrm{int}(P')$. By Lemma 7.8, Q is a simplex with precisely one point of $\pi(\Lambda)$, say p, in the relative interior. Let k be the dimension of Q and let $p_0, \dots, p_k$ be the vertices of Q with $p_0 := o$. By Theorem 7.6, if $p = \sum_{j=0}^{k} \lambda_j p_j$ with $\lambda_0, \dots, \lambda_k > 0$ and $\lambda_0 + \cdots + \lambda_k = 1$, then[3] one has $\lambda_j \geq \lambda^*(k, s) \geq \lambda^*(d-1, s)$ for every $j = 0, \dots, k$ (remember that $\lambda^*(d, s)$ in Theorem 7.6 is assumed to be non-increasing in d).

For a point $x \in P'$, let $\tau(x)$ denote the length of the line segment $\pi^{-1}(x) \cap P$ (and thus, τ represents an "X-ray picture" of P). Employing the convexity of P we see that τ is concave on P'. Consequently,

$$\tau(p) = \tau\left(\sum_{j=0}^{k} \lambda_j p_j\right) \geq \sum_{j=0}^{k} \lambda_j \tau(p_j) \geq \lambda_0 \tau(p_0) \geq \lambda^*(d-1, s) s M > s,$$

where the first inequality follows from the concavity of τ, the second inequality from $\lambda_j > 0$ and $\tau(p_j) \geq 0$ for all $j = 1, \dots k$, and the third inequality from $\lambda_0 \geq \lambda^*(d-1, s)$ and the fact that $\{j \cdot s e_d : j = 0, \dots, M\} \subseteq \pi^{-1}(p_0)$.

On the other hand, since $p \in \operatorname{int}(P') \cap \pi(\Lambda)$, we have $\tau(p) \leq s$ as otherwise P would not be Λ-free. Thus, this gives a contradiction to our assumption on $|P \cap \Lambda|$. It follows that P contains at most M^d points in Λ and we can choose $N(d, s) := M^d$. □

Proof of Theorem 7.5. Let $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ be a polytope. In the following, we enlarge P to a polytope $Q \in \mathcal{P}^d_{\mathrm{i}}$ such that $P \subseteq Q$ and $\emptyset \neq \operatorname{int}(Q) \cap \Lambda \subseteq P \cap \Lambda$. By Lemma 7.10, this implies $1 \leq |\operatorname{int}(Q) \cap \Lambda| \leq |P \cap \Lambda| \leq N(d, s)$. Then, by Theorem 7.7, $\operatorname{vol}(P) \leq \operatorname{vol}(Q) \leq W(d, s, N(d, s))$. Consequently, $\operatorname{vol}(P) \leq V(d, s) := W(d, s, N(d, s))$.

Let us now construct the polytope Q. For that, we consider a sequence of polytopes P^i which we define iteratively. Choose an arbitrary $p_1 \in \Lambda$ such that $p_1 \notin P$ and let $P^1 := \operatorname{conv}(P \cup \{p_1\})$. For $i \geq 1$, we proceed as follows. If $\operatorname{int}(P^i) \cap \Lambda \subseteq P \cap \Lambda$, then we stop and define $Q := P^i$. Otherwise, we select $p_{i+1} \in (\operatorname{int}(P^i) \cap \Lambda) \setminus (P \cap \Lambda)$ and set $P^{i+1} := \operatorname{conv}(P \cup \{p_{i+1}\})$. Note that $P^{i+1} \subsetneq P^i$ for all $i \geq 1$ and that the sequence is finite since P is a polytope. Eventually, we construct a polytope $Q \in \mathcal{P}^d_{\mathrm{i}}$ such that $P \subseteq Q$ and $\operatorname{int}(Q) \cap \Lambda \subseteq P \cap \Lambda$. Furthermore, $\operatorname{int}(Q) \cap \Lambda \neq \emptyset$ since P is properly contained in Q and P is maximal within $\mathcal{P}^d_{\mathrm{if}}(s)$ with respect to inclusion. □

[3] Theorem 7.6 deals with full-dimensional simplices, but one can show that the statement is also true for a lower-dimensional simplex Q as in the proof of Lemma 7.10. To see this, observe that there exists an integer vector l such that $\operatorname{aff}(Q) - l$ is a linear space L of dimension $k \in \{1, \dots, d-1\}$. In particular, $L \cap \mathbb{Z}^d$ is a lattice of rank k. We have $L \cap (s\mathbb{Z}^d) = s(L \cap \mathbb{Z}^d)$. Indeed, let $y \in L \cap (s\mathbb{Z}^d)$. Then clearly $y \in s(L \cap \mathbb{Z}^d)$. On the other hand, if $y \in s(L \cap \mathbb{Z}^d)$, then $y \in sL = L$ and $y \in s\mathbb{Z}^d$. Thus, $y \in L \cap (s\mathbb{Z}^d)$. This means that, by considering $\operatorname{aff}(Q)$, we are concerned with the lattices $L \cap \mathbb{Z}^d$ and $s(L \cap \mathbb{Z}^d)$ instead of the lattices $\mathbb{Z}^d$ and $s\mathbb{Z}^d$, respectively (which is the same, up to a lattice transformation).

Proof of Theorem 7.2. Follows immediately from Proposition 7.3 and Theorems 7.4 and 7.5. □

Remark 7.11. *Let us analyze the inherent reason for the finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$. Theorem 7.2 results from the combination of three statements: Proposition 7.3 and Theorems 7.4 and 7.5. Proposition 7.3 is used to reduce the proof of Theorem 7.2 to the consideration of polytopes, but it does not guarantee finiteness. Theorem 7.4 holds true for every set of integral polytopes, but it does not imply finiteness for the specific set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$. However, Theorem 7.5 exploits the intrinsic properties of polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$. Therefore, finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ comes from the fact that we are able to bound the volume of polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ from above in terms of d and s alone. As can be seen from the proof of Theorem 7.5, the crucial ingredients are the Theorems 7.6 and 7.7. All the other statements which we presented in this section are, in principle, concatenations of arguments to combine the Theorems 7.6 and 7.7 in an appropriate way.*

By digging a bit further in the papers of Hensley [Hen83] and Lagarias and Ziegler [LZ91], one sees that the proof of Theorem 7.7 basically relies on Theorem 7.6. The reason is that Theorem 7.7 is proved first for simplices and then the arguments are used for polytopes which are no simplices. In turn, the key ingredient in the proof of Theorem 7.6 is a certain Diophantine approximation lemma (see [Hen83, Lemmas 2.1 and 3.1] and [LZ91, Lemma 2.1]).

Thus, after a couple of arguments, finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ follows from the fact that, for every d-dimensional integral simplex S in $\mathbb{R}^d$ with precisely one interior point p in $s\mathbb{Z}^d$, each barycentric coordinate of p with respect to S is bounded from below by a (strictly) positive constant which is only dependent on d and s. In other words, finiteness is implied by the property that p has a minimum distance from the boundary of S.

Remark 7.12. *To the best of our knowledge, the finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ for every $d, s \in \mathbb{N}$ was not known so far. However, important partial results[4] were derived by Lawrence [Law91] and Treutlein [Tre08, Tre10]. Lawrence's results imply finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(1)/\operatorname{Aff}(\mathbb{Z}^d)$ (see [Law91, Theorem 4]). His proof technique differs widely from ours and it is not immediately clear how his proof can be generalized to $s \geq 1$. Treutlein shows a more general result which implies the finiteness of the*

[4] The author is aware of the possibility that further literature on finiteness of the considered sets exists. It may well be that the question of finiteness is answered somewhere in terms of a different mathematical formulation which is not immediately associated with optimization, polyhedra, or integrality. We tried hard to identify the relevant sources, but cannot guarantee that some documents are not mentioned here.

set $\mathcal{P}^3_{\text{ifm}}(1)/\operatorname{Aff}(\mathbb{Z}^d)$ (see [Tre08, Proposition 4.5] and [Tre10, Proposition 6.6.4]). His proof technique is similar to ours in the sense that he also obtains an upper bound on the maximum volume of a polytope in $\mathcal{P}^3_{\text{ifm}}(1)$, but a generalization to $s \geq 1$ does not leap out directly from his proof. Moreover, Treutlein mentioned that his result should hold for every $d \in \mathbb{N}$ (this was later verified by Nill and Ziegler [NZ11], see p. 109). Unfortunately, Treutlein's proofs contain some gaps[5] which makes it difficult to improve his volume bound.

7.3 Remarks on the volume bound

In this section, we analyze the constants in our statements and their growth in d. Let $d, s \in \mathbb{N}$ and let $P \in \mathcal{P}^d_{\text{ifm}}(s)$ be a polytope.

As a result of the investigations of Lagarias and Ziegler [LZ91] and Pikhurko [Pik01], it turns out that the constant $W(d, s, k)$ in Theorem 7.7 must be at least double exponential in d. It can be chosen to be $B_1 := ks^d \cdot (7(1 + sk))^{d\cdot 2^{d+1}}$ (see [LZ91, Theorem 1]) or to be $B_2 := k(8ds)^d \cdot (8s + 7)^{d\cdot 2^{2d+1}}$ (see [Pik01, p. 17, formula (9)])[6]. From the proof of Theorem 7.5, it follows that $\operatorname{vol}(P) \leq W(d, s, N(d, s))$, where $N(d, s)$ can be chosen to be $\lceil(\lambda^*(d-1, s))^{-1} + 1\rceil^d$, by Lemma 7.10. The best known lower bounds on the constant $\lambda^*(d, s)$ are $(7(s + 1))^{-2^{d+1}}$ (see [LZ91, Lemmas 2.1 and 2.2]) and $(8(8s+7)^{2^{d+1}})^{-1}$ (see [Pik01, Theorem 2]). It is easy to show that for all $d, s \in \mathbb{N}$ it holds $(7(s + 1))^{-2^{d+1}} \geq (8(8s + 7)^{2^{d+1}})^{-1}$. Thus, we use the first bound as lower bound for $\lambda^*(d, s)$ and thus obtain $\lambda^*(d-1, s) \geq (7(s+1))^{-2^d}$. Substituting this into $\lceil(\lambda^*(d-1, s))^{-1} + 1\rceil^d$ yields

$$N(d, s) \leq \left(1 + \left(7(s + 1)\right)^{2^d}\right)^d.$$

[5] We do not dispute Treutlein's conclusions, but rather point out that his proofs are incomplete. For instance, in the proof of Proposition 4.5 in [Tre08, p. 11] the author writes “we receive after some computation $d \leq 2$” (similarly, in the proof of Proposition 6.6.4 in [Tre10, p. 147] the author writes “folgt nach einigen Abschätzungen $l \leq 2$”). In our opinion, the correct bound on d (resp. l) should be $1 + 2 \cdot (\sqrt{3})^{-1}$ which would mean that the case distinction in the proof of Proposition 4.5 in [Tre08] (resp. Proposition 6.6.4 in[Tre10]) is incomplete. We contacted the author by e-mail to ask how he computed his bounds and whether the missing case can be excluded with other arguments. Unfortunately, Treutlein works now as a consultant and is not involved in these topics anymore, but he was kind enough to send us a copy of his computations. From that copies, it seems to follow that the case $d > 2$ (resp. $l > 2$) can indeed be neglected.

[6] We point out that better bounds are known for simplices and refer to [LZ91] and [Pik01] for details.

Hence, using B_1 gives

$$\begin{aligned}\mathrm{vol}(P) &\le N(d,s)\cdot s^d\cdot\Big(7\big(1+s\cdot N(d,s)\big)\Big)^{d\cdot 2^{d+1}}\\ &\le s^d\cdot\Big(1+\big(7(s+1)\big)^{2^d}\Big)^d\cdot\\ &\qquad\Bigg(7\bigg(1+s\Big(1+\big(7(s+1)\big)^{2^d}\Big)^d\bigg)\Bigg)^{d\cdot 2^{d+1}} \end{aligned}\tag{7.2}$$

and using B_2 leads to

$$\begin{aligned}\mathrm{vol}(P) &\le N(d,s)\cdot(8ds)^d\cdot(8s+7)^{d\cdot 2^{2d+1}}\\ &\le \Big(1+\big(7(s+1)\big)^{2^d}\Big)^d\cdot(8ds)^d\cdot(8s+7)^{d\cdot 2^{2d+1}}.\end{aligned}\tag{7.3}$$

It is straightforward to show that the bound in (7.2) is larger than the bound in (7.3) for all $d\in\mathbb{N}$, $d\ge 2$ and $s\in\mathbb{N}$. Thus, the bound in (7.3) is "best possible"[7], provided that our proof technique is applied and the (so far) best known bounds on $W(d,s,k)$ and $\lambda^*(d,s)$ are used. In the asymptotic notation this bound can be expressed as $\mathrm{vol}(P)=(s+1)^{O(d\cdot 4^d)}$.

Let us now present a family of polytopes which shows that the maximum volume over all polytopes in $\mathcal{P}^d_{\mathrm{fmi}}(s)$ (and thus in $\mathcal{P}^d_{\mathrm{ifm}}(s)$) is at least of order $(s+1)^{\Omega(2^d)}$. We use the following sequence considered in [LZ91, Lemma 2.1]. Let $y_1:=s+1$ and $y_j:=1+s\prod_{i=1}^{j-1}y_i$ for $j\ge 2$ (equivalently one can use the recurrence $y_j:=y_{j-1}^2-y_{j-1}+1$). For every $d\ge 2$, we introduce the simplex $S_{(d,s)}:=\mathrm{conv}(\{o,y_1e_1,\dots,y_{d-1}e_{d-1},(y_d-1)e_d\})$. It is straightforward to show that $S_{(d,s)}$ belongs to $\mathcal{P}^d_{\mathrm{fmi}}(s)$. The volume of $S_{(d,s)}$ can be written as

$$\mathrm{vol}(S_{(d,s)})=\frac{1}{d!}\left(\prod_{i=1}^{d-1}y_i\right)(y_d-1)=\frac{1}{d!}\frac{1}{s}(y_d-1)^2.$$

It is noticed in [LZ91, p. 1026] that one has $y_d\ge(s+1)^{2^{d-2}}$ for all $d\ge 2$. This implies that $\mathrm{vol}(S_{(d,s)})=(s+1)^{\Omega(2^d)}$.

Unfortunately, the bound in (7.3) does not help to determine all polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ for fixed values of d and s since it is tremendously large. For instance, if $d=3$ and $s=1$, then (7.3) gives $\mathrm{vol}(P)\le(24\cdot 15^{128}\cdot(1+14^8))^3\approx$

[7] The author is convinced that there is a huge potential for an improvement of this bound. Such an improvement could be due to another proof technique as well as a sharpening of the bounds on $W(d,s,k)$ and $\lambda^*(d,s)$. We refer to Section 7.4 for an improved bound on the maximum volume of a polytope in $\mathcal{P}^d_{\mathrm{ifm}}(s)$.

$1.85 \cdot 10^{483}$. Thus, even for small dimensions, an enumeration which is based on our volume bound and a computer search is intractable[8].

7.4 The relation between $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$

In this section, we investigate the relation between the two sets $\mathcal{P}^d_{\mathrm{fmi}}(s)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)$. In particular, we focus on the case $s = 1$ which is of interest in cutting plane theory. We recall that two polyhedra are called *equivalent* if they coincide up to a Λ-preserving transformation, and *distinct* if they do not.

By (7.1), we have $\mathcal{P}^d_{\mathrm{fmi}}(s) \subseteq \mathcal{P}^d_{\mathrm{ifm}}(s)$ for all $d, s \in \mathbb{N}$. The complete characterization of all pairs of $d, s \in \mathbb{N}$ for which the equality $\mathcal{P}^d_{\mathrm{fmi}}(s) = \mathcal{P}^d_{\mathrm{ifm}}(s)$ holds true is unknown. Let us summarize the current state of research. For $d = 1, s \geq 1$ and $d = 2, s = 1$ equality can be verified in a straightforward way. Indeed, if $d = 1, s \geq 1$, then every element in $\mathcal{P}^1_{\mathrm{fmi}}(s) = \mathcal{P}^1_{\mathrm{ifm}}(s)$ is equivalent to the interval $[0, s]$. If $d = 2, s = 1$, then every element in $\mathcal{P}^2_{\mathrm{fmi}}(1) = \mathcal{P}^2_{\mathrm{ifm}}(1)$ is either a triangle of type 1 and thus equivalent to $\mathrm{conv}(\{o, 2e_1, 2e_2\})$, or it is a split and thus equivalent to $\mathrm{conv}(\{o, e_1\}) + \mathrm{lin}(\{e_2\})$. On the other hand, for $d \geq 2, s \geq 3$ the inclusion is strict. For instance, consider the polyhedron $Q^d_s := \mathrm{conv}(\{o, (2s+1)e_1, (2s+1)e_1 + e_2, (2s-1)(e_1+e_2)\}) + \mathrm{lin}(\{e_3, \ldots, e_d\})$. It is easy to verify that $Q^d_s \in \mathcal{P}^d_{\mathrm{ifm}}(s) \setminus \mathcal{P}^d_{\mathrm{fmi}}(s)$.

