Carsten Trunk

Spectral Properties of a Class of Analytic Operator Functions and Their Linearizations

Cuvillier Verlag Göttingen

Spectral Properties of a Class of Analytic Operator Functions and Their Linearizations

vorgelegt von

Diplom–Mathematiker Carsten Trunk

aus Berlin

Von der Fakultät II – Mathematik und Naturwissenschaften

der Technischen Universität Berlin

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

– Dr. rer. nat. –

genehmigte Dissertation.

Promotionsausschuss:

Vorsitzender:	Prof. Dr. Rainer Wüst
Berichter:	Priv.–Doz. Dr. Peter Jonas
Berichter:	Prof. Dr. Karl–Heinz Förster

Tag der wissenschaftlichen Aussprache: 19. November 2002

Berlin 2003

D 83

Bibliografische Information Der Deutschen Bibliothek
Die Deutsche Bibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.ddb.de abrufbar.
1. Aufl. - Göttingen : Cuvillier, 2003
Zugl.: Berlin, Univ., Diss., 2003
ISBN 3-89873-718-7

Nonnenstieg 8, 37075 Göttingen
Telefon: 0551-54724-0
Telefax: 0551-54724-21
www.cuvillier.de

1. Auflage, 2003
Gedruckt auf säurefreiem Papier

ISBN 3-89873-718-7

Zusammenfassung

In dieser Arbeit untersuchen wir eine Operatorfunktion T in einem Kreinraum, welche formal geschrieben werden kann als

$$T(\lambda) = \lambda - A + B^{+}(D - \lambda)^{-1}B,$$

wobei anstelle des letzten Terms auf der rechten Seite eine bezüglich A relativ formkompakte Störung ähnlicher Gestalt steht. Die Operatorfunktion $-T^{-1}$ kann dann mittels der Resolvente eines in einem Kreinraum selbstadjungierten Operators $\mathbf{M}$, der eine relativ formkompakte Störung von $A \times D$ ist, dargestellt werden. Wir beschreiben die Beziehungen zwischen der Operatorfunktion T und dem Operator $\mathbf{M}$, insbesondere untersuchen wir das Spektrum, das Punktspektrum und das Spektrum positiven bzw. negativen Typs.

Unter bestimmten Voraussetzungen an die Operatoren A, B und D ist $\mathbf{M}$ ein definisierbarer Operator und $-T^{-1}$ eine definisierbare Operatorfunktion. In diesem Fall beschreiben wir die Beziehungen zwischen dem Spektrum positiven bzw. negativen Typs, einschließlich der entsprechenden Vielfachheiten, von $\mathbf{M}$ und $-T^{-1}$.

Die dabei gewonnenen Ergebnisse werden auf ein Sturm–Liouville–Problem angewandt, bei dem die Koeffizienten rational vom Eigenwertparameter abhängen. In diesem Fall entspricht die Operatorfunktion T dem Differentialausdruck

$$py'' + \lambda y + \sum_{j=1}^{n_+} \frac{q_j^+}{u_j^+ - \lambda} y + \sum_{j=1}^{n_-} \frac{q_j^-}{u_j^- - \lambda} y$$

auf dem Interval $I := [-1, 1]$. Dabei ist λ eine komplexe Zahl, p ein einfaches indefinites Gewicht, und $q_j^{\pm}$, $u_j^{\pm}$ sind reellwertige meßbare Funktionen, die bestimmten Voraussetzungen genügen. Weiterhin betrachten wir den Fall, daß der obige Differentialausdruck auf der Halbachse $I = [0, \infty)$ mit $p \equiv 1$ erklärt ist.

Acknowledgements

I would like to express my deep gratitude to my supervisor Priv.–Doz. Dr. P. Jonas for his help and invaluable suggestions and to Prof. Dr. K.–H. Förster for his constant support.

Furthermore, I am grateful to the Hanns–Seidel–Stiftung, München, and to the Technische Universität Berlin for the financial support I received.

Finally, I would like to thank my family.

Contents

Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces, let A and D be selfadjoint operators in $\mathcal{H}$ and $\mathcal{K}$, respectively, with nonempty resolvent sets and let B be a bounded operator in $\mathcal{L}(\mathcal{H},\mathcal{K})$. For all λ in the resolvent set $\rho(D)$ we define an operator function

$$T(\lambda) = \lambda - A + B^{+}(D-\lambda)^{-1}B, \tag{0.1}$$

where B^{+} denotes the Krein space adjoint of B. Then for all $\lambda \in \rho(D)$, for which $0 \in \rho(T(\lambda))$, the operator function $-T^{-1}$ can be represented in the form

$$-T(\lambda)^{-1} = P_1(\mathbf{M}-\lambda)^{-1}I_1 \tag{0.2}$$

where $\mathbf{M}$ is given by the operator matrix

$$\mathbf{M} = \begin{bmatrix} A & B^{+} \\ B & D \end{bmatrix} \tag{0.3}$$

in $\mathcal{H}\times\mathcal{K}$, I_1 is the embedding of $\mathcal{H}$ in $\mathcal{H}\times\mathcal{K}$ and P_1 the projection on the first component in $\mathcal{H}\times\mathcal{K}$.

In this thesis we consider operator functions T which can formally be written as in (0.1). We relax the boundedness condition on B. The last term on the right of (0.1) is replaced by a term of a similar form which is a relatively compact perturbation in form sense with respect to A. This compactness assumption includes the case when A has a compact resolvent and B is bounded.

Analogously to the case of a bounded operator B, the operator function $-T^{-1}$ can then be represented as in (0.2) where the operator $\mathbf{M}$ arises from $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ by a relatively compact perturbation in form sense.

In this thesis we express relatively compact perturbations in form sense with the help of operators in riggings. In Chapter 1 we review some facts on riggings in Krein spaces. We also give a brief introduction to the theory of definitizable and locally definitizable selfadjoint operators in Krein spaces. In particular we discuss relatively form–compact perturbations of definitizable selfadjoint operators and compact perturbations of fundamentally reducible operators in Krein spaces.

Our main objective is to describe relations between spectral properties of the holomorphic operator function T and the operator $\mathbf{M}$. In Section 2.1 we introduce the notions of resolvent set, spectrum, point spectrum and Jordan chains of the operator function T. Then (cf. Section 2.3) a point λ where the function T is holomorphic, that is $\lambda \in \rho(D)$, belongs to the resolvent set of T if and only if λ belongs to the resolvent set of the operator $\mathbf{M}$. The same equivalence holds for the point spectrum. Special attention is given to the spectrum of positive and negative type of T, resp. $\mathbf{M}$. As the domain of the operator $T(\lambda)$ may depend on λ, we define the sign types of spectral points of T (i.e. spectral points of positive or negative type of T) via some rational function $f(T(\lambda))$ of $T(\lambda)$ which has values in $\mathcal{L}(\mathcal{H})$. This definition generalizes the usual one for $\mathcal{L}(\mathcal{H})$-valued functions (see [LMaM2]). It turns out that the sign types of spectral points of T can be characterized by the sign types of an extension of T to an operator of the space of positive norm to the space of negative norm of some rigging which has a domain independent of λ. It then follows that they coincide with the sign types with respect to $\mathbf{M}$ (Sections 2.1–2.3).

In Sections 2.4 and 2.5 we assume that A and D are definitizable selfadjoint operators and fulfil some further conditions such that by a perturbation result from [J3] the operator $\mathbf{M}$ is definitizable. The sign types of spectral points of T, first defined only for points λ of holomorphy of T, that is for $\lambda \in \rho(D) \cap \mathbb{R}$, can be extended to arbitrary real λ by making use of the (boundary behaviour near $\mathbb{R}$ of the) function $-T^{-1}$, which is a so–called definitizable operator function ([J4]). For points outside of $\rho(D) \cap \mathbb{R}$ the so defined sign type coincides with that of $\mathbf{M}$ if $\mathbf{M}$ satisfies some minimality condition (Proposition 2.18). Lemma 2.19 provides a simple criterion for this minimality. Similar relations hold if the sign types are replaced by the so–called intervals of type π_+ and type π_- (Proposition 2.18, Theorem 2.22).