Recently, Nill and Ziegler showed in a more general context that for every $d \geq 4$ and $s \geq 1$ it holds $\mathcal{P}^d_{\mathrm{fmi}}(s) \subsetneq \mathcal{P}^d_{\mathrm{ifm}}(s)$ (see [NZ11, Theorem 1.4]). In particular, for every $d \geq 4$, they give an explicit example of a d-dimensional polytope which belongs to $\mathcal{P}^d_{\mathrm{ifm}}(1) \setminus \mathcal{P}^d_{\mathrm{fmi}}(1)$ (see [NZ11, Section 3]). These examples might be extendable to any $s \geq 1$.

In [NZ11, Theorem 2.1] it is shown that every d-dimensional Λ-free integral polytope in $\mathbb{R}^d$ either admits a lattice projection onto a $(d-1)$-dimensional Λ-free integral polytope, or it belongs to a set of finitely many (i.e. up to a Λ-preserving transformation) “exceptional polytopes”. This result is clearly a generalization of our Theorem 7.2. Its proof relies on the combination of results from Kannan and Lovász [KL88] and Pikhurko [Pik01] and is thus based on a quite heavy machinery. The method of proof which is used by Nill and Ziegler is similar to our proof of Theorem 7.2 in the sense that it provides a volume bound for the exceptional polytopes. More precisely, from [NZ11, Theorem 2.1], it follows that for every polytope $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ it holds

$$\mathrm{vol}(P) \leq s^d \cdot \left(1 + 8(d-1)(8s+7)^{2^{2d-1}}\right)^d. \tag{7.4}$$

[8] This is the reason why our proof of Theorem 8.1 (presented in the next chapter) does not rely on (7.3).

It is easy to check that the bound in (7.4) is better (i.e. smaller) than the bound in (7.3). Nevertheless, both bounds show the same asymptotic behavior with respect to d.

The remaining cases, i.e. $d = 2, s = 2$ and $d = 3, s \in \{1, 2\}$ are still open. In the following table, we summarize the current state of research on the relation between $\mathcal{P}^d_{\text{fmi}}(s)$ and $\mathcal{P}^d_{\text{ifm}}(s)$ for all pairs of $d, s \in \mathbb{N}$. The entry "=" means that the sets $\mathcal{P}^d_{\text{fmi}}(s)$ and $\mathcal{P}^d_{\text{ifm}}(s)$ coincide, the entry "$\subsetneq$" indicates the strict inclusion, and "?" means that the relation is unknown.

$s \setminus d$	1	2	3	≥ 4
1	=	=	?	$\subsetneq$
2	=	?	?	$\subsetneq$
≥ 3	=	$\subsetneq$	$\subsetneq$	$\subsetneq$

Particular attention has been paid to the case $d = 3, s = 1$. For that case, Nill and Ziegler [NZ11, Proposition 1.6] could sharpen their volume bound by proving that the volume of a polytope in $\mathcal{P}^3_{\text{ifm}}(1)$ is at most 4106 (observe that (7.4) leads to roughly $3.29 \cdot 10^{116}$ and our bound (7.3) gives roughly $1.85 \cdot 10^{483}$, see p. 109). This is a vast improvement, but the volume bound is still too large for a brute force computer search.

There is some indication that $\mathcal{P}^3_{\text{fmi}}(1) = \mathcal{P}^3_{\text{ifm}}(1)$ holds true (see, for instance, the discussions in [Ave11]). In Chapter 8, we classify all elements of $\mathcal{P}^3_{\text{fmi}}(1)/\operatorname{Aff}(\mathbb{Z}^3)$. This can be seen as a partial result on the way to prove the equality.

Until now, we attached great weight to volume bounds of polytopes in $\mathcal{P}^d_{\text{ifm}}(s)$. There are (at least) two more questions which are of interest:

(1) How can one test whether a certain polytope belongs to the set $\mathcal{P}^d_{\text{fmi}}(s)$ (resp. $\mathcal{P}^d_{\text{ifm}}(s)$)?

(2) How many elements are contained in the set $\mathcal{P}^d_{\text{fmi}}(s)/\operatorname{Aff}(\Lambda)$ (resp. $\mathcal{P}^d_{\text{ifm}}(s)/\operatorname{Aff}(\Lambda)$)?

Let us discuss the first question. Membership in the set $\mathcal{P}^d_{\text{fmi}}(s)$ can be checked easily with the help of a result of Lovász: let $P \in \mathcal{P}^d$ be a full-dimensional polytope. Then P belongs to $\mathcal{P}^d_{\text{fmi}}(s)$ if and only if P is integral, Λ-free, and every facet of P contains a point of Λ in its relative interior (see [Lov89, Proposition 3.3]). Thus, P must satisfy three properties in order to belong to $\mathcal{P}^d_{\text{fmi}}(s)$. Integrality of P can be verified by checking the vertices of P. The other two required properties can be verified by solving a couple of feasibility problems. Hence, checking membership in the set $\mathcal{P}^d_{\text{fmi}}(s)$ is algorithmically easy (for fixed d). In contrast, it is not clear at all how membership in the set $\mathcal{P}^d_{\text{ifm}}(s)$ can be verified or disproved. If $P \in \mathcal{P}^d$ is given, then the most natural way is probably to identify a (preferably small) finite set T of integer points in the neighborhood of P (where *neighborhood* is not yet specified) and then to check for each $x \in T$ separately whether the convex hull of $P \cup \{x\}$ is Λ-free or not. Such a "test set" T must have the property that $P \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ if and only if $\mathrm{conv}(P \cup \{x\}) \cap \Lambda \neq \emptyset$ for every $x \in T$. So far, it is not known how such a set T of candidate points can be constructed. In particular, the lack of knowledge of a test for membership in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ makes it difficult to come up with an efficient computer code for the enumeration of the polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(s)$ based on volume bounds as discussed above.

Let us now discuss our second question. From volume bounds alone one cannot infer anything about the cardinalities of the sets $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$. Therefore, we need further tools to attack the second question. In the following we show that the set $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$ has exponentially many elements (for growing d) and we present an easy way to construct elements of this set. For that, let us introduce the following notation. Assume that $d, s \in \mathbb{N}$. If $a \in \mathbb{R}^d$ is a vector with $a_j > 0$ for all $j = 1, \dots, d$, then a can be used to define a simplex $S^d_s(a) := \mathrm{conv}(\{o, sa_1e_1, \dots, sa_de_d\})$ in the positive orthant. Moreover, let

$$\delta^d(a) := \frac{1}{a_1} + \cdots + \frac{1}{a_d}$$

and let $\Delta^d := \{a \in \mathbb{N}^d : \delta^d(a) = 1 \text{ and } a_1 \leq \ldots \leq a_d\}$ be the set of all vectors whose entries are monotonically non-decreasingly ordered natural numbers and whose sum of the corresponding unit fractions is equal to 1. Furthermore, for $a \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$ with $x_j > 0$ for all $j = 1, \dots, d$ we define

$$\frac{a}{x} := \left(\frac{a_1}{x_1}, \dots, \frac{a_d}{x_d}\right).$$

The following lemma[9] shows that for any $a \in \Delta^d$ the simplex $S^d_s(a)$ belongs to $\mathcal{P}^d_{\mathrm{fmi}}(s)$.

Lemma 7.13. *Let $d, s \in \mathbb{N}$ and $a \in \mathbb{N}^d$. Then the following statements hold.*

I. $S^d_s(a)$ is Λ-free if and only if $\delta^d(a) \geq 1$.

II. $S^d_s(a)$ is maximal Λ-free if and only if $\delta^d(a) = 1$.

[9] Lemma 7.13 was found independently by both, the author and Gennadiy Averkov, with two different proof techniques. The stated proof is a modification of the proof of Averkov.

Proof. The facets of $S_s^d(a)$ are the d trivial inequalities $x_j \geq 0$ for $j = 1, \ldots, d$ and the inequality $\delta^d(\frac{a}{x}) \leq s$. Thus, $x \in \operatorname{int}(S_s^d(a))$ if and only if $x_j > 0$ for all $j = 1, \ldots, d$ and $\delta^d(\frac{a}{x}) < s$.

Part I. If $\delta^d(a) < 1$, then $s\mathbb{1} \in \operatorname{int}(S_s^d(a))$ since each component of $s\mathbb{1}$ is strictly positive and $\delta^d(\frac{a}{s\mathbb{1}}) = s\delta^d(a) < s$. Thus, $S_s^d(a)$ is not Λ-free. Now let $\delta^d(a) \geq 1$. Assume there exists some $x \in \operatorname{int}(S_s^d(a)) \cap \Lambda$. Then we must have $x_j \geq s$ for all $j = 1, \ldots, d$. However, this implies $\delta^d(\frac{a}{x}) \geq \delta^d(\frac{a}{s\mathbb{1}}) = s\delta^d(a) \geq s$, which is a contradiction. Hence, $S_s^d(a)$ is Λ-free.

Part II. If $\delta^d(a) > 1$, then Part I ensures that $S_s^d(a)$ is Λ-free. Furthermore, $S_s^d(a)$ is strictly contained in the Λ-free simplex $S_s^d(\delta^d(a) \cdot a)$, where the Λ-freeness of $S_s^d(\delta^d(a) \cdot a)$ follows from Part I and the fact that $\delta^d(\delta^d(a) \cdot a) = 1$. Thus, $S_s^d(a)$ is Λ-free, but not maximal Λ-free. Now let $\delta^d(a) = 1$. By Part I, $S_s^d(a)$ is Λ-free. In order to show that $S_s^d(a)$ is maximal Λ-free, we show that every facet of $S_s^d(a)$ has a point of Λ in its relative interior, which then finishes the proof by a result of Lovász (see [Lov89, Proposition 3.3]). Consider the facet $F_0 := \operatorname{conv}(\{sa_1e_1, \ldots, sa_de_d\})$ and the point $p_0 := s\mathbb{1}$. It holds $p_0 \in \operatorname{relint}(F_0)$ since p_0 is the convex combination of the points $sa_1e_1, \ldots, sa_de_d$ with the strictly positive coefficients $(a_1)^{-1}, \ldots, (a_d)^{-1}$. Now consider the facet $F_i := \operatorname{conv}(\{o, sa_1e_1, \ldots, sa_de_d\} \setminus \{sa_ie_i\})$ for an $i \in \{1, \ldots, d\}$. Then the point $p_i := s(\mathbb{1} - e_i)$ lies in the relative interior of F_i since it is the convex combination of the points o and sa_je_j, $j \in \{1, \ldots, d\} \setminus \{i\}$, with the coefficients $(a_i)^{-1}$ and $(a_j)^{-1}$, $j \in \{1, \ldots, d\} \setminus \{i\}$. □

Given any natural numbers d and s, then it follows from Lemma 7.13 that every $a \in \mathbb{N}^d$ with $\delta^d(a) = 1$ defines a maximal Λ-free integral simplex $S_s^d(a)$. The set Δ^d describes the distinct simplices of this type, i.e. those simplices which differ by a Λ-preserving transformation. A result of Sándor states that there exists a constant $C > 1$ such that for every $d \geq 3$ it holds

$$|\Delta^d| \geq C^{\frac{d^3}{\log(d)}},$$

where $\log(x)$ denotes the logarithm of $x > 0$ to a base which is greater than 1 (see [Sán03, Theorem 1]). This gives an (asymptotically) exponential lower bound on the number of elements in $\mathcal{P}^d_{\text{fmi}}(s)/\operatorname{Aff}(\Lambda)$. We refer to [Guy04, p. 257] for a discussion of the set Δ^d.

In a recent manuscript, Averkov [Ave11] investigates the set $\mathcal{P}^d_{\text{ifm}}(1) \setminus \mathcal{P}^d_{\text{fmi}}(1)$. His main message is that this set is highly complex. Averkov shows that, asymptotically, the number of elements of $(\mathcal{P}^d_{\text{ifm}}(1) \setminus \mathcal{P}^d_{\text{fmi}}(1))/\operatorname{Aff}(\Lambda)$ is exponential in d (see [Ave11, Theorem 2.1]), and that it contains "small" as well as "large" polytopes (see [Ave11, Theorem 2.2]), where smallness (resp. largeness) is measured with respect to both, the lattice diameter and the *lattice size* (we refer to [Ave11, p. 3] for the definition of the notion lattice size). A remarkable feature of Averkov's method of proof is that he provides a technique to explicitly construct polytopes in $\mathcal{P}^d_{\mathrm{ifm}}(1) \setminus \mathcal{P}^d_{\mathrm{fmi}}(1)$ for every $d \geq 4$. It seems that Averkov's results can be extended to $s \geq 1$.

CHAPTER 8

THREE-DIMENSIONAL MAXIMAL LATTICE-FREE INTEGRAL POLYHEDRA

In this chapter, we classify the elements of the set $\mathcal{P}^3_{\mathrm{fmi}}(1)/\operatorname{Aff}(\mathbb{Z}^3)$. By Proposition 7.3, we can restrict our attention to (full-dimensional) polytopes. Let $\mathcal{M}^d$ denote the set of maximal lattice-free integral polytopes in $\mathbb{R}^d$. Then $\mathcal{M}^d$ is the subclass of $\mathcal{P}^d_{\mathrm{fmi}}(1)$ which contains all polytopes of $\mathcal{P}^d_{\mathrm{fmi}}(1)$. In Section 7.4, we showed that the cardinality of the set $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$ grows rapidly in d. From the proof technique used there, it follows that this is also true for the cardinality of the set $\mathcal{M}^d/\operatorname{Aff}(\mathbb{Z}^d)$. Moreover, the proof of Theorem 7.2 does not imply a constructive procedure for an enumeration of the elements of $\mathcal{M}^d/\operatorname{Aff}(\mathbb{Z}^d)$.

Having applications in mixed-integer cutting plane theory in mind, it is desirable to provide a precise classification of $\mathcal{M}^d/\operatorname{Aff}(\mathbb{Z}^d)$ for small dimensions. The explicit description of $\mathcal{M}^1$ and $\mathcal{M}^2$ is easy. Indeed, $\mathcal{M}^1$ is the set of all intervals $[k, k+1]$ for an integer k. Thus, up to a unimodular transformation, the interval $[0,1]$ is the only maximal lattice-free integral polytope. In dimension two, it is easy to see that every element of $\mathcal{M}^2$ is equivalent to $\operatorname{conv}(\{o, 2e_1, 2e_2\})$. However, already the set $\mathcal{M}^3$ is rather complex. Thus, the complete enumeration of the elements of $\mathcal{M}^d$ for an arbitrary $d \geq 3$ is challenging. The main result in this chapter is that we will present the complete list of all distinct elements of $\mathcal{M}^3$.

In [Tre08, p. 3] and [Tre10, pp. 134–135], Treutlein exhibits several examples of polytopes in $\mathcal{M}^3/\operatorname{Aff}(\mathbb{Z}^3)$. We complete his classification by proving the following theorem.