Making an additional assumption on A and D and using a minimal representing operator for an N_κ–function we determine a minimal representing operator for $-T^{-1}$ such that this operator is unitarily equivalent to $\mathbf{M}$, if $\mathbf{M}$ is minimal (Theorem 2.17, Proposition 2.18). Here unitary equivalence is understood with respect to the inner products of the Krein spaces. For non–minimal $\mathbf{M}$ there is a local variant of this fact (Theorem 2.20).

Connections between T and $\mathbf{M}$ in the case where $\mathcal{H}$ and $-\mathcal{K}$ are Hilbert spaces have been studied in the articles [LMeM], [FM], [AL], [MS]. In these articles, in the Krein space setting, it is always assumed that $\sigma(A) \cap \sigma(D)$ is empty or a finite set and, on the other hand, that either the resolvent of A is compact or B is, in some sense, small with respect to A and D. In the publications mentioned above also completeness problems for the eigenfunctions and associated functions of T were investigated. In the present thesis we do not deal with completeness questions for T.

In [LMeM] $T(\lambda)$ is the operator in $L_2([0,1])$ corresponding to the differential expression

$$y'' + \lambda y + \frac{q}{u-\lambda} y, \tag{0.4}$$

and the boundary conditions

$$y(0) = y(1) = 0. \tag{0.5}$$

Here q and u belong to $L_\infty([0,1])$ and $\operatorname{ess\,sup} q < 0$. Then

$$\mathbf{M} := \begin{bmatrix} -\frac{d^2}{dx^2} & -\sqrt{-q} \\ \sqrt{-q} & u \end{bmatrix}$$

in $\mathcal{G} := L_2([0,1]) \times L_2([0,1])$ is of the form (0.3) and satisfies (0.2) (here $\mathcal{G}$ is considered as a Krein space with fundamental symmetry $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). In [LMeM] it is proved that $\mathbf{M}$ is a definitizable selfadjoint operator and that, if u is a step function, the eigenvectors and associated vectors of $\mathbf{M}$ form a Riesz basis.

In Chapter 3 we apply the results of Chapter 2 to Sturm–Liouville operators which are similar to (0.4). In Chapter 3 the relations between the sign types of T and $\mathbf{M}$ considered in Chapter 2 play an essential role.

In Section 3.1 we consider the case that $T(\lambda)$ is the operator in $L_2([-1,1])$ corresponding to the differential expression

$$py'' + \lambda y + \sum_{j=1}^{n_+} \frac{q_j^+}{u_j^+ - \lambda} y + \sum_{j=1}^{n_-} \frac{q_j^-}{u_j^- - \lambda} y, \tag{0.6}$$

with $\lambda \in \mathbb{C}$, on the interval $I := [-1,1]$ with boundary conditions

$$y(-1) = y(1) = 0. \tag{0.7}$$

The function p is identically equal to 1 or a simple indefinite weight. The functions $q_j^\pm$, $u_j^\pm$ are real valued measurable functions, $q_j^+ \geq 0$, $j = 1, \dots, n_+$, $q_j^- \leq 0$, $j = 1, \dots, n_-$, a.e. such that $q_j^\pm (1 + |u_j^\pm|)^{-1} \in L_1(I)$, $j = 1, \dots, n_\pm$. Let D be the diagonal matrix multiplication operator

$$D = \operatorname{diag}(u_1^+, \dots, u_{n_+}^+, u_1^-, \dots, u_{n_-}^-),$$

in $\mathcal{K} := L_2(I)^{n_+} \times L_2(I)^{n_-}$, where $\mathcal{K}$ is considered as a Krein space with fundamental symmetry $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the operator $\mathbf{M}$ arises from $(-p\frac{d^2}{dx^2}) \times D$ by a relatively compact perturbation in form sense and satisfies (0.2)(for a definition of $\mathbf{M}$ see page 59). It is a consequence of Section 2.3 that a point $\lambda \in \rho(D)$ belongs to the resolvent set (point spectrum, spectrum of positive type, spectrum of negative type) of T if and only if it belongs to the resolvent set (point spectrum, spectrum of positive type, spectrum of negative type) of $\mathbf{M}$.

Under some additional assumptions on the functions $u_j^{\pm}$, $j = 1, \ldots, n_{\pm}$ (which, in essence, imply that D is a definitizable selfadjoint operator in $\mathcal{K}$ such that D has no finite critical points), it follows that $\mathbf{M}$ is a definitizable selfadjoint operator and $T(\lambda)^{-1}$ is a definitizable operator function. In addition we prove a simple criterion for the minimality of $\mathbf{M}$ with respect to $-T^{-1}$ (cf. Theorem 3.3). If $\mathbf{M}$ is minimal with respect to $-T^{-1}$, then, by the considerations of Section 2.5, an open subset of $\overline{\mathbb{R}}$ is of positive type (negative type, type π_+, type π_-) with respect to $\mathbf{M}$ if and only if it is of the same type with respect to $-T^{-1}$. Finally, if we assume that all the functions $u_j^{\pm}$, $j = 1, \ldots, n_{\pm}$, are step functions, we can show that there exists a Riesz basis consisting of eigenvectors and associated vectors of $\mathbf{M}$.

In Section 3.2 $T(\lambda)$ is again the operator corresponding to the expression (0.6). Now we assume that $p \equiv 1$ and $I = [0, \infty)$. Instead of (0.7) we consider the boundary condition

$$y(0) = 0.$$

In this case we obtain the same relations between the various kinds of spectra of T and $\mathbf{M}$ as in Section 3.1. Moreover, under some additional assumptions on the functions $u_j^{\pm}$, $j = 1, \ldots, n_{\pm}$, the operator $\mathbf{M}$ is a definitizable operator and, again, $T(\lambda)^{-1}$ is a definitizable operator function. In Proposition 3.7 we give an example for a situation where results on the absence of positive eigenvalues for Sturm–Liouville operators can be used, in combination with the relations between the spectra of T and $\mathbf{M}$, to exclude critical points of $\mathbf{M}$ on the positive half–axis.

In Section 3.3 $T(\lambda)y$ is given by (0.4) on the interval $I = [-1, 1]$ with the boundary condition (0.7). In contrast to [LMeM], we allow q to change its sign. For simplicity, we assume that q is a real valued piecewise continuous function and that u is a real valued measurable function. Now, roughly speaking, $q(u-\lambda)^{-1}$ can be considered as a sum of two quotients $q_+(u_+-\lambda)^{-1}$ and $q_-(u_- - \lambda)^{-1}$, where the first one is defined on $\Delta_+ := \{x \in I : q(x) > 0\}$, the second one on $\Delta_- := \{x \in I : q(x) < 0\}$, and $q_{\pm}$ and $u_{\pm}$ are the restrictions of q and u to $\Delta_{\pm}$. Then $\mathbf{M}$ arises from $(-\frac{d^2}{dx^2}) \times u_+ \times u_-$

by a compact perturbation (in the resolvent sense). It follows that $\mathbf{M}$ is definitizable over the set

$$\overline{\mathbb{C}} \setminus ((\{\infty\} \cup \sigma_e(u_+)) \cap \sigma_e(u_-)).$$

If the functions q and u belong to $C^1(I)$ such that $u' > 0$ and q has finitely many zeros, we are able to prove that $\mathbf{M}$ is a definitizable operator in the space $L_2(I) \times L_2(\Delta_+) \times L_2(\Delta_-)$.

Finally, we consider the case of the half–axis, where $T(\lambda)$ is given by (0.4) with boundary condition $y(0) = 0$ (cf. Section 3.4). Then the operator $\mathbf{M}$ is definitizable over $\overline{\mathbb{C}} \setminus \{([0, \infty] \cup \sigma_e(u_+)) \cap \sigma_e(u_-)\}$ and, if q and u fulfil some further conditions, $\mathbf{M}$ is a definitizable operator. Moreover, the absence of eigenvalues of T can be used to locate the position of critical points of $\mathbf{M}$.

1. Riggings and Perturbations of Selfadjoint Operators in Krein Spaces

1.1. The Scale of Spaces Associated with a Selfadjoint Operator in a Hilbert Space

We recall some well-known facts on the scale of spaces associated with a selfadjoint operator H in a Hilbert space $(\mathcal{H}, (\cdot,\cdot))$. We equip $\mathcal{H}_s(H) := \mathcal{D}(|H|^s)$, $s \in [0,\infty)$, with the Hilbert space scalar product

$$(x,y)_s := ((1+H^2)^{\frac{s}{2}}x, (1+H^2)^{\frac{s}{2}}y), \quad x,y \in \mathcal{H}_s(H). \tag{1.1}$$

Put $\|x\|_s := (x,x)_s^{\frac{1}{2}}$, $x \in \mathcal{H}_s(H)$. By $\mathcal{H}_{-s}(H)$, $s \in [0,\infty)$, we denote the completion of $\mathcal{H}$ with respect to the quadratic norm $\|\cdot\|_{-s}$ defined by

$$\|x\|_{-s} = \|(1+H^2)^{-\frac{s}{2}}x\|, \quad x \in \mathcal{H}.$$

Evidently,

$$\|x\|_{-s} = \sup\{|(x,y)| : y \in \mathcal{H}_s(H), \|y\|_s \le 1\}.$$

The extension by continuity of the form $(\cdot,\cdot)$ to $\mathcal{H}_s(H) \times \mathcal{H}_{-s}(H)$, $s \in \mathbb{R}$, is also denoted by $(\cdot,\cdot)$. The mapping $x \mapsto g_x$, $x \in \mathcal{H}_{-s}(H)$, $g_x \in \mathcal{H}_s(H)$, $s \in [0,\infty)$, defined by

$$(y,x) = (y,g_x)_s, \quad y \in \mathcal{H}_s(H),$$

is an isometry of $\mathcal{H}_{-s}(H)$ onto $\mathcal{H}_s(H)$. For arbitrary $z \in \mathbb{C}$ the operator $H-z$ can be extended by continuity to a continuous linear operator $(H-z)^\sim$ from $\mathcal{H}_{\frac{1}{2}}(H)$ into $\mathcal{H}_{-\frac{1}{2}}(H)$. $(H-z)^\sim$ is an isomorphism of $\mathcal{H}_{\frac{1}{2}}(H)$ onto $\mathcal{H}_{-\frac{1}{2}}(H)$ if and only if $z \in \rho(H)$. In this case we have $((H-z)^\sim)^{-1} = \widetilde{R}(z,H)$ where $\widetilde{R}(z,H)$ is the extension by continuity of $R(z,H) := (H-z)^{-1}$ to a continuous linear operator of $\mathcal{H}_{-\frac{1}{2}}(H)$ onto $\mathcal{H}_{\frac{1}{2}}(H)$. The operator $\widetilde{H}$ is $(\cdot,\cdot)$-symmetric, i. e.

$$(\widetilde{H}x, y) = (x, \widetilde{H}y), \;\; x, y \in \mathcal{H}_{\frac{1}{2}}(H).$$

1.2. The Scale of Spaces Associated with a Selfadjoint Operator in a Krein Space

Assume now that $(\mathcal{H}, [\cdot,\cdot])$ is a Krein space and J is a fundamental symmetry of $\mathcal{H}$, $(x, y) := [Jx, y]$, $x, y \in \mathcal{H}$. Let A be a selfadjoint operator in the Krein space $\mathcal{H}$ with $\rho(A) \neq \emptyset$. We consider the selfadjoint operators

$$H := JA, \quad K := AJ$$

in $(\mathcal{H}, (\cdot,\cdot))$. By the relation $HJ = JK$ the operator J maps $\mathcal{H}_s(H)$ ($\mathcal{H}_s(K)$) isometrically onto $\mathcal{H}_s(K)$ ($\mathcal{H}_s(H)$, resp.), $s \in [0, \infty)$. Therefore, J can be extended by continuity to an isometric operator of $\mathcal{H}_{-s}(H)$ ($\mathcal{H}_{-s}(K)$) onto $\mathcal{H}_{-s}(K)$ ($\mathcal{H}_{-s}(H)$, resp.), $s \in [0, \infty)$. Both of these extensions will be denoted by $\widetilde{J}$. We set

$$[x, y] := (Jx, y) = (x, \widetilde{J}y), \;\; [y, x] = \overline{[x, y]}, \;\; x \in \mathcal{H}_s(H), y \in \mathcal{H}_{-s}(K).$$

We define a scale of Hilbert spaces by

$$\mathcal{H}_s(A, J) := \mathcal{H}_s(H), \;\; \mathcal{H}_{-s}(A, J) := \mathcal{H}_{-s}(K), \;\; s \in [0, \infty).$$

It is easy to see that the spaces $\mathcal{H}_s(A, J)$, $s \in \mathbb{R}$, regarded as (Hilbertable) linear topological spaces, and the duality $[\cdot,\cdot]$ between $\mathcal{H}_s(A, J)$ and $\mathcal{H}_{-s}(A, J)$ do not depend on the choice of J. For simplicity of notation we write $\mathcal{H}_s$ or $\mathcal{H}_s(A)$ instead of $\mathcal{H}_s(A, J)$ when no confusion can arise.

The operator A may be extended by continuity to a continuous operator $\widetilde{A}$ from $\mathcal{H}_{\frac{1}{2}}(H)$ to $\mathcal{H}_{-\frac{1}{2}}(K)$. We have $\widetilde{A} = \widetilde{J}\widetilde{H}$ (see Section 1.1). It will cause no confusion if we denote the adjoint with respect to the $[\cdot,\cdot]$-duality in the same way as the usual Krein space adjoint, by "+". We have $(\widetilde{A})^+ = \widetilde{A}$, i.e. $\widetilde{A}$ is $[\cdot,\cdot]$-symmetric,

$$[\widetilde{A}x, y] = [x, \widetilde{A}y], \;\; x, y \in \mathcal{H}_{\frac{1}{2}}(A, J).$$

From $\rho(A) \neq \emptyset$ we conclude that

$$A = \widetilde{A} \,|\, \{x \in \mathcal{H}_{\frac{1}{2}}(A, J) : \widetilde{A}x \in \mathcal{H}\}. \tag{1.2}$$

The identity on $\mathcal{H}_{\frac{1}{2}}(A,J)$ regarded as an operator into $\mathcal{H}_{-\frac{1}{2}}(A,J)$ will be denoted by E.

Lemma 1.1. *The range $\mathcal{R}(A-z)$ of $A-z$, $z\in\mathbb{C}$, is closed if and only if $\mathcal{R}(\widetilde{A}-zE)$ is closed. If this holds, then $A-z$ and $\widetilde{A}-zE$ have the same nullity and deficiency:*

$$\text{nul}(A-z)=\text{nul}(\widetilde{A}-zE),\ \ \text{def}(A-z)=\text{def}(\widetilde{A}-zE). \tag{1.3}$$

In particular, we have $z\in\rho(A)$ if and only if $\widetilde{A}-zE$ is an isomorphism.