Theorem 8.1. *Let $P \in \mathcal{M}^3$. Then P is equivalent to one of the following polytopes (see Fig. 8.1):*

- *one of the seven simplices*

$$\begin{aligned}
M_1 &= \operatorname{conv}(\{o, 2e_1, 3e_2, 6e_3\}),\\
M_2 &= \operatorname{conv}(\{o, 2e_1, 4e_2, 4e_3\}),\\
M_3 &= \operatorname{conv}(\{o, 3e_1, 3e_2, 3e_3\}),\\
M_4 &= \operatorname{conv}(\{o, e_1, 2e_1 + 4e_2, 3e_1 + 4e_3\}),\\
M_5 &= \operatorname{conv}(\{o, e_1, 2e_1 + 5e_2, 3e_1 + 5e_3\}),\\
M_6 &= \operatorname{conv}(\{o, 3e_1, e_1 + 3e_2, 2e_1 + 3e_3\}),\\
M_7 &= \operatorname{conv}(\{o, 4e_1, e_1 + 2e_2, 2e_1 + 4e_3\}),
\end{aligned}$$

- *the pyramid $M_8 = \operatorname{conv}(B \cup \{a\})$ with the base $B = \operatorname{conv}(\{\pm 2e_1, \pm 2e_2\})$ and the apex $a = (1, 1, 2)$,*

- *the pyramid $M_9 = \operatorname{conv}(B \cup \{a\})$ with the base $B = \operatorname{conv}(\{2e_1, -e_1, 2e_2, -e_2\})$ and the apex $a = (1, 1, 3)$,*

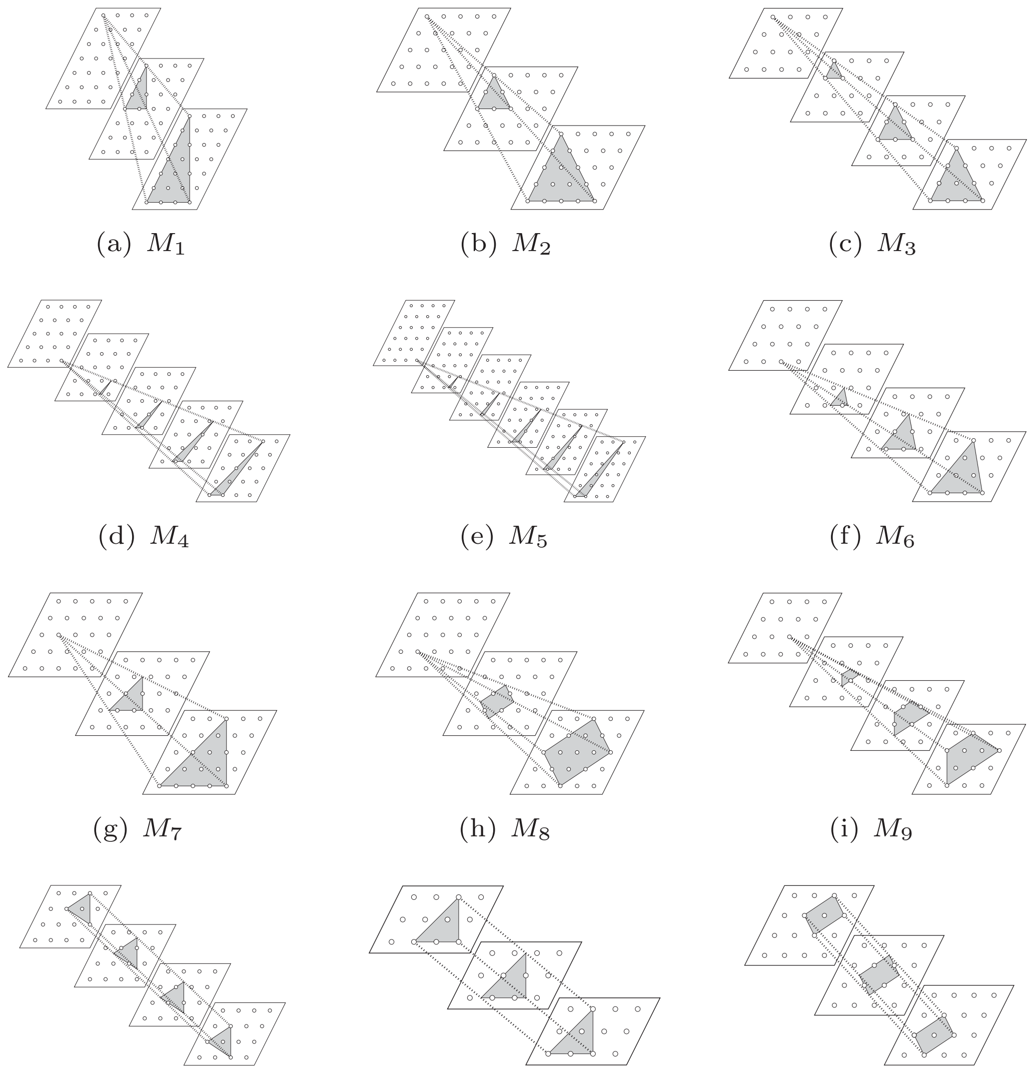
- *the prism $M_{10} = \operatorname{conv}(B \cup (B + u))$ with the two bases B and $B + u$, where $B = \operatorname{conv}(\{e_1, e_2, -(e_1 + e_2)\})$ and $u = (1, 2, 3)$,*

- *the prism $M_{11} = \operatorname{conv}(B \cup (B + u))$ with the two bases B and $B + u$, where $B = \operatorname{conv}(\{\pm e_1, 2e_2\})$ and $u = (1, 0, 2)$,*

- *the parallelepiped $M_{12} = \operatorname{conv}(\{\sigma_1 u_1 + \sigma_2 u_2 + \sigma_3 u_3 : \sigma_1, \sigma_2, \sigma_3 \in \{0, 1\}\})$ where $u_1 = (-1, 1, 0)$, $u_2 = (1, 1, 0)$, and $u_3 = (1, 1, 2)$.*

We remark that the polytopes M_1 to M_{12} in Theorem 8.1 are indeed distinct in the sense that no two of them coincide by applying a unimodular transformation.

In Section 8.1, we introduce the tools that we need for proving Theorem 8.1 and we explain the idea of the proof. The proof of Theorem 8.1 is given in Sections 8.2–8.5. It is based on a case distinction on the number of facets of a polytope in $\mathcal{M}^3$. We first show that at most six facets are possible. The analysis of polytopes with six facets is given in Section 8.2. Section 8.3 deals with polytopes that have five facets. The investigation of polytopes with four facets (i.e. of simplices) can be found in Section 8.4. Section 8.5 concludes the proof of Theorem 8.1 with remarks on the computer enumeration which we use.



(a) M_1 (b) M_2 (c) M_3

(d) M_4 (e) M_5 (f) M_6

(g) M_7 (h) M_8 (i) M_9

(j) M_{10} (k) M_{11} (l) M_{12}

Figure 8.1: All maximal lattice-free integral polytopes in dimension three.

8.1 Preliminaries and proof outline

Throughout this chapter, we fix $d = 3$ as the underlying dimension. A two-dimensional polytope is said to be a *polygon*. If P is an integral polygon, then we use $i(P)$ and $b(P)$ to denote the number of integer points in the relative interior and on the relative boundary of P, respectively. *Pick's formula* (see, for instance, [Gru07, Theorem 19.2]) relates the area of P to the quantities $i(P)$ and $b(P)$ and states that

$$A(P) = i(P) + \frac{b(P)}{2} - 1.$$

For $h \in \mathbb{Z}$ the set $\mathbb{Z}^2 \times \{h\}$ in the affine space $\mathbb{R}^2 \times \{h\}$ can be identified with the lattice $\mathbb{Z}^2$ in $\mathbb{R}^2$. Such an identification will be used several times when we analyze the facets of polytopes in $\mathcal{M}^3$. Therefore, we will need results about integral polygons with a small number of interior integer points. In particular, we need the following result of Rabinowitz.

Theorem 8.2. ([Rab89, Theorem 5].) *Let $P \subseteq \mathbb{R}^2$ be an integral polygon with exactly one interior integer point. Then P is equivalent to one of the polygons in Fig. 8.2.*

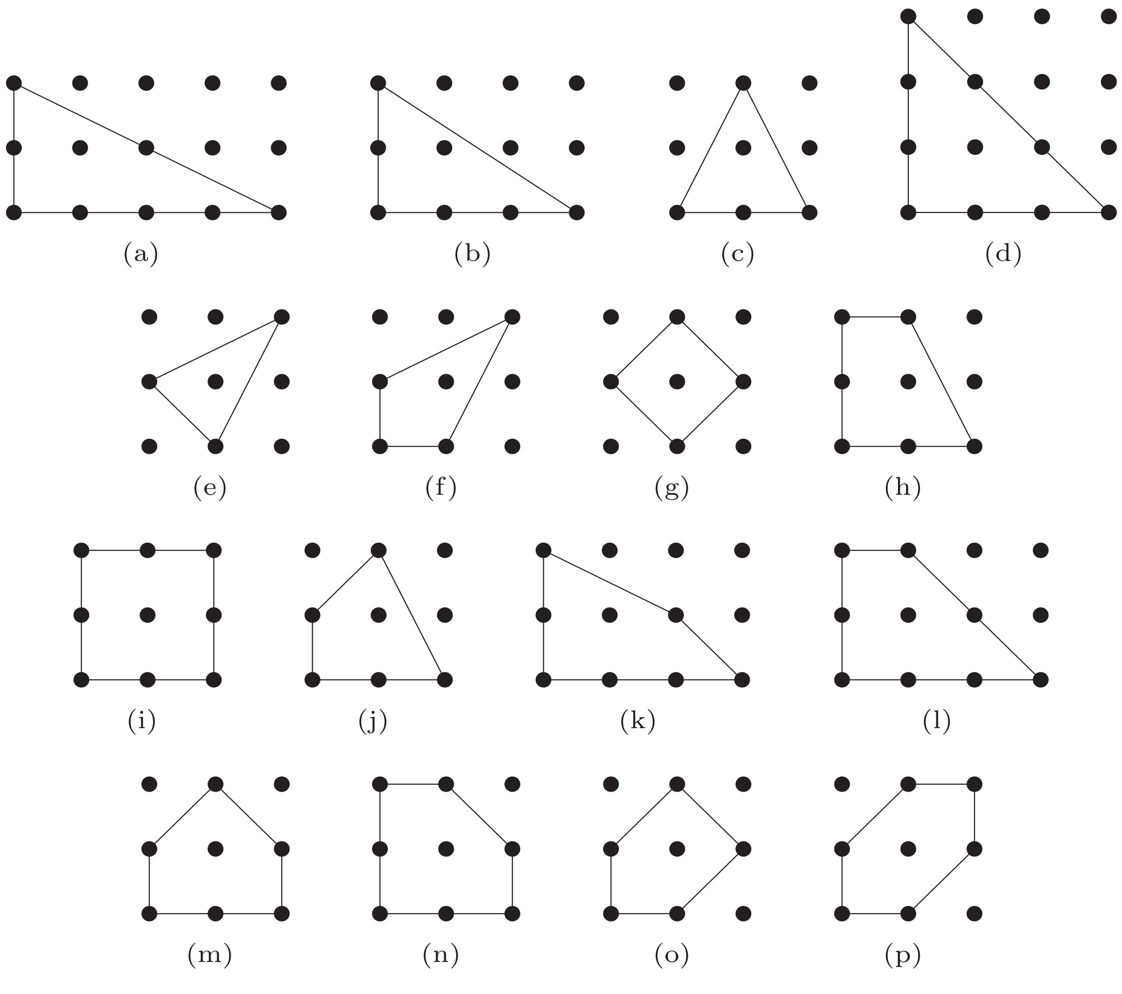


Figure 8.2: All integral polygons with one interior integer point.

Remark 8.3. *With the help of Theorem 8.2 and Lemma 7.8, the set $\mathcal{R}^2(a)$ (see p. 102 for the definition) can be computed for a given $a \in \mathbb{Z}^2$. Let us assume that $a = o$, since, by a unimodular transformation, the choice of a is not important. Then, up to a unimodular transformation, every element of $\mathcal{R}^2(o)$ coincides with one of the following sets:*

$$\begin{aligned} R_1 &:= \operatorname{conv}(\{o, 2e_1\}), \\ R_2 &:= \operatorname{conv}(\{o, 3e_1, 2e_2\}), \\ R_3 &:= \operatorname{conv}(\{o, 2e_1, e_1 + 2e_2\}), \\ R_4 &:= \operatorname{conv}(\{o, 2e_1 + e_2, 2e_2 + e_1\}). \end{aligned}$$

This can be seen as follows. By Lemma 7.8 I and III, all elements of $\mathcal{R}^2(o)$ are simplices with precisely one relative interior integer point. Thus, up to a unimodular transformation, all two-dimensional elements of $\mathcal{R}^2(o)$ appear in Fig.s 8.2(a)–8.2(e). Using Lemma 7.8 II and IV, we end up with R_2, R_3, and R_4. Obviously, R_1 is the only one-dimensional element of $\mathcal{R}^2(o)$.

Let $u, v \in \mathbb{R}^3$ and let $U, V \subseteq \mathbb{R}^3$ be linear subspaces. Then the two affine subspaces $u + U$ and $v + V$ are said to be *parallel* if $U \subseteq V$ or $V \subseteq U$. We define two polyhedra $P, Q \in \mathcal{P}^3$ to be parallel if $\operatorname{aff}(Q)$ and $\operatorname{aff}(P)$ are parallel. Furthermore, let $\mathcal{F}(P)$ and $\mathcal{F}(Q)$ be the sets of all faces of P and Q, respectively. Then P and Q are said to be *combinatorially equivalent* (or *of the same combinatorial type*) if there exists a bijection $T : \mathcal{F}(P) \to \mathcal{F}(Q)$ satisfying $T(F_1) \subseteq T(F_2)$ for all $F_1, F_2 \in \mathcal{F}(P)$ with $F_1 \subseteq F_2$. The following polytopes will be relevant:

- A polytope $P \in \mathcal{P}^3$ is called a *pyramid* if $P = \operatorname{conv}(F \cup \{a\})$, where F is a polygon and $a \in \mathbb{R}^3 \setminus \operatorname{aff}(F)$. In this case F is called the *base* of P, and a is the *apex* of P.

- A polytope $P \in \mathcal{P}^3$ is called a *prism* if $P = F + I$, where F is a polygon and I is a line segment which is not parallel to F. In this case the two polygons $F + v$ with $v \in \operatorname{vert}(I)$ are called the *bases* of P.

- A polytope $P \in \mathcal{P}^3$ is called a *parallelepiped* if $P = I_1 + I_2 + I_3$, where I_1, I_2, I_3 are line segments whose directions form a basis of $\mathbb{R}^3$.

We will start our analysis of $\mathcal{M}^3$ by showing that every element of $\mathcal{M}^3$ has at most six facets (see Lemma 8.4). This yields a quite short list of possible combinatorial types for elements of $\mathcal{M}^3$. Our analysis proceeds by the distinction of the different combinatorial types. More precisely, we first distinguish the elements of $\mathcal{M}^3$ by the number of their facets – which can be four, five or six. Then we further subdivide our analysis with respect to the different combinatorial types that can occur for a given number of facets. Let us now explain the structure of the proof of Theorem 8.1 in more detail. The proof is based on the following two ideas.

The first idea is to apply the "parity argument", a rather common tool in the geometry of numbers. Two integer points $x, y \in \mathbb{Z}^d$ are said to have the same *parity* if each component of $x-y$ is even, i.e. if $x \equiv y \pmod 2$. Obviously, $\frac{1}{2}(x+y)$ is integer if and only if x and y have the same parity. For $P \in \mathcal{M}^3$, we will apply this argument to the integer points on the boundary of P by exploiting the fact that each facet of P contains an integer point in its relative interior (see Proposition 2.1 II). Clearly, P contains at most $2^3 = 8$ integer points of different parity on its boundary. Proofs based on the parity argument are presented in Section 8.2.

The second idea is to apply the "slicing argument". Let $P \in \mathcal{M}^3$. We take an arbitrary facet F of P and assume without loss of generality that $F \subseteq \mathbb{R}^2 \times \{0\}$ and $P \subseteq \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$. Then we consider the section $F' = P \cap (\mathbb{R}^2 \times \{1\})$. By assumption, F is an integral polygon with at least one integer point in its relative interior. Moreover, F' is lattice-free in $\mathbb{R}^2 \times \{1\}$ with respect to the lattice $\mathbb{Z}^2 \times \{1\}$. It follows that either P is "not too high" with respect to F or that F contains a bounded number of integer points. Proofs based on the slicing argument are presented in Sections 8.3 and 8.4.

8.2 Elements in $\mathcal{M}^3$ with six facets

In this section, we show that there exists, up to a unimodular transformation, only one polytope in $\mathcal{M}^3$ with six facets.