Proof. 1. We first prove the last assertion. If $z\in\rho(A)$ then $\overline{z}\in\rho(A)$ and $A-\overline{z}$ considered as operator from $\mathcal{H}_1(A,J)$ into $\mathcal{H}$ is an isomorphism. Then the same is true for its adjoint $(A-\overline{z})^+\in\mathcal{L}(\mathcal{H},\mathcal{H}_{-1}(A,J))$ with respect to the $[\cdot,\cdot]$-duality. $(A-\overline{z})^+$ is an extension by continuity of $A-z$. Hence, by interpolation, $\widetilde{A}-zE$ is an isomorphism. If $\widetilde{A}-zE$ is an isomorphism, then evidently $z\in\rho(A)$.
2. Assume that $\mathcal{R}(\widetilde{A}-zE)$ is closed. Let (x_n) be a sequence from $\mathcal{D}(A)$ such that $((A-z)x_n)$ converges in $\mathcal{H}$ to some $y\in\mathcal{H}$. Then $(\widetilde{A}-zE)x_n$ converges in $\mathcal{H}_{-\frac{1}{2}}(A,J)$ to y. Hence there exists an $x\in\mathcal{H}_{\frac{1}{2}}(A,J)$ with $(\widetilde{A}-zE)x=y$. Then, by (1.2), $x\in\mathcal{D}(A)$ and $(A-z)x=y$. $\mathcal{R}(A-z)$ is closed.
3. Assume now that $\mathcal{R}(A-z)$ is closed. Let (x'_n) be a sequence from $\mathcal{H}_{\frac{1}{2}}(A,J)$ such that $\lim_{n\to\infty}(\widetilde{A}-zE)x'_n =: y$ exists in $\mathcal{H}_{-\frac{1}{2}}(A,J)$. For $\lambda\in\rho(A)$, $(\widetilde{A}-\lambda E)^{-1}$ is an isomorphism and we have

$$\lim_{n\to\infty}(\widetilde{A}-\lambda E)^{-1}(\widetilde{A}-zE)x'_n=\lim_{n\to\infty}(A-z)(A-\lambda)^{-1}x'_n=(\widetilde{A}-\lambda E)^{-1}y.$$

Hence $(\widetilde{A}-\lambda E)^{-1}y\in\mathcal{R}(A-z)$, i.e. there exists an $x'\in\mathcal{D}(A)$ with $(A-z)x'=(\widetilde{A}-\lambda E)^{-1}y$. It follows that

$$\begin{aligned}(\widetilde{A}-zE)(A-z)x'&=(\widetilde{A}-zE)(\widetilde{A}-\lambda E)^{-1}y=\\&=y+(\lambda-z)(\widetilde{A}-\lambda E)^{-1}y=y+(\lambda-z)(A-z)x'\end{aligned}$$

and, hence, $y\in\mathcal{R}(\widetilde{A}-zE)$.
4. The first relation of (1.3) is a consequence of (1.2). The second relation of (1.3) follows from the first by duality. □

We remark that if $z\in\rho(A)$ then $(\widetilde{A}-zE)^{-1}\in\mathcal{L}(\mathcal{H}_{-\frac{1}{2}}(A,J),\mathcal{H}_{\frac{1}{2}}(A,J))$ coincides with the extension $\widetilde{R}(z;A)$ by continuity of $R(z;A):=(A-z)^{-1}$.

For brevity we set $\mathcal{L}^{(A)} := \mathcal{L}(\mathcal{H}_{\frac{1}{2}}(A,J), \mathcal{H}_{-\frac{1}{2}}(A,J))$. If $\mathcal{H}'$ and $\mathcal{H}''$ are Hilbert spaces, $\mathfrak{S}_\infty(\mathcal{H}', \mathcal{H}'')$ denotes the linear space of compact operators of $\mathcal{H}'$ in $\mathcal{H}''$. We set $\mathfrak{S}_\infty := \mathfrak{S}_\infty(\mathcal{H}, \mathcal{H})$ and $\mathfrak{S}_\infty^{(A)} := \mathfrak{S}_\infty(\mathcal{H}_{\frac{1}{2}}(A,J), \mathcal{H}_{-\frac{1}{2}}(A,J))$. The following lemma will be needed below.

Lemma 1.2. *Let $\rho(A) \neq \emptyset$. The following assertions are equivalent:*

(1) $(A-z)^{-1}$ *is compact for some (and hence for all) $z \in \rho(A)$.*

(2) $(H-z)^{-1}$ *is compact for some (and hence for all) $z \in \rho(H)$.*

(3) *The natural embedding of $\mathcal{H}_1(H) = \mathcal{H}_1(A,J)$ into $\mathcal{H}$ is compact.*

(4) *For every $s_1, s_2 \in [0,\infty)$ with $s_1 > s_2$ the natural embedding of $\mathcal{H}_{s_1}(A,J)$ into $\mathcal{H}_{s_2}(A,J)$ is compact.*

(5) *For every $s_1, s_2 \in [0,\infty)$ with $s_1 > s_2$ the natural embedding of $\mathcal{H}_{-s_2}(A,J)$ into $\mathcal{H}_{-s_1}(A,J)$ is compact.*

Proof. For $z \in \rho(A) \cap \rho(JA)$ we have

$$(A-z)^{-1} - (JA-z)^{-1} = (A-z)^{-1}(1-J)JA(JA-z)^{-1} = \\ = (JA-z)^{-1}(J-1)A(A-z)^{-1}.$$

Hence the assertions (1) and (2) are equivalent. For non–real z, the natural embedding of $\mathcal{H}_1(H)$ into $\mathcal{H}$ is the composition of $H-z$ regarded as an operator from $\mathcal{H}_1(H)$ into $\mathcal{H}$, which is an isomorphism, and $(H-z)^{-1}$ regarded as an operator in $\mathcal{H}$. Therefore, the statements (2) and (3) are equivalent. Similarly, (3) is equivalent to $(1+H^2)^{-\frac{1}{2}} \in \mathfrak{S}_\infty$. This is equivalent to $(1+H^2)^{-\frac{s}{2}} \in \mathfrak{S}_\infty$ for all $s > 0$. Similarly to the reasoning above, one shows that $(1+H^2)^{-\frac{s}{2}} \in \mathfrak{S}_\infty$ for all $s > 0$ is equivalent to (4). That (4) and (5) are equivalent is a consequence of the duality of the scales. □

In the rest of Section 1.2 we recall the definition of sign types of spectral points of selfadjoint operators in Krein spaces (see [LMaM1]). With the help of the scale considered above we give a characterization of the sign types which will be needed in Chapter 2. First we recall that a point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* of a densely defined closed operator C, $\lambda_0 \in \sigma_{ap}(C)$, if there exists a sequence $(x_n) \subset \mathcal{D}(C)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(C-\lambda_0)x_n\| \to 0$ if $n \to \infty$.

Definition 1.3. For a selfadjoint operator C in a Krein space $\mathcal{H}$ with $\rho(C) \neq \emptyset$, a point $\lambda_0 \in \sigma(C)$ is called a spectral point of *positive* (*negative*) *type of* C if $\lambda_0 \in \sigma_{ap}(C)$ and for each sequence $(x_n) \subset \mathcal{D}(C)$ with $\|x_n\| = 1$ and $\|(C - \lambda_0)x_n\| \to 0$ for $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all spectral points of positive (negative) type of C by $\sigma_{++}(C)$ (resp. $\sigma_{--}(C)$). We shall say that an open subset Δ of $\overline{\mathbb{R}}$ is of *positive type* (*negative type*) with respect to C if $\Delta \setminus \{\infty\} \subset \rho(C) \cup \sigma_{++}(C)$ (resp. $\Delta \setminus \{\infty\} \subset \rho(C) \cup \sigma_{--}(C)$). An open set Δ of $\overline{\mathbb{R}}$ is called of *definite type* if Δ is of positive or negative type with respect to C.

The sets $\sigma_{++}(C)$ and $\sigma_{--}(C)$ are contained in $\mathbb{R}$. Indeed, for $\lambda \in \sigma_{++}(C)$ and for (x_n) as above we have $-(\operatorname{Im}\lambda)[x_n, x_n] = \operatorname{Im}[(C-\lambda)x_n, x_n] \to 0$ for $n \to \infty$.

For operators which are, in a sense, subordinated to a scale as above (see Section 1.3 below) the spectra of positive and negative type can be characterized as follows.