Lemma 8.4. *Let $P \in \mathcal{M}^3$. Then P has at most six facets. Furthermore, if P has six facets, then each facet is either a parallelogram as in Fig. 8.2(g) or a triangle as in Fig. 8.2(c).*

Proof. We first show that P has at most six facets. Let $\mathcal{F}$ be the set of all facets of P. We choose two integer points p_1, p_2 on an edge of P with $[p_1, p_2] \cap \mathbb{Z}^3 = \{p_1, p_2\}$. For each $F \in \mathcal{F}$ we fix an integer point p_F in the relative interior of F in the following way. If $F \in \mathcal{F}$ is a facet with $p_1, p_2 \in F$, then let p_F be a point in $\operatorname{relint}(F) \cap \mathbb{Z}^3$ such that the triangle with vertices p_1, p_2, p_F has minimal area. This ensures that $[p_F, p_i] \cap \mathbb{Z}^3 = \{p_F, p_i\}$ for $i = 1, 2$. If $F \in \mathcal{F}$ and $F \cap \{p_1, p_2\} = \{p_i\}$ for some $i \in \{1, 2\}$, then let p_F be a point in $\operatorname{relint}(F) \cap \mathbb{Z}^3$ with $[p_F, p_i] \cap \mathbb{Z}^3 = \{p_F, p_i\}$. If $F \in \mathcal{F}$ and $F \cap \{p_1, p_2\} = \emptyset$, then let p_F be any point in $\operatorname{relint}(F) \cap \mathbb{Z}^3$. Let $X := \{p_1, p_2\} \cup \{p_F : F \in \mathcal{F}\}$. By construction, all points in X have different parity. Hence, $|\mathcal{F}| = |X| - 2 \leq 2^3 - 2 = 6$.

Let us now show the second part of the assertion. For that, we first show that each facet of P contains exactly one integer point in its relative interior. Assume, by contradiction, that there exists a facet F_1 containing at least two integer points in its relative interior. Choose a vertex v_1 of F_1 and two integer points $p_1, p_2 \in \operatorname{relint}(F_1) \cap \mathbb{Z}^3$ such that the triangle with vertices v_1, p_1, p_2 has minimal area. Let $e = [v_1, v_2]$ be an edge of P which is not contained in F_1 and let $\bar{v}_2$ be the integer point on the edge e which is closest to v_1. Let F_2 and F_3 be the two facets containing both v_1 and $\bar{v}_2$. Let p_3 (resp. p_4) be an integer point in the relative interior of F_2 (resp. F_3) such that $[v_1, p_i] \cap \mathbb{Z}^3 = \{v_1, p_i\}$ and $[\bar{v}_2, p_i] \cap \mathbb{Z}^3 = \{\bar{v}_2, p_i\}$ for $i = 3, 4$ (this can again be achieved by choosing triangles with minimal area). In the remaining three facets choose arbitrary relative interior integer points p_5, p_6, p_7 such that $[\bar{v}_2, p_i] \cap \mathbb{Z}^3 = \{\bar{v}_2, p_i\}$ for $i = 5, 6, 7$. By construction, the nine points $v_1, \bar{v}_2, p_1, \ldots, p_7$ must have different parity which is a contradiction.

Let now F be an arbitrary facet of P. It follows that F is one of the polygons shown in Fig. 8.2. If F is different from the quadrilateral 8.2(g) and the triangle 8.2(c), then it contains four integer points with different parity. These four integer points together with the five interior integer points of the other five facets of P are nine points of different parity which is a contradiction. □

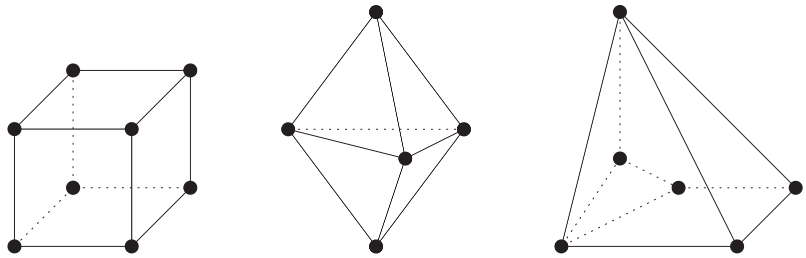
The next lemma shows that all facets of a polytope P in $\mathcal{M}^3$ with six facets are quadrilaterals as pictured in Fig. 8.2(g) and thus, the shape of P is uniquely determined.

Lemma 8.5. *Let $P \in \mathcal{M}^3$ be a polytope with six facets. Then P is a parallelepiped and each of the six facets of P is a parallelogram as in Fig. 8.2(g). In particular, P is equivalent to M_{12}.*

Proof. By Lemma 8.4, P has only two types of facets. Since quadrangular facets do not contain edges with relative interior integer points, it follows that P has an even number of triangular facets and that these facets are pairwise attached. In [Grü03, Sections 6.2 and 6.3] all possible combinatorial types[1] of three-dimensional polytopes with six facets are enumerated (there are exactly seven such types). Since each of the six facets of P is either a quadrilateral as in Fig. 8.2(g) or a triangle as in Fig. 8.2(c), and since triangular facets occur pairwise, we deduce that P is one of the three combinatorial types in Fig. 8.3.

First assume that P is of combinatorial type B, having only triangular facets. Since all facets contain exactly one edge with exactly one relative interior integer point, only two different arrangements of the three additional integer points on the edges of P are possible. The gray nodes in Fig. 8.4(a) represent these integer points. Let us now argue that P has nine points of different parity. In both cases in Fig. 8.4(a), the three gray nodes lie on three

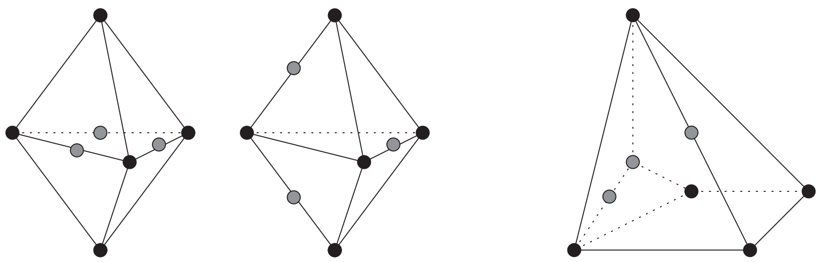
[1] Section 6.2 in [Grü03] deals with d-dimensional polytopes with $d+3$ vertices, and Section 6.3 in [Grü03] discusses their corresponding Gale diagrams (see [Grü03, Section 5.4] for a definition) and exhibits all combinatorial types of them (see [Grü03, Fig. 6.3.1]). By choosing $d = 3$ and considering the dual polytopes, we obtain the desired combinatorial types of three-dimensional polytopes with six facets.



(a) Type A. (b) Type B. (c) Type C.

Figure 8.3: Possible combinatorial types of P.

different edges such that no two of the edges belong to the same facet. Thus, the interior of a line segment joining any two of the gray nodes is contained in the interior of P. This implies that the integer points represented by the three gray nodes have different parity. Furthermore, since every facet of P contains exactly one integer point in its relative interior, every line segment with one endpoint being a relative interior integer point of a facet of P and the other endpoint being another integer point in P does not contain an integer point in its relative interior. It follows that the three gray integer points together with the six relative interior integer points of the six facets of P are nine points of different parity which is a contradiction. Thus, P cannot be of combinatorial type B.



(a) Polytopes of type B. (b) Polytope of type C.

Figure 8.4: Polytopes P of combinatorial types B and C.

Now assume that P is of combinatorial type C, having two quadrangular and four triangular facets. Then the location of the two additional integer points on its edges is already determined by the structure of the facets of P as illustrated in Fig. 8.4(b). These two points together with a particular vertex of P (the gray nodes in Fig. 8.4(b)) and the six relative interior integer points of the six facets of P sum up to nine points of different parity. Thus, P cannot be of combinatorial type C.

It follows that P must be of combinatorial type A. This implies that all facets of P are quadrangular and have the shape of the parallelogram which is shown in Fig. 8.2(g). Thus, P is equivalent to the parallelepiped M_{12}. $\square$

8.3 Elements in $\mathcal{M}^3$ with five facets

By [Grü03, Section 6.1], there are exactly two combinatorial types[2] of three-dimensional polytopes with five facets, namely *quadrangular pyramids* (i.e. polytopes which are combinatorially equivalent to a pyramid with a quadrangular base) and *triangular prisms* (i.e. polytopes which are combinatorially equivalent to a prism with triangular bases). We will analyze both combinatorial types separately.

8.3.1 Quadrangular pyramids

Let $P \in \mathcal{M}^3$ be a quadrangular pyramid with base F and apex $a = (a_1, a_2, a_3)$. By a unimodular transformation, we assume that $F \subseteq \mathbb{R}^2 \times \{0\}$ and $a_3 > 0$. We can further assume that $a_3 \geq 2$ since for $a_3 = 1$, P is contained in $\mathbb{R}^2 \times [0, 1]$ which is a contradiction to its maximality.

We first show that there is, up to a unimodular transformation, only one quadrangular pyramid $P \in \mathcal{M}^3$ with $a_3 = 2$ and $a_3 = 3$, respectively.

Lemma 8.6. *Let $P \in \mathcal{M}^3$ be a quadrangular pyramid with base $F \subseteq \mathbb{R}^2 \times \{0\}$ and apex $a = (a_1, a_2, a_3)$, where $a_3 = 2$. Then P is equivalent to the pyramid M_8.*

Proof. Let $F' := P \cap (\mathbb{R}^2 \times \{1\})$. Since each triangular facet of P contains an integer point in its relative interior, it follows that F' is a maximal lattice-free quadrilateral. Thus, F' contains precisely four integer points, one in the relative interior of each of its edges. Without loss of generality we assume that $F' \cap \mathbb{Z}^3 = \{0, 1\}^2 \times \{1\}$. By convexity, $\operatorname{vert}(F')$ lies in the union of $(0, 1) \times \mathbb{R} \times \{1\}$ and $\mathbb{R} \times (0, 1) \times \{1\}$. On the other hand $\operatorname{vert}(F') = \frac{1}{2}a + \frac{1}{2}\operatorname{vert}(F) \subseteq \frac{1}{2}\mathbb{Z}^3$. Hence $\operatorname{vert}(F')$ lies in the union of $\{\frac{1}{2}\} \times \frac{1}{2}\mathbb{Z} \times \{1\}$ and $\frac{1}{2}\mathbb{Z} \times \{\frac{1}{2}\} \times \{1\}$. Clearly, $\operatorname{vert}(F')$ is disjoint with $[0, 1]^2 \times \{1\}$. It follows that F' contains the set $B := \frac{1}{2}e_1 + \frac{1}{2}e_2 + e_3 + \operatorname{conv}(\{\pm e_1, \pm e_2\})$. If B were a proper subset of F', then one of the points from the set $\{0, 1\}^2 \times \{1\}$ would be in the relative interior of F', a contradiction. Hence $F' = B$. We have determined that, up to a unimodular transformation, F is a translate of $\operatorname{conv}(\{\pm 2e_1, \pm 2e_2\})$ and F' is a translate of B by an integer vector. This implies the assertion. $\square$

[2] Section 6.1 in [Grü03] deals with d-dimensional polytopes with $d+2$ vertices. By choosing $d = 3$ and considering the dual polytopes, we obtain the desired combinatorial types of three-dimensional polytopes with five facets.

Lemma 8.7. *Let $P \in \mathcal{M}^3$ be a quadrangular pyramid with base $F \subseteq \mathbb{R}^2 \times \{0\}$ and apex $a = (a_1, a_2, a_3)$, where $a_3 = 3$. Then P is equivalent to the pyramid M_9.*

Proof. If $p \in P \cap (\mathbb{R}^2 \times \{2\})$ is an integer point in the relative interior of a facet of P, then $2p - a \in P \cap (\mathbb{R}^2 \times \{1\})$ is also an integer point in the relative interior of the same facet of P. Consequently, $F' := P \cap (\mathbb{R}^2 \times \{1\})$ contains precisely four integer points, one in the relative interior of each of its edges. Without loss of generality we assume that $F' \cap \mathbb{Z}^3 = \{0,1\}^2 \times \{1\}$. By convexity, $\mathrm{vert}(F')$ lies in the union of $(0,1) \times \mathbb{R} \times \{1\}$ and $\mathbb{R} \times (0,1) \times \{1\}$. On the other hand $\mathrm{vert}(F') = \frac{1}{3}a + \frac{2}{3}\mathrm{vert}(F) \subseteq \frac{1}{3}\mathbb{Z}^3$. Hence $\mathrm{vert}(F')$ lies in the union of $\{\frac{1}{3}, \frac{2}{3}\} \times \frac{1}{3}\mathbb{Z} \times \{1\}$ and $\frac{1}{3}\mathbb{Z} \times \{\frac{1}{3}, \frac{2}{3}\} \times \{1\}$. Clearly, $\mathrm{vert}(F')$ is disjoint with $[0,1]^2 \times \{1\}$. A simple analysis of all possible cases reveals that, by a unimodular transformation, only one F' is possible and we can assume that $F' = \frac{1}{3}e_1 + \frac{1}{3}e_2 + e_3 + \mathrm{conv}(\{\frac{4}{3}e_1, -\frac{2}{3}e_1, \frac{4}{3}e_2, -\frac{2}{3}e_2\})$. Thus, up to a unimodular transformation, F is a translate of $\mathrm{conv}(\{2e_1, -e_1, 2e_2, -e_2\})$. This implies the assertion. $\square$

In the following we assume that $a_3 \geq 4$. Our aim is to show that no further quadrangular pyramid in $\mathcal{M}^3$ exists. The proof consists of the following steps. First, we construct all bases which are possible for such a pyramid. Second, we argue that for $a_3 \geq 11$ only two of them can appear as bases. In a third step, we analyze these two separately. Finally, the other bases are ruled out by a computer enumeration.

We start with a lemma which will be used later for simplices in Section 8.4 again.

Lemma 8.8. *Let $P \in \mathcal{M}^3$ be a simplex or a quadrangular pyramid with base $F \subseteq \mathbb{R}^2 \times \{0\}$ and apex $a = (a_1, a_2, a_3)$, where $h := a_3 \geq 4$. Then $w(F) = 2$ and the following inequalities hold:*

$$2i(F) + b(F) \leq \left\lfloor \frac{6h-4}{h-2} \right\rfloor \leq 10. \tag{8.1}$$

If P is a simplex (resp. a quadrangular pyramid), then $(i(F), b(F)) \in Z_S$ (resp. Z_Q), where

$$Z_S := \{(1,j) : j = 3, \dots, 8\} \cup \{(2,j) : j = 3, \dots, 6\},$$
$$Z_Q := \{(1,j) : j = 4, \dots, 8\} \cup \{(2,j) : j = 4, \dots, 6\}.$$

Proof. Let $F' := P \cap (\mathbb{R}^2 \times \{1\})$. Since F contains an integer point in its relative interior we have $w(F) \geq 2$. Assume that $w(F) \geq 3$. Then $h \geq 4$ implies $w := w(F') = w(F)\frac{h-1}{h} \geq \frac{9}{4} > 1 + 2 \cdot (\sqrt{3})^{-1}$. Hence, by Theorem 5.4,

F' is not lattice-free which is a contradiction. Thus, we have $w(F) = 2$ and it follows $2 > w = w(F)\frac{h-1}{h} \geq \frac{3}{2}$. Applying (5.3) to F', we obtain

$$A(F) = \left(\frac{h}{h-1}\right)^2 A(F') \leq \left(\frac{h}{h-1}\right)^2 \frac{w^2}{2(w-1)} = \frac{2h}{h-2}, \tag{8.2}$$

where the last equality follows from $w = 2\frac{h-1}{h}$. Consequently, combining (8.2) and Pick's formula we arrive at

$$2i(F) + b(F) = 2A(F) + 2 \leq \frac{6h-4}{h-2}.$$

Using that $\left\lfloor \frac{6h-4}{h-2} \right\rfloor$ is monotonically non-increasing for $h \geq 4$ yields the stated inequalities.

We now show $i(F) \leq 2$. Assume, by contradiction, that $i(F) \geq 3$. Let π denote the projection onto the first two coordinates, i.e. the mapping $\pi(x) := (x_1, x_2)$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Since $w(F) = 2$, by performing an appropriate unimodular transformation to P, we can assume that $\pi(F) = [o, 2e_1]$ (the unimodular transformation is chosen such that $F \subseteq [o, 2e_1] \times \{0\} \times \mathbb{R}$). For $x \in \pi(P)$ let $\tau(x)$ be the length of the line segment $\pi^{-1}(x) \cap P$. By the convexity of P, it follows that τ is concave on $\pi(P)$. Furthermore, we have $\tau(\pi(a)) = 0$. Our assumptions $w(F) = 2$ and $i(F) \geq 3$ imply $\tau(e_1) \geq 3$. By Lemma 7.9, $\pi(P)$ contains an integer point in its interior. The relative interior of $[e_1, \pi(a)]$ does not contain integer points. Indeed, let y be the integer point in $[e_1, \pi(a)] \setminus \{e_1\}$ closest to e_1. Then, by the concavity of τ, we have $\tau(y) \geq \tau(\frac{1}{2}e_1 + \frac{1}{2}\pi(a)) \geq \frac{1}{2}\tau(e_1) + \frac{1}{2}\tau(\pi(a)) \geq \frac{3}{2} + 0 > 1$ yielding a contradiction to the lattice-freeness of P. Thus, the interior of $\operatorname{conv}(\{o, e_1, \pi(a)\})$ or the interior of $\operatorname{conv}(\{e_1, 2e_1, \pi(a)\})$ contains an integer point. By symmetry reasons, we may assume that for $T := \operatorname{conv}(\{o, e_1, \pi(a)\})$ one has $\operatorname{int}(T) \cap \mathbb{Z}^2 \neq \emptyset$.

Let R be an element of $\mathcal{R}^2(e_1)$ which is contained in T and such that the relative interior of R contains an interior integer point of T. Then R is equivalent to one of the polygons R_1 to R_4 in Remark 8.3.

Case 1: $R \equiv R_1 \pmod{\operatorname{Aff}(\mathbb{Z}^2)}$. Then $R = [e_1, p]$ for some $p \in T \cap \mathbb{Z}^2$ and such that the point $\frac{1}{2}(e_1 + p)$ is integer and in the interior of T. By the concavity of τ, one has $\tau(\frac{1}{2}(e_1 + p)) \geq \frac{1}{2}\tau(e_1) + \frac{1}{2}\tau(p) \geq \frac{3}{2} + 0 > 1$. Thus, a contradiction to the lattice-freeness of P.

Case 2: $R \equiv R_4 \pmod{\operatorname{Aff}(\mathbb{Z}^2)}$. Then $R = \operatorname{conv}(\{e_1, p, q\})$ for some $p, q \in T \cap \mathbb{Z}^2$ and such that the point $\frac{1}{3}(e_1 + p + q)$ is integer and in the interior of T. By the concavity of τ, we have $\tau(\frac{1}{3}(e_1 + p + q)) \geq \frac{1}{3}(\tau(e_1) + \tau(p) + \tau(q)) \geq \frac{1}{3}\tau(e_1) \geq 1$. It follows that $\tau(p) = \tau(q) = 0$, since otherwise one has $\tau(\frac{1}{3}(e_1 + p + q)) > 1$ yielding a contradiction to the lattice-freeness of P. Then, in view of the choice of T, we have $p, q \in [o, \pi(a)]$. The equality $\{p, q\} = \{o, \pi(a)\}$ would imply that $a_3 = 3$ contradicting the assumption $a_3 \geq 4$. Thus, one of the points p, q (say p) lies in the relative interior of $[o, \pi(a)]$. We consider the point $2p - q$, which is the integer point on $[o, \pi(a)] \setminus [p, q]$ closest to p.

We will use the following property of R_4. Let r_1, r_2, r_3 be the vertices of R_4. Then the segment joining r_1 and $2r_2 - r_3$ (the reflection of r_3 with respect to r_2) contains precisely two integer points in its relative interior. Consider the subcase that the point $2p - q$ lies in the relative interior of $[o, \pi(a)]$. Then the relative interior of $[e_1, 2p - q]$ is contained in the interior of T. Taking into account the indicated property of R_4 we see that the relative interior of $[e_1, 2p - q]$ contains two integer points. Thus, applying the same arguments as in Case 1, we arrive at a contradiction. For the subcase that the point $2p - q$ coincides with o or $\pi(a)$, the fact that the relative interior of $[e_1, 2p - q]$ contains two integer points contradicts the fact that the segments $[o, e_1]$ and $[e_1, \pi(a)]$ do not contain integer points in their relative interiors.

Case 3: $R \equiv R_i \pmod{\mathrm{Aff}(\mathbb{Z}^2)}$ for $i \in \{2, 3\}$. Then there exists an edge e of R incident to e_1 which contains at least three integer points. Since the edge $[o, 2e_1]$ of $\pi(P)$ contains three integer points and the integer point e_1 is between the two remaining integer points, it follows that the edge e is not contained in the boundary of $\pi(P)$. Thus, on e we can find an integer point p such that $\frac{1}{2}(e_1 + p)$ is integer and in the interior of $\pi(P)$. But then, applying the same arguments as in Case 1, we arrive at a contradiction.

So far, we have shown that $i(F) \in \{1, 2\}$ and that $2i(F) + b(F) \leq 10$. If P is a simplex, then $b(F) \geq 3$. Thus, it follows $(i(F), b(F)) \in Z_S$. If P is a quadrangular pyramid, then $b(F) \geq 4$. Thus, we have $(i(F), b(F)) \in Z_Q$. $\square$

In order to continue our analysis of quadrangular pyramids $P \in \mathcal{M}^3$, we need a list of all integral quadrilaterals Q in the plane with $w(Q) = 2$ and $(i(Q), b(Q)) \in Z_Q$ since these quadrilaterals are candidates for the base of P. By (8.1), it follows that for $a_3 \geq 11$ it holds $2i(F) + b(F) \leq 6$. Therefore, the base F of such a pyramid has exactly one integer point in its relative interior and exactly the four vertices as the only integer points on its boundary. From Fig. 8.2, it follows that in this case only the two quadrilaterals in Fig.s 8.2(f) and 8.2(g) qualify as a base for P. In Lemmas 8.10 and 8.11, we will analyze these two possible bases separately from the others. However, let us first prove the following lemma.

Lemma 8.9. *Let $Q \subseteq \mathbb{R}^2$ be an integral quadrilateral with $w(Q) = 2$, $i(Q) = 2$, and $b(Q) \in \{4, 5, 6\}$. Then Q is equivalent to one of the quadrilaterals in Fig. 8.5.*

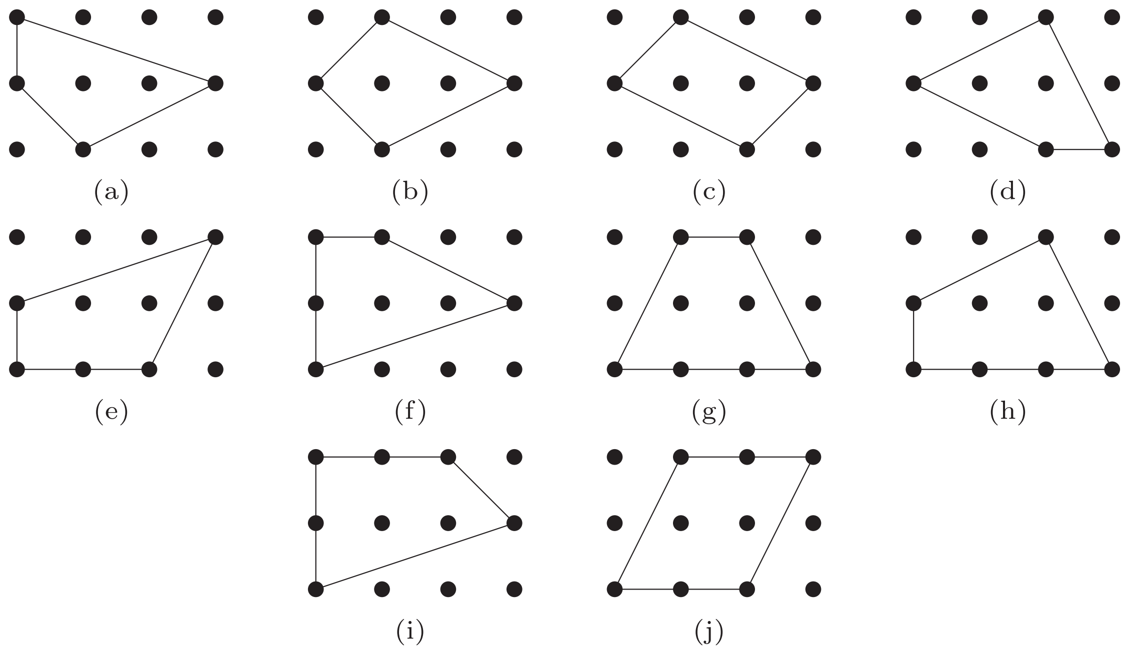


Figure 8.5: Quadrilaterals Q with $w(Q) = i(Q) = 2$ and $b(Q) \in \{4, 5, 6\}$.

Proof. Let Q be an integral quadrilateral in the plane satisfying $w(Q) = 2$ and $i(Q) = 2$. We divide the proof according to the number of integer points on the boundary of Q.

Case 1: $b(Q) = 4$. Pick's formula gives $A(Q) = 3$. By a unimodular transformation, we assume that the two interior integer points are placed at $(1, 0)$ and $(2, 0)$. This implies that for every $u \in \mathbb{Z}^2 \setminus \{o, \pm e_2\}$ we have $w(Q, u) \geq 3$ and therefore it must hold $v_2 \in \{0, \pm 1\}$ for every vertex $v = (v_1, v_2)$ of Q. We distinguish three subcases based on the number of vertices of Q that lie on the line $\mathrm{lin}(\{e_1\})$.

Subcase 1a: Two vertices of $Q = \mathrm{conv}(\{a, b, c, d\})$ lie on the line $\mathrm{lin}(\{e_1\})$. Then one vertex is $a = (0, 0)$ and the other is $c = (3, 0)$. Let the remaining two vertices b and d satisfy $d_2 = 1 = -b_2$. We can assume that $d = (0, 1)$. Indeed, if $d = (d_1, 1)$, then we apply the unimodular transformation $(x, y) \mapsto (x - d_1 y, y)$ which maps d to $(0, 1)$, but leaves the points a, $(1, 0)$, $(2, 0)$, and c untouched. For convexity reasons, it follows that $b \in \{(1, -1), (2, -1), (3, -1), (4, -1), (5, -1)\}$. Choices $b = (1, -1)$ and $b = (5, -1)$ are equivalent and lead to the quadrilateral in Fig. 8.5(a), $b = (2, -1)$ and $b = (4, -1)$ lead to Fig. 8.5(b), and $b = (3, -1)$ leads to Fig. 8.5(c).

Subcase 1b: One vertex of $Q = \mathrm{conv}(\{a, b, c, d\})$ lies on the line $\mathrm{lin}(\{e_1\})$. By a unimodular transformation, we assume that $a = (0, 0)$ and that b, c, and d satisfy $b_2 = 1 = -c_2 = -d_2$. From $b(Q) = 4$, it follows that $c_1 = d_1 + 1$. Without loss of generality we can place b at $(0, 1)$. By the convexity of Q

and since $(1,0)$ and $(2,0)$ are the only interior integer points of Q we obtain $c = (5,-1)$ and $d = (4,-1)$. This gives the quadrilateral which is shown in Fig. 8.5(d).

Subcase 1c: No vertex of $Q = \operatorname{conv}(\{a,b,c,d\})$ lies on the line $\operatorname{lin}(\{e_1\})$. Without loss of generality let $a_2 = b_2 = 1 = -c_2 = -d_2$. It follows that $b_1 = a_1 + 1$ and $c_1 = d_1 + 1$. Hence, $A(Q) = 2$ which contradicts Pick's formula.

Case 2: $b(Q) = 5$. Pick's formula gives $A(Q) = 3.5$. Placing the two interior integer points of Q at $(1,0)$ and $(2,0)$ as above implies again that $v_2 \in \{0,\pm1\}$ for each vertex $v = (v_1,v_2)$ of Q. If two vertices of Q lie on the line $\operatorname{lin}(\{e_1\})$, then Q has no edge with a relative interior integer point, a contradiction to $b(Q) = 5$. If no vertex of Q lies on the line $\operatorname{lin}(\{e_1\})$, then $A(Q) = 3$, a contradiction to Pick's formula. Thus, precisely one vertex of $Q = \operatorname{conv}(\{a,b,c,d\})$ lies on the line $\operatorname{lin}(\{e_1\})$. Place it at $a = (0,0)$. Without loss of generality let $b_2 = c_2 = -1 = -d_2$. Using an appropriate unimodular transformation we can assume that $d = (0,1)$. Thus, either the edge connecting b and c or the edge connecting c and d has a relative interior integer point which is $\frac{1}{2}(b+c)$ or $\frac{1}{2}(c+d)$, respectively. In the first case we end up with $b = (3,-1)$ and $c = (5,-1)$ (Fig. 8.5(e)), whereas the latter leads to $b = (5,-1)$ and $c = (6,-1)$ (Fig. 8.5(f)).

Case 3: $b(Q) = 6$. Pick's formula gives $A(Q) = 4$. Placing the two interior integer points of Q at $(1,0)$ and $(2,0)$ as above implies again that $v_2 \in \{0,\pm1\}$ for each vertex $v = (v_1,v_2)$ of Q. If two vertices of Q lie on the line $\operatorname{lin}(\{e_1\})$, then Q has no edge with a relative interior integer point, a contradiction to $b(Q) = 6$. We consider two subcases.

Subcase 3a: No vertex of $Q = \operatorname{conv}(\{a,b,c,d\})$ lies on the line $\operatorname{lin}(\{e_1\})$. Without loss of generality let $a_2 = b_2 = 1 = -c_2 = -d_2$. We either have $b_1 = a_1+2$ and $c_1 = d_1+2$ or $b_1 = a_1+1$ and $c_1 = d_1+3$. Using an appropriate unimodular transformation we can assume that $a = (0,1)$. Then, the first case leads to $b = (2,1)$, $c = (3,-1)$, and $d = (1,-1)$ (Fig. 8.5(j)), whereas the latter leads to $b = (1,1)$, $c = (4,-1)$, and $d = (1,-1)$ (Fig. 8.5(g)).

Subcase 3b: One vertex of $Q = \operatorname{conv}(\{a,b,c,d\})$ lies on the line $\operatorname{lin}(\{e_1\})$. Without loss of generality let $a = (0,0)$ and let b, c, and d satisfy $b_2 = 1 = -c_2 = -d_2$. Using an appropriate unimodular transformation we can assume that $b = (0,1)$. Then the edge connecting c and d has either two or one relative interior integer points. In the first case we obtain $c = (5,-1)$ and $d = (2,-1)$ (Fig. 8.5(h)). In the second case both edges, the one connecting c and d and the one connecting b and c have each one relative interior integer point and it follows $c = (6,-1)$ and $d = (4,-1)$ (Fig. 8.5(i)). $\square$

Lemma 8.9 completes the list of the possible bases of a quadrangular pyramid $P \in \mathcal{M}^3$: precisely the quadrilaterals shown in Fig.s 8.2(f)–8.2(l) and 8.5

qualify for a base of P. We will now show that there is no quadrangular pyramid $P \in \mathcal{M}^3$ with $a_3 \geq 11$.

Lemma 8.10. *Let $P \subseteq \mathbb{R}^3$ be a pyramid with base* $\operatorname{conv}(\{\pm e_1, \pm e_2\})$ *and apex $a = (a_1, a_2, a_3) \in \mathbb{Z}^3$, where $a_3 \geq 4$. Then P is not maximal lattice-free.*

Proof. By applying an appropriate unimodular transformation, we assume that $0 \leq a_i < a_3$ for $i = 1, 2$. We represent the base by $F := \operatorname{conv}(\{\pm e_1, \pm e_2\}) = \{y \in \mathbb{R}^3 : |y_1| + |y_2| \leq 1,\ y_3 = 0\}$. It follows that $P = \{x \in \mathbb{R}^3 : x = (1-\lambda)y + \lambda a$ for some $0 \leq \lambda \leq 1$ and some $y \in F\}$ and therefore

$$\begin{aligned}
\operatorname{int}(P) &= \{x \in \mathbb{R}^3 : x = (1-\lambda)y + \lambda a \\
&\qquad\qquad \text{for some } 0 < \lambda < 1 \text{ and some } y \in \operatorname{relint}(F)\} \\
&= \{x \in \mathbb{R}^3 : \frac{1}{1-\lambda}x - \frac{\lambda}{1-\lambda}a \in \operatorname{relint}(F) \text{ for some } 0 < \lambda < 1\} \\
&= \{x \in \mathbb{R}^3 : |x_1 - \lambda a_1| + |x_2 - \lambda a_2| < 1 - \lambda \\
&\qquad\qquad \text{and } x_3 = \lambda a_3 \text{ for some } 0 < \lambda < 1\}.
\end{aligned}$$

This implies

$$\begin{aligned}
&\operatorname{int}(P) \cap \mathbb{Z}^3 = \{x \in \mathbb{Z}^3 : \\
&\quad |a_3 x_1 - a_1 x_3| + |a_3 x_2 - a_2 x_3| < a_3 - x_3,\ x_3 \in \{1, \ldots, a_3 - 1\}\}.
\end{aligned} \tag{8.3}$$

From (8.3) we derive the following equivalences:

- $(0,0,1) \in \operatorname{int}(P)$ if and only if $a_1 + a_2 < a_3 - 1$,
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_1 + a_2 > a_3 + 1$,
- $(1,0,1) \in \operatorname{int}(P)$ if and only if $a_1 - a_2 > 1$,
- $(0,1,1) \in \operatorname{int}(P)$ if and only if $a_2 - a_1 > 1$.