Lemma 1.4. *Let $\mathcal{H}$, J, A be as above and let C be a selfadjoint operator in $\mathcal{H}$, $\rho(C) \neq \emptyset$, with $\mathcal{D}(C) \subset \mathcal{H}_{\frac{1}{2}}(A, J)$ such that C can be extended by continuity to a mapping $\widehat{C} \in \mathcal{L}(\mathcal{H}_{\frac{1}{2}}(A, J), \mathcal{H}_{-\frac{1}{2}}(A, J))$ and, for some $z \in \mathbb{C}$, $\widehat{C} - zE$ is an isomorphism. If $\|\cdot\|_{\frac{1}{2}}$ and $\|\cdot\|_{-\frac{1}{2}}$ denote the norms of $\mathcal{H}_{\frac{1}{2}}(A, J)$ and $\mathcal{H}_{-\frac{1}{2}}(A, J)$, the following assertions are equivalent.*

(1) $\lambda_0 \in \sigma_{++}(C)$ ($\lambda_0 \in \sigma_{--}(C)$).

(2) $\lambda_0 \in \sigma_{ap}(C)$ *and for each sequence* $(x_n) \subset \mathcal{H}_{\frac{1}{2}}(A, J)$ *with* $\|x_n\|_{\frac{1}{2}} = 1$ *and* $\|(\widehat{C} - \lambda_0 E)x_n\|_{-\frac{1}{2}} \to 0$ *for* $n \to \infty$, *we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp. } \limsup_{n\to\infty} [x_n, x_n] < 0).$$

Proof. For $x \in \mathcal{D}(C)$ we have $\|(\widehat{C} - \lambda_0 E)x\|_{-\frac{1}{2}} \leq \|(\widehat{C} - \lambda_0 E)x\|$ and $\|x\| \leq \|x\|_{\frac{1}{2}}$. Hence assertion (2) implies assertion (1).

Assume that (1) holds. Let $(x_n) \subset \mathcal{H}_{\frac{1}{2}}(A, J)$ be a sequence with

$$\|x_n\|_{\frac{1}{2}} = 1, \quad \|(\widehat{C} - \lambda_0 E)x_n\|_{-\frac{1}{2}} \to 0 \quad \text{for } n \to \infty. \tag{1.4}$$

Evidently, we have $z \in \rho(C)$. Let $y_n := -(z - \lambda_0)(C - z)^{-1}x_n$. Then

$$\lim_{n\to\infty} \|x_n - y_n\|_{\frac{1}{2}} = \lim_{n\to\infty} \|(\widehat{C} - zE)^{-1}(\widehat{C} - \lambda_0 E)x_n\|_{\frac{1}{2}} = 0.$$

Hence, by (1.4),

$$\lim_{n\to\infty} \|y_n\|_{\frac{1}{2}} = 1, \quad \lim_{n\to\infty} \|(\widehat{C} - \lambda_0 E)y_n\|_{-\frac{1}{2}} = 0.$$

Then, making use of the fact that $(\widehat{C} - zE)^{-1}$ is an isomorphism we obtain

$$\begin{aligned} 1 = \lim_{n\to\infty} \|y_n\|_{\frac{1}{2}} = |z-\lambda_0| \lim_{n\to\infty} \|(C-z)^{-1}x_n\|_{\frac{1}{2}} \leq \\ \leq M \liminf_{n\to\infty} \|x_n\|_{-\frac{1}{2}} = M \liminf_{n\to\infty} \|y_n\|_{-\frac{1}{2}} \leq M \liminf_{n\to\infty} \|y_n\|. \end{aligned}$$

for some constant M, and

$$\begin{aligned} \lim_{n\to\infty} \|(C-\lambda_0)y_n\| = |z-\lambda_0| \lim_{n\to\infty} \|(C-\lambda_0)(C-z)^{-1}x_n\| = \\ = |z-\lambda_0| \lim_{n\to\infty} \|(\widehat{C} - zE)^{-1}(\widehat{C} - \lambda_0 E)x_n\| = 0. \end{aligned}$$

Therefore, by condition (1)

$$0 < \liminf_{n\to\infty} [y_n, y_n] = \liminf_{n\to\infty} [x_n, x_n].$$

Hence (2) holds and the lemma is proved. □

1.3. A Class of Perturbations of Selfadjoint Operators in Krein Spaces

Let $\mathcal{H}$, J, A, be as in Section 1.2. We consider the following class of perturbations of A (see e.g. [JL2], [J3]). These perturbations can also be defined with the help of sesquilinear forms.

Definition 1.5. Assume that the operator $Z \in \mathcal{L}^{(A)}$ can be written as a sum $Z = Z_1 + Z_2$, $Z_1, Z_2 \in \mathcal{L}^{(A)}$ such that the following holds:

(i) There exists a $z_0 \in \mathbb{C}$ such that $\widetilde{A} - z_0 E + Z_1$ is an isomorphism of $\mathcal{H}_{\frac{1}{2}}(A, J)$ onto $\mathcal{H}_{-\frac{1}{2}}(A, J)$.

(ii) The range $\mathcal{R}(Z_2)$ of Z_2 is contained in $\mathcal{H}$ and Z_2 can be extended by continuity to a bounded operator in $\mathcal{H}$.

Then the restriction of $\widetilde{A} + Z$ to

$$\mathcal{D}(A \dot{+} Z) := \{x \in \mathcal{H}_{\frac{1}{2}}(A, J) : (\widetilde{A} + Z)x \in \mathcal{H}\}$$

regarded as an operator in $\mathcal{H}$ is denoted by $A\dot{+}Z$ (cf. [KY]).

Evidently, $A\dot{+}Z_1$ is densely defined and closed, and we have $z_0 \in \rho(A\dot{+}Z_1)$. Therefore, $A\dot{+}Z$ is a densely defined closed operator. If for some $z \in \mathbb{C}$ the operator $\widetilde{A} + Z - zE$ is an isomorphism, then we have $z \in \rho(A\dot{+}Z)$. If $Z = Z^+$ and for some $z \in \mathbb{C}$ the operator $\widetilde{A} + Z - zE$ is an isomorphism, then $A\dot{+}Z$ is a selfadjoint operator with nonempty resolvent set.

Lemma 1.6. ([J3]) *Let $V \in \mathfrak{S}_\infty^{(A)}$. Then $A\dot{+}V$ is defined. If, additionally, $V = V^+$, then $A\dot{+}V$ is selfadjoint.*

Lemma 1.7. *Let $V \in \mathfrak{S}_\infty^{(A)}$ and assume that $\rho(A) \cap \rho(A\dot{+}V) \neq \emptyset$. Then*

$$R(\lambda) - R(\lambda; A\dot{+}V) \in \mathfrak{S}_\infty, \quad \lambda \in \rho(A) \cap \rho(A\dot{+}V). \tag{1.5}$$

For any $z \in \mathbb{C}$ the range $\mathcal{R}(A\dot{+}V - z)$ is closed if and only if $\mathcal{R}(\widetilde{A} + V - zE)$ is closed. In this case

$$\operatorname{nul}(A\dot{+}V - z) = \operatorname{nul}(\widetilde{A} + V - zE), \ \ \operatorname{def}(A\dot{+}V - z) = \operatorname{def}(\widetilde{A} + V - zE).$$

In particular, we have $z \in \rho(A\dot{+}V)$ if and only if $\widetilde{A} + V - zE$ is an isomorphism.

Proof. Relation (1.5) was proved in [J3]. Assume that $\mathcal{R}(\widetilde{A} + V - zE)$ is closed. Then the closedness of $\mathcal{R}(A\dot{+}V - z)$ can be proved as in part 2 of the proof of Lemma 1.1.

To prove the converse we first claim that for $\lambda \in \rho(A) \cap \rho(A\dot{+}V)$ the operator $\widetilde{A} + V - \lambda E$ is an isomorphism. Indeed, by Lemma 1.1, $\widetilde{A} - \lambda E$ is an isomorphism and, since $V \in \mathfrak{S}_\infty^{(A)}$, $\widetilde{A} + V - \lambda E$ is a Fredholm operator of index zero. Thus, by $\lambda \in \rho(A\dot{+}V)$, the operator $\widetilde{A} + V - \lambda E$ is an isomorphism. If $\mathcal{R}(A\dot{+}V - z)$ is closed then by a reasoning similar to part 3 of the proof of Lemma 1.1 we find that $\mathcal{R}(\widetilde{A} + V - zE)$ is closed.

The remaining assertions of Lemma 1.7 are proved as in Lemma 1.1. □

We define the *essential spectrum* $\sigma_{ess}(C)$ of a densely defined closed operator C to be the complement of its Fredholm domain.

If then A and V fulfil the assumptions of Lemma 1.7 it follows from (1.5) that

$$\sigma_{ess}(A) = \sigma_{ess}(A\dot{+}V). \tag{1.6}$$

1.4. Definitizable Operators

For any subspace $\mathcal{H}'$ of the Krein space $(\mathcal{H}, [\cdot,\cdot])$ the supremum $(\leq \infty)$ of the dimensions of the subspaces $\mathcal{L}$ of $\mathcal{H}'$ such that $[\cdot,\cdot]$ is positive (negative) definite on $\mathcal{L}$ is denoted by $\kappa_+(\mathcal{H}')$ (resp. $\kappa_-(\mathcal{H}')$).

Let A be a definitizable selfadjoint operator in $\mathcal{H}$, i.e. $\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the spectrum of A is real or its non–real part consists of a finite number of points. Moreover, A has a spectral function $E(\cdot\,; A)$ defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not coincide with the points of some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [L3]).

In the remainder of Section 1.4 we assume, unless otherwise stated, that A is the direct orthogonal sum of two definitizable operators A_1 and A_2 in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $\mathcal{H} = \mathcal{H}_1 [\dot{+}] \mathcal{H}_2$. This situation will occur in Chapter 2 below. Let $E(\cdot\,)$ denote the direct sum of the spectral functions of A_1 and A_2. We do not exclude the case $\mathcal{H}_2 = \{0\}$.

The classification of spectral points in Definition 1.3 was originally introduced with the help of the spectral function. For the convenience of the reader, we will give a proof for the known equivalence of these two descriptions of sign types under our assumptions on A.

Lemma 1.8. *A point $\lambda \in \sigma(A) \cap \mathbb{R}$ belongs to $\sigma_{++}(A)$ ($\sigma_{--}(A)$) if and only if there exists an open interval δ of $\mathbb{R}$, $\lambda \in \delta$, for which $E(\delta)$ is defined, with $\kappa_-(E(\delta)\mathcal{H}) = 0$ (resp. $\kappa_+(E(\delta)\mathcal{H}) = 0$).*

Proof. It is sufficient to prove the lemma for $\sigma_{++}(A)$. A similar reasoning applies for $\sigma_{--}(A)$.

Let $\lambda \in \sigma(A) \cap \mathbb{R}$ and let Δ be a bounded open interval, $\lambda \in \Delta$, such that $E(\Delta)$ is nonnegative. If (x_n) is a sequence in $\mathcal{D}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(A - \lambda)x_n\| \to 0$ for $n \to \infty$, then $\|(A - \lambda)(1 - E(\Delta))x_n\| \to 0$. This implies $\|(1 - E(\Delta))x_n\| \to 0$ and, hence, $\lim_{n\to\infty} \|E(\Delta)x_n\| = 1$. Since $(E(\Delta)\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space we obtain

$$\liminf_{n\to\infty} [x_n, x_n] = \liminf_{n\to\infty} [E(\Delta)x_n, E(\Delta)x_n] > 0.$$

Assume now that there does not exist an open subset δ with the properties mentioned in the lemma. Let Δ_0 be a bounded open interval, $\lambda \in \Delta_0$, such that $E(\Delta_0)$ is defined, and let $A_0 := A|E(\Delta_0)\mathcal{H}$. $E(\Delta_0)$ is not nonnegative. Then by [L2] there exists a maximal nonpositive A_0-invariant subspace $\mathcal{M}_-$

of $E(\Delta_0)\mathcal{H}$. It is easy to see that $\sigma(A_0|\mathcal{M}_-) \subset \overline{\Delta_0}$. We claim that

$$\lambda \in \sigma(A_0|\mathcal{M}_-). \tag{1.7}$$

If (1.7) holds, let (y_n) be a sequence in $\mathcal{M}_-$ with $\|y_n\| = 1$, $n = 1, 2, \dots$, and $\lim_{n\to\infty} \|(A_0 - \lambda)y_n\| = 0$. We have $[y_n, y_n] \le 0$ and $\liminf_{n\to\infty}[y_n, y_n] \le 0$.

It remains to prove (1.7). Suppose $\lambda \notin \sigma(A_0|\mathcal{M}_-)$. Let Δ be an open interval such that $E(\Delta)$ is defined, with $\lambda \in \Delta \subset \Delta_0$ and $\overline{\Delta} \cap \sigma(A_0|\mathcal{M}_-) = \emptyset$. We have $E(\Delta)\mathcal{M}_- \subset \mathcal{M}_-$ and $E(\Delta)\mathcal{M}_- \subset E(\Delta)\mathcal{H}$. Then $\sigma(A_0|E(\Delta)\mathcal{M}_-) \subset \sigma(A_0|\mathcal{M}_-)$ and $\sigma(A_0|E(\Delta)\mathcal{M}_-) \subset \sigma(A_0|E(\Delta)\mathcal{H}) \subset \overline{\Delta}$. This shows that $E(\Delta)\mathcal{M}_- = \{0\}$. Since $E(\Delta)\mathcal{M}_-$ is a maximal nonpositive subspace of $E(\Delta)\mathcal{H}$, $E(\Delta)$ is nonnegative, a contradiction to our assumption on λ. Hence (1.7) holds, which completes the proof. □

Hence, by Lemma 1.8, an open subset Δ of $\overline{\mathbb{R}}$ is of positive type (negative type) with respect to A if and only if $\kappa_-(E(\delta)\mathcal{H}) = 0$ (resp. $\kappa_+(E(\delta)\mathcal{H}) = 0$) for every (in $\overline{\mathbb{R}}$) connected subset δ of Δ with $\overline{\delta} \subset \Delta$.

We shall say that an open subset Δ of $\overline{\mathbb{R}}$ is of *type* π_+ (*type* π_-) with respect to A if $\kappa_-(E(\delta)\mathcal{H}) < \infty$ (resp. $\kappa_+(E(\delta)\mathcal{H}) < \infty$) for every (in $\overline{\mathbb{R}}$) connected subset δ of Δ with $\overline{\delta} \subset \Delta$ such that $E(\delta)$ is defined. The open set Δ is called of *type* π if Δ is of type π_+ or type π_- with respect to A. We remark that for an interval (a, b), $a, b \in \rho(A)$, of type π the space $E((a, b))\mathcal{H}$ is a Pontryagin space.

Let $\sigma_e(A)$ be the extended spectrum of A, i.e. $\sigma_e(A) = \sigma(A)$ if A is bounded and $\sigma_e(A) = \sigma(A) \cup \{\infty\}$ otherwise, let $\rho_e(A) = \overline{\mathbb{C}} \setminus \sigma_e(A)$. We denote by $\widetilde{\sigma}_{e,+}(A)$ ($\widetilde{\sigma}_{e,-}(A)$) the set of all $\lambda \in \sigma_e(A) \cap \overline{\mathbb{R}}$ such that for every open connected subset δ of $\overline{\mathbb{R}}$, $\lambda \in \delta$, for which $E(\delta)$ is defined, we have $\kappa_+(E(\delta)\mathcal{H}) = \infty$ (resp. $\kappa_-(E(\delta)\mathcal{H}) = \infty$). The sets $\widetilde{\sigma}_{e,+}(A)$ and $\widetilde{\sigma}_{e,-}(A)$ are closed in $\overline{\mathbb{R}}$.