If one of the above mentioned conditions is fulfilled, then P is not lattice-free. We can therefore assume that the following two inequalities are satisfied:

$$|a_1 + a_2 - a_3| \leq 1, \tag{8.4}$$
$$|a_1 - a_2| \leq 1. \tag{8.5}$$

It is straightforward to show that for any a_1, a_2, a_3 which satisfy (8.4) and (8.5) one has $|a_3 - 2a_1| + |a_3 - 2a_2| \leq 2$. In view of (8.3), $(1,1,2) \in \operatorname{int}(P)$ if and only if $|a_3 - 2a_1| + |a_3 - 2a_2| < a_3 - 2$. Hence, when (8.4) and (8.5) are fulfilled, then the point $(1,1,2)$ is in the interior of P for every apex a with $a_3 > 4$. It remains to exclude the case $a_3 = 4$. The only integer vectors $a = (a_1, a_2, a_3)$ which satisfy (8.4), (8.5), $a_3 = 4$, and $0 \leq a_i < a_3$ for $i = 1, 2$ are precisely the vectors in the set $\{(2,2,4), (2,1,4), (3,2,4), (1,2,4), (2,3,4)\}$. All these vectors do not correspond to maximal lattice-free pyramids. $\square$

Lemma 8.11. *Let $P \subseteq \mathbb{R}^3$ be a pyramid with base* $\operatorname{conv}(\{e_1, e_2, \pm(e_1+e_2)\})$ *and apex $a = (a_1, a_2, a_3) \in \mathbb{Z}^3$, where $a_3 \geq 4$. Then P is not maximal lattice-free.*

Proof. By applying an appropriate unimodular transformation, we assume that $0 \leq a_i < a_3$ for $i = 1, 2$. The set $\operatorname{conv}(\{e_1, e_2, \pm(e_1 + e_2)\})$ is the set of all $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ which satisfy

$$\begin{aligned} &y_1 \leq 1, && y_1 - 2y_2 \leq 1, && y_3 = 0, \\ &y_2 \leq 1, && y_2 - 2y_1 \leq 1. \end{aligned}$$

By this, using the same arguments as in the proof of Lemma 8.10, we infer that $\operatorname{int}(P)$ is the set of all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ which satisfy

$$\begin{aligned} &x_1 - \lambda a_1 < 1 - \lambda, && x_1 - \lambda a_1 - 2(x_2 - \lambda a_2) < 1 - \lambda, && x_3 = \lambda a_3, \\ &x_2 - \lambda a_2 < 1 - \lambda, && x_2 - \lambda a_2 - 2(x_1 - \lambda a_1) < 1 - \lambda, \end{aligned}$$

for some $0 < \lambda < 1$. Consequently, $\operatorname{int}(P) \cap \mathbb{Z}^3$ is the set of all vectors $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy

$$\begin{aligned} &a_3x_1 + (1 - a_1)x_3 < a_3, && a_3x_1 - 2a_3x_2 + (1 - a_1 + 2a_2)x_3 < a_3, \\ &a_3x_2 + (1 - a_2)x_3 < a_3, && a_3x_2 - 2a_3x_1 + (1 - a_2 + 2a_1)x_3 < a_3, \\ & && x_3 \in \{1, \ldots, a_3 - 1\}. \end{aligned}$$

From these inequalities we obtain that $(1, 1, 1) \in \operatorname{int}(P)$ if and only if $a_1 > 1$ and $a_2 > 1$. Hence, lattice-freeness requires that $a_1 \in \{0, 1\}$ or $a_2 \in \{0, 1\}$. By symmetry, it suffices to consider the cases $a_1 = 0$ and $a_1 = 1$.

Case 1: $a_1 = 0$. If $a_2 > 1$, then $(0, 1, 1) \in \operatorname{int}(P)$. Otherwise $(0, 0, 1) \in \operatorname{int}(P)$.

Case 2: $a_1 = 1$. If $a_2 > 3$, then $(0, 1, 1) \in \operatorname{int}(P)$. Hence, let us assume that $a_2 \leq 3$. Now, if $2a_2 < a_3$, then $(0, 0, 1) \in \operatorname{int}(P)$. So we must have $2a_2 \geq a_3$ and it follows $a_3 \in \{4, 5, 6\}$. Thus, we have $a \in \{(1, 2, 4), (1, 3, 4), (1, 3, 5), (1, 3, 6)\}$. All these vectors do not correspond to maximal lattice-free pyramids. □

Lemmas 8.10 and 8.11 restrict potential quadrangular pyramids $P \in \mathcal{M}^3$ to satisfy $4 \leq a_3 \leq 10$. Since, in addition, the set of possible bases is known from Fig.s 8.2(h)–8.2(l) and 8.5 we are left with a finite list of quadrangular candidate pyramids. Computer enumeration shows that none of them is maximal lattice-free (see Section 8.5).

8.3.2 Triangular prisms

Let $P \in \mathcal{M}^3$ be a triangular prism. We note that our definition of a triangular prism (see p. 123) does not necessarily imply that P is a prism (as defined on p. 119) in the sense that it has two bases which are parallel or even translates. Therefore, our first lemma shows that P has indeed two bases which are translates.

Lemma 8.12. *Let $P \in \mathcal{M}^3$ be a triangular prism. Then its two triangular facets are translates.*

Proof. Let H_1, H_2, and H_3 be the hyperplanes containing the quadrilateral facets of P. We show that H_1, H_2, and H_3 do not share a point. Assume the contrary and choose $p \in H_1 \cap H_2 \cap H_3$. Let T_2 be the triangular facet of P such that the pyramid S with base T_2 and apex p contains P. Let T_1 be the triangular facet of P distinct from T_2. Let q be a vertex of T_2 closest to $\mathrm{aff}(T_1)$ and let H be the hyperplane parallel to $\mathrm{aff}(T_1)$ and passing through q. If T_1 and T_2 are not parallel, then the relative interior of $P \cap H$ is contained in the interior of P. On the other hand $T_1 + q - r$, where r is the integer point $r = T_1 \cap [p, q]$, is contained in $P \cap H$. Hence the relative interior of $P \cap H$ contains an integer point, a contradiction. Thus, T_1 and T_2 are parallel. Then, since T_2 is a base of P and T_1 is a section of S parallel to T_2, we infer that T_1 and T_2 are homothetic. By construction, T_1 is strictly smaller than T_2. Since T_1 is an integral triangle which contains at least one integer point in its relative interior we have $w(T_1) \geq 2$. Therefore, since T_2 is integer and strictly larger, $w(T_2) \geq 3$. Without loss of generality we assume that $T_2 \subseteq \mathbb{R}^2 \times \{0\}$ and $T_1 \subseteq \mathbb{R}^2 \times \{h\}$ with $h \geq 2$ ($h = 1$ do not need to be considered since the quadrangular facets of P must have integer points in their relative interior). Let now $T' := P \cap (\mathbb{R}^2 \times \{1\})$. It follows that

$$w(T') = \frac{h-1}{h} w(T_2) + \frac{1}{h} w(T_1) \geq \frac{3(h-1)+2}{h} = 3 - \frac{1}{h} \geq \frac{5}{2} > 1 + \frac{2}{\sqrt{3}},$$

a contradiction to (5.1) in Theorem 5.4, since T' is a lattice-free polygon in $\mathbb{R}^2 \times \{1\}$ with respect to the lattice $\mathbb{Z}^2 \times \{1\}$. Hence H_1, H_2, and H_3 do not share a point and P is a prism. $\square$

By Lemma 8.12, it suffices to investigate triangular prisms $P \in \mathcal{M}^3$ whose triangular facets are translates. Without loss of generality we assume that the two triangular facets of P, denoted T_1 and T_2, satisfy $T_2 \subseteq \mathbb{R}^2 \times \{0\}$ and $T_1 \subseteq \mathbb{R}^2 \times \{h\}$ with $h \geq 2$. From Theorem 5.4 and the fact that P is lattice-free, it follows that the hyperplane $H := \mathbb{R}^2 \times \{1\}$ satisfies $w(P \cap H) \leq 1 + 2 \cdot (\sqrt{3})^{-1}$. Hence, $1 + 2 \cdot (\sqrt{3})^{-1} \geq w(P \cap H) = w(T_2) \geq 2$ and since $w(T_2) \in \mathbb{Z}$ we obtain $2 = w(T_2) = w(P \cap H)$. Using (5.3) yields $2 \geq A(P \cap H) = A(T_2)$

and applying Pick's formula gives $2i(T_2)+b(T_2) = 2A(T_2)+2 \leq 2\cdot 2+2 = 6$. This implies that $i(T_2) = 1$ and $b(T_2) \in \{3,4\}$. Thus, by Fig. 8.2, P has two triangular facets which are translates and which are either the triangle in Fig. 8.2(e) or the triangle in Fig. 8.2(c). We prove that for each of these two cases there exists, up to a unimodular transformation, exactly one maximal lattice-free triangular prism.

Lemma 8.13. *Let $P \in \mathcal{M}^3$ be a triangular prism whose triangular facets are triangles as in Fig. 8.2(e). Then P is equivalent to M_{10}.*

Proof. Without loss of generality we assume that the two triangular facets of P, denoted F and F', are given by $F := \operatorname{conv}(\{e_1, e_2, -(e_1+e_2)\})$ and $F' := a + F$, where $a = (a_1, a_2, a_3)$ is the integer point in the relative interior of F'. By applying an appropriate unimodular transformation, we further assume that $0 \leq a_i < a_3$ for $i = 1, 2$. Since the quadrangular facets of P need to contain integer points in their relative interior it holds $a_3 \geq 2$. By symmetry, we assume $a_1 \leq a_2$. In particular, we have $a_2 \geq 1$, otherwise $(0,0,1) \in \operatorname{int}(P)$. We now set up the facet description of P which is only dependent on the parameters a_1, a_2, and a_3. Using the same arguments as in the proof of Lemma 8.10, it follows that $\operatorname{int}(P) \cap \mathbb{Z}^3$ is the set of all $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy

$$\begin{array}{ll} a_3x_1 - 2a_3x_2 + (2a_2 - a_1)x_3 < a_3, & a_3x_1 + a_3x_2 - (a_1+a_2)x_3 < a_3, \\ a_3x_2 - 2a_3x_1 + (2a_1 - a_2)x_3 < a_3, & x_3 \in \{1, \dots, a_3 - 1\}. \end{array}$$

From these inequalities we obtain the following equivalences:

- $(0,0,1) \in \operatorname{int}(P)$ if and only if $-a_1 + 2a_2 < a_3$,
- $(0,1,1) \in \operatorname{int}(P)$ if and only if $2a_1 < a_2$,
- $(1,1,1) \in \operatorname{int}(P)$ if and only if $a_3 < a_1 + a_2$.

Thus, lattice-freeness of P implies that the following inequalities must hold:

$$a_1 + a_3 \leq 2a_2, \tag{8.6}$$

$$a_2 \leq 2a_1, \tag{8.7}$$

$$a_1 + a_2 \leq a_3. \tag{8.8}$$

Adding (8.6) and (8.8) yields $2a_1 \leq a_2$ and together with (8.7) we obtain $a_2 = 2a_1$. Substituting this into (8.6) and (8.8) leads to $a_3 \leq 3a_1$ and $3a_1 \leq a_3$ which means that $a_3 = 3a_1$. It follows that $a = (a_1, 2a_1, 3a_1)$ for some $a_1 \geq 1$. If $a_1 \geq 2$, then $(1,2,3) \in \operatorname{int}(P)$. Thus, we choose $a_1 = 1$ and end up with the prism M_{10}. ∎

Lemma 8.14. *Let $P \in \mathcal{M}^3$ be a triangular prism whose triangular facets are triangles as in Fig. 8.2(c). Then P is equivalent to M_{11}.*

Proof. Without loss of generality we assume that the two triangular facets of P, denoted F and F', are given by $F := \operatorname{conv}(\{\pm e_1, 2e_2\})$ and $F' := (a - e_2) + F$, where $a = (a_1, a_2, a_3)$ is the integer point in the relative interior of F'. By applying an appropriate unimodular transformation, we further assume that $0 \leq a_i < a_3$ for $i = 1, 2$. Since the quadrangular facets of P need to contain integer points in their relative interior it holds $a_3 \geq 2$. We now set up the facet description of P which is only dependent on the parameters a_1, a_2, and a_3. Using the same arguments as in the proof of Lemma 8.10, it follows that $\operatorname{int}(P) \cap \mathbb{Z}^3$ is the set of all $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy

$$\begin{aligned} 2a_3x_1 + a_3x_2 - (2a_1 + a_2 - 1)x_3 &< 2a_3, & -a_3x_2 + (a_2 - 1)x_3 &< 0, \\ -2a_3x_1 + a_3x_2 + (2a_1 - a_2 + 1)x_3 &< 2a_3, & x_3 &\in \{1, \ldots, a_3 - 1\}. \end{aligned}$$

From these inequalities we obtain the following equivalences:

- $(0, 1, 1) \in \operatorname{int}(P)$ if and only if $2a_1 + 1 < a_2 + a_3$,
- $(1, 1, 1) \in \operatorname{int}(P)$ if and only if $a_3 + 1 < 2a_1 + a_2$.

Thus, lattice-freeness of P implies that the following inequalities must hold:

$$a_2 + a_3 \leq 2a_1 + 1, \tag{8.9}$$
$$2a_1 + a_2 \leq a_3 + 1. \tag{8.10}$$

Adding (8.9) and (8.10) yields $a_2 \leq 1$ and therefore $a_2 \in \{0, 1\}$. We distinguish into two cases.

Case 1: $a_2 = 0$. If $2a_1 > 1$, then $(1, 0, 1) \in \operatorname{int}(P)$. Thus, we have $2a_1 \leq 1$ implying $a_1 = 0$. Substituting this into (8.9) leads to $a_3 \leq 1$ which is a contradiction.

Case 2: $a_2 = 1$. From (8.9) and (8.10), we obtain $a_3 = 2a_1$ and therefore $a = (a_1, 1, 2a_1)$ for some $a_1 \geq 1$. If $a_1 \geq 2$, then $(1, 1, 2) \in \operatorname{int}(P)$. Thus, it holds $a = (1, 1, 2)$ which leads to the prism M_{11}. □

8.4 Elements in $\mathcal{M}^3$ with four facets

Let $P \in \mathcal{M}^3$ be a simplex and let F be an arbitrary facet of P. By a unimodular transformation, we assume that $F \subseteq \mathbb{R}^2 \times \{0\}$. Throughout this section we refer to F as the base of P and denote the vertex $a = (a_1, a_2, a_3)$ of P which is not contained in $\operatorname{aff}(F)$ as the apex of P, where we assume $a_3 > 0$. We can further assume that $a_3 \geq 2$ since for $a_3 = 1$, P is contained in $\mathbb{R}^2 \times [0, 1]$ which is a contradiction to its maximality.

We first consider simplices $P \in \mathcal{M}^3$ with $a_3 = 2$ and $a_3 = 3$, respectively. For that, let $F' := P \cap (\mathbb{R}^2 \times \{1\})$. Since each facet of P contains an integer point in its relative interior, it follows that F' is a maximal lattice-free triangle. Indeed, if $a_3 = 2$, then any integer point $w = (w_1, w_2, w_3)$ in the relative interior of one of the three facets different from F satisfies $w_3 = 1$. On the other hand, if $a_3 = 3$, then any integer point $w = (w_1, w_2, w_3)$ in the relative interior of one of the three facets different from F with $w_3 = 2$ guarantees that the point $2w - a \in F'$ is also an integer point in the relative interior of the same facet as w. It follows that F' is a triangle of type 1, type 2, or type 3 (see Proposition 5.3 II on p. 37).

Lemma 8.15. *Let $P \in \mathcal{M}^3$ be a simplex with base $F \subseteq \mathbb{R}^2 \times \{0\}$ and apex $a = (a_1, a_2, a_3)$, where $a_3 \in \{2, 3\}$. Then P is equivalent to one of the simplices M_1, M_2, M_3, M_6, or M_7.*

Proof. We distinguish into three cases according to the type of triangle of $F' := P \cap (\mathbb{R}^2 \times \{1\})$.

Case 1: F' is a triangle of type 1. Without loss of generality, we can assume that F' is given by $F' = \operatorname{conv}(\{e_3, 2e_1 + e_3, 2e_2 + e_3\})$. Thus, if $a_3 = 2$, F is a translate of $\operatorname{conv}(\{o, 4e_1, 4e_2\})$ which leads to M_2. If $a_3 = 3$, F is a translate of $\operatorname{conv}(\{o, 3e_1, 3e_2\})$ which leads to M_3.

Case 2: F' is a triangle of type 2. Then there exists an edge of F' which contains at least two integer points in its relative interior. Without loss of generality we assume that this edge contains the points $(0, 0, 1)$ and $(0, 1, 1)$ in its relative interior. Let the vertex $w = (w_1, w_2, 1)$ of F' opposite to this edge satisfy $w_1 > 1$. By an appropriate unimodular transformation, we assume that the remaining two edges pass through the points $(1, 0, 1)$ and $(1, 1, 1)$.

First assume $a_3 = 2$. Then $\operatorname{vert}(F') = \frac{1}{2}a + \frac{1}{2}\operatorname{vert}(F) \subseteq \frac{1}{2}\mathbb{Z}^3$. Hence, the vertex w lies in $\frac{1}{2}\mathbb{Z} \times \{\frac{1}{2}\} \times \{1\}$ and the other two vertices of F' lie in $\{0\} \times \frac{1}{2}\mathbb{Z} \times \{1\}$. It follows that $F' = \operatorname{conv}(\{(0, \frac{3}{2}, 1), (0, -\frac{1}{2}, 1), (2, \frac{1}{2}, 1)\})$ or $F' = \operatorname{conv}(\{(0, 2, 1), (0, -1, 1), (\frac{3}{2}, \frac{1}{2}, 1)\})$. Thus, in the former case, F is a translate of $\operatorname{conv}(\{(0, 3, 0), (0, -1, 0), (4, 1, 0)\})$ leading to M_7, whereas in the latter case F is a translate of $\operatorname{conv}(\{(0, 4, 0), (0, -2, 0), (3, 1, 0)\})$ leading to M_1.

Now assume $a_3 = 3$. Then $\operatorname{vert}(F') = \frac{1}{3}a + \frac{2}{3}\operatorname{vert}(F) \subseteq \frac{1}{3}\mathbb{Z}^3$. Hence, two vertices of F' lie in $\{0\} \times \frac{1}{3}\mathbb{Z} \times \{1\}$ and the vertex w lies either in $\frac{1}{3}\mathbb{Z} \times \{\frac{1}{3}\} \times \{1\}$ or in $\frac{1}{3}\mathbb{Z} \times \{\frac{2}{3}\} \times \{1\}$. By symmetry, we can assume that w lies in $\frac{1}{3}\mathbb{Z} \times \{\frac{2}{3}\} \times \{1\}$. It follows that $F' = \operatorname{conv}(\{(0, 2, 1), (0, -2, 1), (\frac{4}{3}, \frac{2}{3}, 1)\})$ or $F' = \operatorname{conv}(\{(0, \frac{4}{3}, 1), (0, -\frac{2}{3}, 1), (2, \frac{2}{3}, 1)\})$. Therefore, in the former case, F is a translate of $\operatorname{conv}(\{(0, 3, 0), (0, -3, 0), (2, 1, 0)\})$ leading to M_1, whereas in the latter case F is a translate of $\operatorname{conv}(\{(0, 2, 0), (0, -1, 0), (3, 1, 0)\})$ leading to M_6.

Case 3: F' is a triangle of type 3. Without loss of generality let $(0,0,1)$, $(1,0,1)$, and $(0,1,1)$ be the only integer points on the relative boundary of F' and let $F' = \operatorname{conv}(\{u,v,w\})$ with $u_1 < 0$, $1 < u_2$, $1 < v_1$, $0 < v_2 < 1$, $0 < w_1 < 1$, $w_2 < 0$, and $u_3 = v_3 = w_3 = 1$ (see Fig. 8.6).

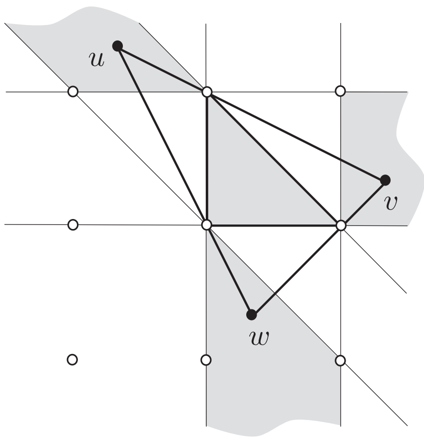


Figure 8.6: Triangle of type 3.

First assume $a_3 = 2$. Then $\operatorname{vert}(F') = \frac{1}{2}a + \frac{1}{2}\operatorname{vert}(F) \subseteq \frac{1}{2}\mathbb{Z}^3$. Thus, it follows $v_2 = w_1 = \frac{1}{2}$ and hence we obtain $v = (\frac{3}{2}, \frac{1}{2}, 1)$ and $w = (\frac{1}{2}, -\frac{1}{2}, 1)$. This implies $u = (-\frac{3}{2}, \frac{3}{2}, 1)$. However, the edge connecting u and w contains the two integer points $(0,0,1)$ and $(-1,1,1)$ in its relative interior which is a contradiction to the fact that F' is of type 3.

Now assume $a_3 = 3$. Then $\operatorname{vert}(F') = \frac{1}{3}a + \frac{2}{3}\operatorname{vert}(F) \subseteq \frac{1}{3}\mathbb{Z}^3$. Thus, it follows $v_2 \in \{\frac{1}{3}, \frac{2}{3}\}$ and $w_1 \in \{\frac{1}{3}, \frac{2}{3}\}$. Since the edge connecting v and w goes through the point $(1,0,1)$, the following cases are possible:

$$\begin{array}{llllll}
v = (\frac{5}{3}, \frac{1}{3}, 1),\ w = (\frac{1}{3}, -\frac{1}{3}, 1) & \Rightarrow & u = (-\frac{5}{3}, \frac{5}{3}, 1) & \Rightarrow & F' \text{ is of type 2,} \\
v = (\frac{7}{3}, \frac{2}{3}, 1),\ w = (\frac{1}{3}, -\frac{1}{3}, 1) & \Rightarrow & u = (-\frac{7}{6}, \frac{7}{6}, 1) & \Rightarrow & u \notin \frac{1}{3}\mathbb{Z}^3, \\
v = (\frac{4}{3}, \frac{1}{3}, 1),\ w = (\frac{1}{3}, -\frac{2}{3}, 1) & \Rightarrow & u = (-\frac{2}{3}, \frac{4}{3}, 1), & & \\
v = (\frac{5}{3}, \frac{2}{3}, 1),\ w = (\frac{1}{3}, -\frac{2}{3}, 1) & \Rightarrow & u = (-\frac{5}{9}, \frac{10}{9}, 1) & \Rightarrow & u \notin \frac{1}{3}\mathbb{Z}^3, \\
v = (\frac{4}{3}, \frac{2}{3}, 1),\ w = (\frac{1}{3}, -\frac{4}{3}, 1) & \Rightarrow & u = (-\frac{4}{15}, \frac{16}{15}, 1) & \Rightarrow & u \notin \frac{1}{3}\mathbb{Z}^3, \\
v = (\frac{4}{3}, \frac{1}{3}, 1),\ w = (\frac{2}{3}, -\frac{1}{3}, 1) & & & \Rightarrow & F' \text{ is no triangle,} \\
v = (\frac{5}{3}, \frac{2}{3}, 1),\ w = (\frac{2}{3}, -\frac{1}{3}, 1) & \Rightarrow & u = (-\frac{10}{3}, \frac{5}{3}, 1) & \Rightarrow & (-1,1,1) \in \operatorname{relint}(F'), \\
v = (\frac{4}{3}, \frac{2}{3}, 1),\ w = (\frac{2}{3}, -\frac{2}{3}, 1) & \Rightarrow & u = (-\frac{4}{3}, \frac{4}{3}, 1) & \Rightarrow & F' \text{ is of type 2.}
\end{array}$$

In seven of the above eight cases, we see that F' is not a valid triangle. In the case where $v = (\frac{4}{3}, \frac{1}{3}, 1)$, $w = (\frac{1}{3}, -\frac{2}{3}, 1)$, and $u = (-\frac{2}{3}, \frac{4}{3}, 1)$ we infer that F is a translate of $B := \operatorname{conv}(\{(2, \frac{1}{2}, 0), (\frac{1}{2}, -1, 0), (-1, 2, 0)\})$. However, it is straightforward to verify that B does not have a translate where all the three vertices are integer. $\square$

In the following we assume that $a_3 \geq 4$. Our strategy to find simplices $P \in \mathcal{M}^3$ with $a_3 \geq 4$ consists of the following steps. First, we construct all bases which are possible for such a simplex. Second, we argue that all simplices satisfy $a_3 \leq 12$. This gives a finite set of simplices that need to be checked for maximal lattice-freeness. Finally, the ultimate list of maximal lattice-free simplices is obtained by computer enumeration.

By Lemma 8.8, all integral triangles T in the plane with $w(T) = 2$ and $(i(T), b(T)) \in Z_S$ are potential candidates for the base of a simplex $P \in \mathcal{M}^3$ with $a_3 \geq 4$. From (8.1), it follows that for $a_3 \geq 11$, one has $2i(F) + b(F) \leq 6$ and therefore $(i(F), b(F)) = (1, 3)$ or $(i(F), b(F)) = (1, 4)$. If $(i(F), b(F)) = (1, 3)$, then F is, up to a unimodular transformation, the triangle shown in Fig. 8.2(e). In Lemma 8.17 we show that $a_3 \leq 12$ in this case since otherwise P is not lattice-free. If $(i(F), b(F)) = (1, 4)$, then F is, up to a unimodular transformation, the triangle shown in Fig. 8.2(c). In Lemma 8.18 we show that $a_3 \leq 8$ in this case since otherwise P is not lattice-free. Thus, we can use computer enumeration to find all simplices $P \in \mathcal{M}^3$ with $a_3 \geq 4$.

Let us now complete the list of potential bases.

Lemma 8.16. *Let $T \subseteq \mathbb{R}^2$ be an integral triangle with $w(T) = 2$, $i(T) = 2$, and $b(T) \in \{3, 4, 5, 6\}$. Then T is equivalent to one of the triangles in Fig. 8.7.*

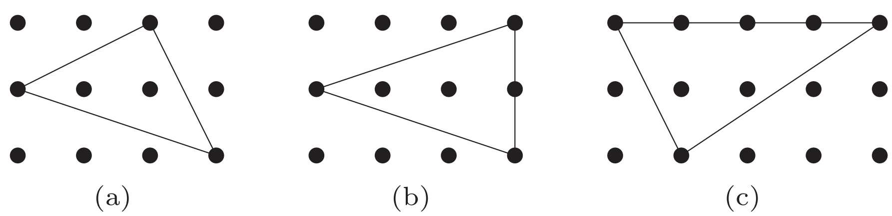


Figure 8.7: Triangles T with $w(T) = i(T) = 2$ and $b(T) \in \{3, 4, 5, 6\}$.

Proof. Let T be an integral triangle in the plane satisfying $w(T) = 2$ and $i(T) = 2$. We divide the proof according to the number of integer points on the boundary of T.

Case 1: $b(T) = 3$. By a unimodular transformation, we assume that the two interior integer points of T are placed at $(1, 0)$ and $(2, 0)$. This implies that for every $u \in \mathbb{Z}^2 \setminus \{o, \pm e_2\}$ we have $w(T, u) \geq 3$ and therefore it must hold $v_2 \in \{0, \pm 1\}$ for every vertex $v = (v_1, v_2)$ of T. Observe that exactly one vertex of $T = \mathrm{conv}(\{a, b, c\})$ lies on the line $\mathrm{lin}(\{e_1\})$, say $a = (0, 0)$. Let the remaining two vertices b and c satisfy $b_2 = 1 = -c_2$. Using an appropriate unimodular transformation we can assume that $b = (0, 1)$. For convexity reasons it follows that $c = (5, -1)$ which leads to the triangle in Fig. 8.7(a).

Case 2: $b(T) = 4$. Pick's formula gives $A(T) = 3$. Placing the two interior integer points of T at $(1, 0)$ and $(2, 0)$ as above implies again that $v_2 \in \{0, \pm 1\}$ for every vertex $v = (v_1, v_2)$ of T. Let $T = \mathrm{conv}(\{a, b, c\})$. Clearly, T cannot have two vertices on the line $\mathrm{lin}(\{e_1\})$. If none of the vertices is on the line $\mathrm{lin}(\{e_1\})$, then assume without loss of generality that $a_2 = b_2 = 1 = -c_2$. It follows that either $b_1 = a_1 + 2$ with $A(T) = 2$, or $b_1 = a_1 + 1$ with $A(T) = 1$. In both cases this is a contradiction to Pick's formula. Thus, exactly one vertex lies on the line $\mathrm{lin}(\{e_1\})$, say $a = (0, 0)$. Let the remaining two vertices b and c satisfy $b_2 = 1 = -c_2$. As above, we can assume that $b = (0, 1)$ which implies $c = (6, -1)$. This gives the triangle in Fig. 8.7(b).

Case 3: $b(T) = 5$. Pick's formula gives $A(T) = 3.5$. Placing the two interior integer points of T at $(1, 0)$ and $(2, 0)$ as above implies again that $v_2 \in \{0, \pm 1\}$ for every vertex $v = (v_1, v_2)$ of T. Clearly, T cannot have two vertices on the line $\mathrm{lin}(\{e_1\})$. If none of the vertices of T lies on the line $\mathrm{lin}(\{e_1\})$, then with similar arguments as above we infer that $A(T) \leq 3$, a contradiction to Pick's formula. Thus, precisely one vertex of $T = \mathrm{conv}(\{a, b, c\})$ lies on the line $\mathrm{lin}(\{e_1\})$, say $a = (0, 0)$. Without loss of generality let $b_2 = 1 = -c_2$. Note that the two edges connecting a and b, resp. connecting a and c, do not have integer points in their relative interior. The edge connecting b and c has at most one relative interior integer point. Therefore, we have at most four integer points on the boundary of T which is a contradiction to $b(T) = 5$.

Case 4: $b(T) = 6$. Pick's formula gives $A(T) = 4$. Placing the two interior integer points of T at $(1, 0)$ and $(2, 0)$ as above implies again that $v_2 \in \{0, \pm 1\}$ for every vertex $v = (v_1, v_2)$ of T. Clearly, T cannot have two vertices on the line $\mathrm{lin}(\{e_1\})$. If exactly one vertex of T lies on the line $\mathrm{lin}(\{e_1\})$, then using the same arguments as above we infer that T has at most four integer points on its boundary, a contradiction to $b(T) = 6$. Thus, none of the vertices of T is on the line $\mathrm{lin}(\{e_1\})$. Without loss of generality let $T = \mathrm{conv}(\{a, b, c\})$ with $a_2 = b_2 = 1 = -c_2$. Then $b_1 = a_1 + 4$, otherwise Pick's formula is violated. Using an appropriate unimodular transformation, we obtain $a = (0, 1)$, $b = (4, 1)$, and $c = (1, -1)$ (see Fig. 8.7(c)). $\square$

From Lemmas 8.8 and 8.16, it follows that the base of a simplex $P \in \mathcal{M}^3$ with $a_3 \geq 4$ has the structure shown in Fig.s 8.2(a)–8.2(e) and 8.7. Furthermore, inequalities (8.1) imply that for $a_3 \geq 11$ only 8.2(c) and 8.2(e) are possible. In the following two lemmas we will show that simplices having those two bases are not lattice-free for $a_3 \geq 13$. Thus, by computer enumeration (see Section 8.5) over all potential bases and values for a_3 ranging from 4 to 12, we obtain a finite list of simplices. Screening those which are not maximal lattice-free we end up with only two additional[3] simplices, namely M_4 and M_5.