A point $t \in \overline{\mathbb{R}}$ is called a *critical point* (an *essential critical point*) of A if there is no open subset Δ of definite type (resp. type π) with $t \in \Delta$. The set of critical (essential critical) points of A is denoted by $c(A)$ (resp. $c_\infty(A)$). We have

$$\begin{aligned} c(A) \setminus \{\infty\} &= \sigma(A) \setminus (\sigma_{++}(A) \cup \sigma_{--}(A)), \\ c_\infty(A) &= \widetilde{\sigma}_{e,+}(A) \cap \widetilde{\sigma}_{e,-}(A). \end{aligned} \tag{1.8}$$

In Chapter 2 and Chapter 3 we will make use of the fact that a critical point which is not an essential critical point is an eigenvalue. A critical point t is called *regular* if there exists an open deleted neighbourhood δ_0 of t such that the set of the projections $E(\delta)$ where δ runs through all intervals δ

with $\overline{\delta} \subset \delta_0$ is bounded. The set of regular critical points of A is denoted by $c_r(A)$. The elements of $c_s(A) := c(A) \setminus c_r(A)$ are called *singular* critical points. The spectral function E can be extended by continuity to all intervals whose endpoints do not belong to $c_s(A)$.

We assume now, in addition, that $E(\overline{\mathbb{R}})\mathcal{H}$ is not a Pontryagin space (i.e. $\kappa_+(E(\overline{\mathbb{R}})\mathcal{H}) = \kappa_-(E(\overline{\mathbb{R}})\mathcal{H}) = \infty$) and either $c_\infty(A) = \emptyset$ or $c_\infty(A) = \{\infty\}$ holds, and that there exist intervals of the form (μ_+, ∞) and $(-\infty, \mu_-)$ of definite type with respect to A. In the following lemma we shall make use of the system of all those open connected components I of $\mathbb{R} \setminus (\widetilde{\sigma}_{e,+}(A) \cup \widetilde{\sigma}_{e,-}(A))$ such that if both endpoints of I are finite then one of them belongs to $\widetilde{\sigma}_{e,+}(A)$ and the other to $\widetilde{\sigma}_{e,-}(A)$. By the second relation of (1.8) and our additional assumption $c_\infty(A) \subset \{\infty\}$ this system is finite. We shall denote it by

$$(a_1, b_1), (a_2, b_2), \ldots, (a_l, b_l), \quad l \geq 1,$$

with

$$b_1 \leq a_2 < b_2 \leq a_3 < \cdots < b_{l-1} \leq a_l \quad \text{if } l > 1.$$

By definition the spectrum of A is discrete in each interval (a_j, b_j), $j = 1, \ldots, l$.

Lemma 1.9. *Assume that $E(\overline{\mathbb{R}})\mathcal{H}$ is not a Pontryagin space and either $c_\infty(A) = \emptyset$ or $c_\infty(A) = \{\infty\}$ holds. Then A is definitizable if and only if there exist intervals of the form (μ_+, ∞) and $(-\infty, \mu_-)$ of definite type with respect to A.*

Moreover, if such points μ_+, μ_- exist, and if $s_j \in \rho(A) \cap (a_j, b_j)$, $j = 1, \ldots, l$, then the following holds.

(1) *We have*

$$\sup_{\nu \in (-\infty, s_1)} \{\kappa_+(E((\nu, s_1))\mathcal{H})\} < \infty \ \text{or} \ \sup_{\nu \in (-\infty, s_1)} \{\kappa_-(E((\nu, s_1))\mathcal{H})\} < \infty.$$

(2) *We have*

$$\sup_{\nu \in (s_l, \infty)} \{\kappa_+(E((s_l, \nu))\mathcal{H})\} < \infty \ \text{or} \ \sup_{\nu \in (s_l, \infty)} \{\kappa_-(E((s_l, \nu))\mathcal{H})\} < \infty.$$

(3) *If $l > 1$ then $E((s_{j-1}, s_j))\mathcal{H}$, $j = 2, \ldots, l$, are Pontryagin spaces.*

(4) *If $\infty \notin c_s(A)$ then $E((-\infty, s_1))\mathcal{H}$ and $E((s_l, \infty))\mathcal{H}$ are Pontryagin spaces.*

Proof. It is well known that if A is definitizable then μ_+ and μ_- as in the lemma exist.

Assume now that μ_+ and μ_- with the above properties exist. The spectrum of the restriction $A_0 := A|(1 - E(\overline{\mathbb{R}}))\mathcal{H}$ consists of a finite set of poles of the resolvent. It is easy to see that there is a rational function q_0 which has no zeros in $\overline{\mathbb{R}}$ such that $q(A_0) = 0$ for $q(z) := q_0(z)\overline{q_0(\overline{z})}$. If we find a definitizing rational function r for $A|E(\overline{\mathbb{R}})\mathcal{H}$, then rq is definitizing for A. Therefore to prove that A is definitizable we may restrict ourselves to the case $\sigma(A) \subset \mathbb{R}$. Since $\mathcal{H}$ is not a Pontryagin space, we have $\widetilde{\sigma}_{e,+}(A) \neq \emptyset$ and $\widetilde{\sigma}_{e,-}(A) \neq \emptyset$. We will prove next that A satisfies (1)–(4).

Suppose the two suprema in (1) are equal to infinity and $(-\infty, \mu_-)$ is of positive type with respect to A. If $(-\infty, \mu_-)$ is of negative type a similar reasoning applies. We have $\kappa_-(E(\delta)\mathcal{H}) = 0$ for each compact subset δ of $(-\infty, \mu_-)$. We choose an $\epsilon > 0$ such that $\kappa_-(E((\mu_- - \epsilon, s_1)))\mathcal{H}) = \infty$ holds. This implies $\widetilde{\sigma}_{e,-}(A) \cap [\mu_- - \epsilon, s_1] \neq \emptyset$ and, hence, that $\min \widetilde{\sigma}_{e,-}(A)$ is the right endpoint of one of the intervals (a_i, b_i) which contradicts the choice of s_1. A similar proof holds for (2). Assertions (3) and (4) are simple consequences from the definition of s_j, $j = 1, \dots, l$, and (1) and (2).

Now, we choose a rational function r having poles only in $\rho(A) \setminus \mathbb{R}$ with the following properties.

(1) $r(\infty) = 0$, r has real zeros only in $s_1, \dots, s_l$.

(2) If $l > 1$, r is positive (negative) on (s_{j-1}, s_j), $j = 2, \dots, l$, if (s_{j-1}, s_j) is of type π_+ (resp. not of type π_+) with respect to A.

(3) The function r is positive on the interval $(-\infty, s_1)$ if we have that $\sup_{\nu \in (-\infty, s_1)}\{\kappa_-(E((\nu, s_1))\mathcal{H})\} < \infty$ and negative otherwise. Similarly for $(-\infty, s_1)$ replaced by (s_l, ∞).

With the help of the spectral function it is easy to see that $[r(A)\cdot, \cdot]$ is positive semidefinite or has a finite number of negative squares. Then by [L3] A is definitizable. □

1.5. Perturbations of Definitizable Operators

Now we consider perturbations of definitizable operators. The following lemma shows, in particular, that for a definitizable operator A and (a not necessarily symmetric) $V \in \mathfrak{S}_\infty^{(A)}$ the assumption $\rho(A) \cap \rho(A \dot{+} V) \neq \emptyset$ of Lemma 1.7 is fulfilled.

Here and in the following we denote, for a closed and densely defined operator A, by $\sigma_{p,norm}(A)$ the set of all $\lambda \in \mathbb{C}$ which are isolated points of $\sigma(A)$ and normal eigenvalues of A, that is, the corresponding Riesz-Dunford projection is of finite rank.