Lemma 8.17. *Let $P \subseteq \mathbb{R}^3$ be a simplex with one facet being* $\operatorname{conv}(\{e_1, e_2, -(e_1+e_2)\})$ *and with apex $a = (a_1, a_2, a_3) \in \mathbb{Z}^3$, where $a_3 \geq 13$. Then P is not lattice-free.*

Proof. By applying an appropriate unimodular transformation, we can assume that $0 \leq a_i < a_3$ for $i = 1, 2$. By symmetry, we further assume that $a_1 \leq a_2$. We now set up the facet description of P which is only dependent on the parameters a_1, a_2, and a_3. Using the same arguments as in the proof of Lemma 8.10, it follows that $\operatorname{int}(P) \cap \mathbb{Z}^3$ is the set of all $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy

$$\begin{aligned} &a_3x_1 - 2a_3x_2 + (1 + 2a_2 - a_1)x_3 < a_3, \quad && a_3x_1 + a_3x_2 + (1 - a_1 - a_2)x_3 < a_3, \\ &a_3x_2 - 2a_3x_1 + (1 + 2a_1 - a_2)x_3 < a_3, && x_3 \in \{1, \dots, a_3 - 1\}. \end{aligned}$$

From these inequalities, it follows that

$$a_3 + 1 \geq a_1 + a_2, \tag{8.11}$$

otherwise $(1, 1, 1) \in \operatorname{int}(P)$. Assume that $a_1 = 0$. If $a_2 \leq 1$, then $(0, 0, 1) \in \operatorname{int}(P)$, otherwise $(0, 1, 1) \in \operatorname{int}(P)$. Therefore, we must have $a_1 \geq 1$. It follows that

$$2a_1 + 1 \geq a_2, \tag{8.12}$$

otherwise $(0, 1, 1) \in \operatorname{int}(P)$. Observe that $(0, 0, 1) \in \operatorname{int}(P)$ if and only if $a_1 + a_3 - 2a_2 > 1$ and $a_2 + a_3 - 2a_1 > 1$. If $a_1 \leq 3$, then $(0, 0, 1) \in \operatorname{int}(P)$ since it holds $a_1 + a_3 - 2a_2 \geq a_3 - 3a_1 - 2 \geq 2 > 1$ (where the first inequality follows from (8.12) and the second from $a_3 \geq 13$) and $a_2 + a_3 - 2a_1 = (a_2 - a_1) + a_3 - a_1 \geq a_3 - a_1 \geq 10 > 1$. Thus, we must have $a_1 \geq 4$. Using (8.11) this implies $a_3 \geq a_2 + 3$ and therefore $a_2 + a_3 - 2a_1 \geq 2(a_2 - a_1) + 3 > 1$. Hence, we have

$$2a_2 + 1 \geq a_1 + a_3, \tag{8.13}$$

otherwise $(0, 0, 1) \in \operatorname{int}(P)$. Consider the point $(1, 2, 3)$. We now show that $(1, 2, 3) \in \operatorname{int}(P)$. It holds

$$\begin{aligned} (1, 2, 3) \in \operatorname{int}(P) \qquad \Longleftrightarrow \qquad & 3a_1 + 3a_2 - 2a_3 > 3, \\ & 3a_1 - 6a_2 + 4a_3 > 3, \\ & -6a_1 + 3a_2 + a_3 > 3. \end{aligned}$$

[3] The computer enumeration delivers not only the simplices M_4 and M_5, but also some of the simplices which are already known from Lemma 8.15.

Using inequalities (8.11)–(8.13), $a_1 \geq 4$, and $a_3 \geq 13$ it follows:

$$
\begin{aligned}
3a_1 + 3a_2 - 2a_3 &\overset{(8.13)}{\geq} 5a_1 - a_2 - 2 \overset{(8.12)}{\geq} 3a_1 - 3 = 3(a_1 - 1) > 3,\\
3a_1 - 6a_2 + 4a_3 &\overset{(8.11)}{\geq} 7a_1 - 2a_2 - 4 \overset{(8.12)}{\geq} 3(a_1 - 2) > 3,\\
-6a_1 + 3a_2 + a_3 &\overset{(8.11)}{\geq} -5a_1 + 4a_2 - 1 \overset{(8.13)}{\geq} 2a_3 - 3a_1 - 3\\
&\overset{(8.11)}{\geq} 2a_2 - a_1 - 5 \overset{(8.13)}{\geq} a_3 - 6 > 3.
\end{aligned}
$$

□

Lemma 8.18. *Let $P \subseteq \mathbb{R}^3$ be a simplex with one facet being* $\operatorname{conv}(\{\pm e_1, 2e_2\})$ *and with apex $a = (a_1, a_2, a_3) \in \mathbb{Z}^3$, where $a_3 \geq 9$. Then P is not lattice-free.*

Proof. By applying an appropriate unimodular transformation, we can assume that $0 \leq a_i < a_3$ for $i = 1, 2$. We now set up the facet description of P which is only dependent on the parameters a_1, a_2, and a_3. Using the same arguments as in the proof of Lemma 8.10, it follows that $\operatorname{int}(P) \cap \mathbb{Z}^3$ is the set of all $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ which satisfy

$$
\begin{aligned}
2a_3x_1 + a_3x_2 + (2 - 2a_1 - a_2)x_3 &< 2a_3, & -a_3x_2 + a_2x_3 &< 0,\\
-2a_3x_1 + a_3x_2 + (2 + 2a_1 - a_2)x_3 &< 2a_3, & x_3 &\in \{1, \dots, a_3 - 1\}.
\end{aligned}
$$

From these inequalities we obtain the following equivalences:

- $(0, 1, 1) \in \operatorname{int}(P)$ if and only if $2a_1 + 2 < a_2 + a_3$,
- $(1, 1, 1) \in \operatorname{int}(P)$ if and only if $a_3 + 2 < 2a_1 + a_2$.

Thus, lattice-freeness of P implies that the following inequalities must hold:

$$
\begin{aligned}
2a_1 + 2 &\geq a_2 + a_3, && (8.14)\\
a_3 + 2 &\geq 2a_1 + a_2. && (8.15)
\end{aligned}
$$

Adding (8.14) and (8.15) yields $a_2 \leq 2$. Now consider the point $(1, 1, 2)$, and let us show that $(1, 1, 2) \in \operatorname{int}(P)$. It holds

$$
(1, 1, 2) \in \operatorname{int}(P) \quad\Longleftrightarrow\quad
\begin{aligned}
4a_1 + 2a_2 - a_3 &> 4,\\
-\,4a_1 + 2a_2 + 3a_3 &> 4,\\
-\,2a_2 + a_3 &> 0.
\end{aligned}
$$

Using (8.14), (8.15), $a_2 \leq 2$, and $a_3 \geq 9$ it follows:

$$\begin{aligned} 4a_1 + 2a_2 - a_3 &\overset{(8.14)}{\geq} 4a_2 + a_3 - 4 > 4, \\ -4a_1 + 2a_2 + 3a_3 &\overset{(8.15)}{\geq} 4a_2 + a_3 - 4 > 4, \\ -2a_2 + a_3 &> 4. \end{aligned}$$

□

8.5 Remarks on the computer enumeration

In this section, we want to discuss the computer enumeration which we used to finish the proof of Theorem 8.1. Computer enumeration were applied for quadrangular pyramids and simplices. The following assumptions are made.

Let Q be a quadrangular pyramid with base Q' and apex $q = (q_1, q_2, q_3) \in \mathbb{Z}^3$. By a unimodular transformation, we assume that $Q' \subseteq \mathbb{R}^2 \times \{0\}$ and $q_3 \geq 2$. We further assume that $0 \leq q_i < q_3$ for $i = 1, 2$. Moreover, let S be a simplex and let S' be an arbitrary facet of S. By a unimodular transformation, we assume that $S' \subseteq \mathbb{R}^2 \times \{0\}$. We call the facet S' the base of S, and the vertex $s = (s_1, s_2, s_3) \in \mathbb{Z}^3$ of S which is not contained in $\operatorname{aff}(S')$ is called the apex of S. We further assume that $s_3 \geq 2$ and that $0 \leq s_i < s_3$ for $i = 1, 2$.

In view of our results in Sections 8.1–8.4 it remains to verify the following.

(1) Let Q be a quadrangular pyramid with base Q' and apex q as defined above. Moreover, let Q' be one of the quadrilaterals in Fig.s 8.2(h)–8.2(l) and 8.5, and let q satisfy $4 \leq q_3 \leq 10$. Then Q does not belong to $\mathcal{M}^3$.

(2) Let S be a simplex with base S' and apex s as defined above. Moreover, let S' be one of the triangles in Fig.s 8.2(a)–8.2(e) and 8.7, and let s satisfy $4 \leq s_3 \leq 12$. Then, if S belongs to $\mathcal{M}^3$, S is equivalent to one of the simplices M_1 to M_7.

Both statements can be verified by a computer enumeration which involves less than 11 000 polytopes. Let us first consider quadrangular pyramids and statement (1). There are only 15 different possible bases. For each of these bases, we check for all values of q_3 ranging from 4 to 10 and for all values of q_1 and q_2 ranging from 0 to $q_3 - 1$ the corresponding pyramid for maximal lattice-freeness. This is done by testing (i) whether the pyramid itself has an integer point in its interior, and (ii) whether each facet has an integer point in its relative interior. Thus, we can test for maximal lattice-freeness by solving a couple of feasibility problems. For each base, there are $4^2 + 5^2 +$

$6^2 + 7^2 + 8^2 + 9^2 + 10^2 = 371$ pyramids to check. Since we have 15 different bases this makes $15 \cdot 371 = 5\,565$ pyramids in total.

Now consider simplices and statement (2). There are only 8 different possible bases. For each of these bases, we check for all values of s_3 ranging from 4 to 12 and for all values of s_1 and s_2 ranging from 0 to $s_3 - 1$ the corresponding simplex for maximal lattice-freeness. For each base, there are $4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 + 12^2 = 636$ simplices to check. Since we have 8 different bases this makes $8 \cdot 636 = 5\,088$ simplices in total.

Remark 8.19. *The number of candidate pyramids and candidate simplices could be reduced further by considering each base separately in order to obtain better bounds on q_3 and s_3, respectively. By doing this, we were able to reduce the number of candidate polytopes to about $3\,000$. Unfortunately, this did not speed up the computer enumeration considerably.*

We did not succeed in getting rid of the computer enumeration completely, but we believe that it should be possible to find for each single base an algebraic argument which makes the computer enumeration superfluous.

OUTLOOK

In this chapter, we propose four topics for future research directions.

The results of this thesis can be viewed as a theoretical starting point for the development of a cutting plane algorithm for mixed-integer linear programs (MILP's). However, it is still a long way to go until practically relevant real-life MILP's can be solved with the help of a cutting plane algorithm which is based on the lattice-free polyhedra presented in this thesis. We think that the following research questions might be of interest in order do the next step. Answers to them would at least nicely complement the results in this thesis.

(1) The evaluation of facet-defining inequalities for $\operatorname{conv}(P_I)$ in Chapters 4 and 6 is based on the strength measure of Goemans [Goe95]. Furthermore, in Chapter 6, a particular probabilistic model for the two-dimensional case is considered and the addition of a single non-split inequality on top of the split closure is analyzed. From the literature, it seems that the evaluation of cutting planes is highly dependent on

   (a) the used strength measure,

   (b) the probabilistic (or non-probabilistic) model,

   (c) whether a single cut or a family of cuts is considered,

   (d) whether one round or several rounds of cuts are applied

   (see, for instance, [AW10], [AWW09b], [BBCM11], [BCM11], [BCM10], [BS08], [DL09], [DPW11], [DPWW11a], [DPWW11b], [HAN10]). Indeed, depending on how the parameters (a)–(d) are chosen, it is inferred in the mentioned literature that certain cuts are strong or weak, important or less important, indispensable or dispensable. In fact, for particular families of cuts it is known that they are needed with respect to one constellation of the parameters (a)–(d), but neglectable for another constellation. We think that a comparison of the different choices of the parameters (a)–(d) could help to get more insight.

(2) In Theorem 5.6, the relation between the area and the lattice width of arbitrary lattice-free sets $K \in \mathcal{K}^2$ is presented. All the stated inequalities are best possible (in the sense that there exist lattice-free sets in $\mathcal{K}^2$ satisfying the inequalities with equality), except for the inequality in (5.5) – which is not tight when $2 < w(K) \leq 1 + 2 \cdot (\sqrt{3})^{-1}$. Unfortunately, we were not able to fill this little gap. Therefore, further research is needed. Of course, to know the exact inequality is not relevant for a cutting plane algorithm, but it is mathematically interesting and would complete the list of inequalities.

(3) In Theorem 7.2, finiteness of the set $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ (and thus of the set $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$) is shown. We think that both sets deserve a more thorough analysis. Let us explain why and let $Q \in \mathcal{P}^d_{\mathrm{ifm}}(s)$ be arbitrary. Then one can construct an MILP such that the feasible region of its linear programming relaxation is a polytope P, and such that adding to P the cut associated with Q gives the X-body of P, but adding to P all the cuts associated with the elements in $\mathcal{P}^d_{\mathrm{ifm}}(s) \setminus \{Q\}$ does not give the X-body of P (see, for instance, [Ave11, Theorem 3.3]). Hence, it is desirable to have a classification of the elements of $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$ at hand. The results in Sections 7.3 and 7.4 imply that this is a very challenging problem since even for the set $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$, the number and the volume of the elements grow dramatically in d. Moreover, in Chapter 8, we saw that already for the case $d = 3, s = 1$ it is quite involved to find all elements of $\mathcal{P}^3_{\mathrm{fmi}}(1)/\operatorname{Aff}(\mathbb{Z}^3)$.

An interesting task for future research could be to classify the elements of the two sets $\mathcal{P}^d_{\mathrm{fmi}}(s)/\operatorname{Aff}(\Lambda)$ and $\mathcal{P}^d_{\mathrm{ifm}}(s)/\operatorname{Aff}(\Lambda)$. For that purpose, it seems necessary to identify further structural properties of both sets, for instance in terms of the integer points on the boundary of their representatives or better volume bounds than the ones we have found. It would also be interesting to know for which of the open cases $d = 2, s = 2$, and $d = 3, s \in \{1, 2\}$ the equality $\mathcal{P}^d_{\mathrm{fmi}}(s) = \mathcal{P}^d_{\mathrm{ifm}}(s)$ holds true.

(4) Apart from the open questions above, the ultimate goal of the presented cutting plane approach is to develop a cutting plane algorithm which

is based on facet-defining inequalities for $\mathrm{conv}(P_I)$. If an MILP as in (1.1) is given, then a corresponding set $\mathrm{conv}(P_I)$ can be defined. Now, any lattice-free polyhedron in the x-variable space that contains the root vertex f in its interior can be used to generate a cutting plane. However, from a practical point of view, it is not clear at all which lattice-free polyhedra one should use to derive strong cutting planes. Within a cutting plane framework they must be computed based on the available data, i.e. the given matrix A, the right hand side vector b, the objective function vector c, the set of integer constrained variables $\mathcal{I}$, and the updated sets P_I which are computed during the algorithm. It seems natural to determine the lattice-free polyhedra with the help of auxiliary MILP's within the algorithm. For instance, let us assume that $m = 2$. Then it is known (see [DL09], [BCM11], and [DPW11]) that split and type 1 inequalities are sufficient to compute an optimal mixed-integer point in a finite number of rounds. Usually, one would start by adding split cuts to the linear programming relaxation of (1.1). After some rounds it might be the case that the improvement in terms of the objective function value diminishes. If it falls below a certain value, then type 1 inequalities should be applied. The problem is now to determine a "good" type 1 triangle, where "good" could, for instance, be defined as having a small sum of coefficients of the corresponding cut. So far, no convincing way is known to generate such a type 1 triangle algorithmically, not to mention more complicated objects than type 1 triangles. Thus, from the practical point of view, future research might focus on the development of techniques to utilize lattice-free polyhedra for cutting plane generation.

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