Lemma 1.10. *Let A be a definitizable selfadjoint operator in $\mathcal{H}$ and let $V \in \mathfrak{S}_\infty^{(A)}$. Then there exists an $\eta_0 > 0$ such that $i\eta \in \rho(A \underset{\smile}{+} V)$ for $|\eta| \geq \eta_0$. Further, we have*

$$\tag{1.9} \begin{aligned} &\sigma(A) \setminus \sigma_{p,norm}(A) = \\ &= \sigma_{ess}(A) = \sigma_{ess}(A \underset{\smile}{+} V) = \sigma(A \underset{\smile}{+} V) \setminus \sigma_{p,norm}(A \underset{\smile}{+} V). \end{aligned}$$

Proof. The first assertion of Lemma 1.10 is a consequence of [J3, Proposition 3.1]. By (1.6) we have to prove only the first and the last equality of (1.9). As A is definitizable, $\rho(A)$ is dense in $\mathbb{C}$, hence, for $\lambda \notin \sigma_{ess}(A) = \sigma_{ess}(A \underset{\smile}{+} V)$, the index of the operator $A - \lambda$ is zero, and, by Lemmas 1.1 and 1.7 the same holds for A replaced by $A \underset{\smile}{+} V$. As the open set $\mathbb{C} \setminus \sigma_{ess}(A)$ $(= \mathbb{C} \setminus \sigma_{ess}(A \underset{\smile}{+} V))$ consists of at most two connected open components each of which contains points of $\rho(A)$ and of $\rho(A \underset{\smile}{+} V)$, it follows that $\mathbb{C} \setminus \sigma_{ess}(A) \subset \rho(A) \cup \sigma_{p,norm}(A)$ and $\mathbb{C} \setminus \sigma_{ess}(A \underset{\smile}{+} V) \subset \rho(A \underset{\smile}{+} V) \cup \sigma_{p,norm}(A \underset{\smile}{+} V)$ hold and (1.9) is proved. □

In the proof of Theorem 1.12 below we will use the subsequent theorem which is a special case of [J3, Theorem 3.6]. We remark that in the Theorems 3.6 and 3.10 of [J3] in all places where unbounded intervals of type π_+ (π_-) occur, it has to be added that $\kappa_-(E(\Delta)\mathcal{H})$ (resp. $\kappa_+(E(\Delta)\mathcal{H})$) is zero for all compact subintervals Δ in some neighbourhood of ∞.

Theorem 1.11. *Let A be a definitizable selfadjoint operator in $\mathcal{H}$ such that $\sigma(A) \setminus \mathbb{R} \subset \sigma_{p,norm}(A)$. Let $V = V^+ \in \mathfrak{S}_\infty^{(A)}$. Further, assume that there exist points $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, $t_1, t_2 \in \rho(A) \cap \rho(A \underset{\smile}{+} V)$ such that*

$$\sup_{\nu \in (-\infty, t_1)} \{\kappa_+(E((\nu, t_1))\mathcal{H})\} < \infty \quad \textit{and} \quad \sup_{\nu \in (t_2, \infty)} \{\kappa_-(E((t_2, \nu))\mathcal{H})\} < \infty,$$

where E denotes the spectral function of A.

Then no point of $\overline{\mathbb{R}} \setminus (t_1, t_2)$ is an accumulation point of the non–real spectrum of $A \underset{\smile}{+} V$. Further, if P' is the Riesz–Dunford–Taylor projection corresponding to $\overline{\mathbb{R}} \setminus (t_1, t_2)$ and $A \underset{\smile}{+} V$, $A \underset{\smile}{+} V | P'\mathcal{H}$ is a definitizable selfadjoint *operator with*

$$\sup_{\nu\in(-\infty,t_1)} \{\kappa_+(E'((\nu,t_1))\mathcal{H})\} < \infty \quad \textit{and} \quad \sup_{\nu\in(t_2,\infty)} \{\kappa_-(E'((t_2,\nu))\mathcal{H})\} < \infty,$$

where E' denotes the spectral function of $A\underset{\smile}{+}V|P'\mathcal{H}$. Moreover, $\infty \in c_s(A)$ if and only if $\infty \in c_s(A\underset{\smile}{+}V)$.

A similar theorem holds with κ_+ and κ_- interchanged.

Theorem 1.12. *Let A be a definitizable selfadjoint operator in $\mathcal{H}$ such that $c_\infty(A) = \emptyset$ or $c_\infty(A) = \{\infty\}$ and assume that $\sigma(A) \setminus \mathbb{R} \subset \sigma_{p,norm}(A)$. Let $V = V^+ \in \mathfrak{S}_\infty^{(A)}$. Then $A\underset{\smile}{+}V$ is a definitizable selfadjoint operator in $\mathcal{H}$ such that $c_\infty(A\underset{\smile}{+}V) = \emptyset$ or $c_\infty(A\underset{\smile}{+}V) = \{\infty\}$; in particular, all finite critical points of $A\underset{\smile}{+}V$ belong to $\sigma_p(A\underset{\smile}{+}V)$. Moreover, the following holds.*

(i) $\sigma(A\underset{\smile}{+}V) \setminus \mathbb{R} \subset \sigma_{p,norm}(A\underset{\smile}{+}V)$.

(ii) $\widetilde{\sigma}_{e,+}(A) = \widetilde{\sigma}_{e,+}(A\underset{\smile}{+}V)$, $\widetilde{\sigma}_{e,-}(A) = \widetilde{\sigma}_{e,-}(A\underset{\smile}{+}V)$. *Hence any open interval is of type π_+ (π_-) with respect to A if and only if it is of type π_+ (π_-, resp.) with respect to $A\underset{\smile}{+}V$.*

(iii) *If, for some $m \in \mathbb{R}$, (m,∞) is of positive (negative) type with respect to A, then there exists an $m' \in (m,\infty)$ such that (m',∞) is of positive (resp. negative) type with respect to $A\underset{\smile}{+}V$. A similar statement holds for intervals of the form $(-\infty, m)$.*

(iv) $\infty \notin c_s(A)$ *if and only if* $\infty \notin c_s(A\underset{\smile}{+}V)$.

Proof. 1. Relation (i) is a consequence of Lemma 1.10. Let E be the spectral function of A. If $E(\overline{\mathbb{R}})\mathcal{H}$ is a Pontryagin space then, by the assumption $\sigma(A) \setminus \mathbb{R} \subset \sigma_{p,norm}(A)$, $\mathcal{H}$ is a Pontryagin space and all assertions of the theorem are valid. Assume for the rest of the proof that $E(\overline{\mathbb{R}})\mathcal{H}$ is not a Pontryagin space.
2. We shall assume that the system $\{(a_j, b_j) : j = 1, \ldots, l\}$ considered in Lemma 1.9 consists of at least two intervals. If that system consists of only one interval, a similar reasoning applies but some of the following considerations are irrelevant in this case. By the definition of the intervals (a_j, b_j) and Lemma 1.10 we may choose some points $s_j \in (a_j, b_j) \cap \rho(A) \cap \rho(A\underset{\smile}{+}V)$, $j = 1, \ldots, l$. Let G_j be the open disc with center on $\mathbb{R}$ such that $s_j, s_{j+1} \in$

∂G_j, $j = 1, \ldots, l-1$, and let G_0 be the open disc with center on $\mathbb{R}$ such that $s_1, s_l \in \partial G_0$. It is no restriction to assume that

$$\partial G_0 \cup \partial G_1 \cup \ldots \cup \partial G_{l-1} \subset \rho(A) \cap \rho(A \underset{\smile}{+} V).$$

Let E_j (E_j'), $j = 0, 1, \ldots, l-1$, be the Riesz-Dunford projection corresponding to G_j and A (resp. $A \underset{\smile}{+} V$). Then Lemma 1.7 implies

$$E_j' - E_j \in \mathfrak{S}_\infty, \quad j = 0, 1, \ldots, l-1.$$

As E_j, $j = 0, 1, \ldots, l-1$, is a selfadjoint projection in $\mathcal{H}$ there exists a fundamental symmetry of $\mathcal{H}$ which commutes with E_j. On account of [J1, Theorem 3.1], we have $\widetilde{\sigma}_{e,+}(E_j) = \widetilde{\sigma}_{e,+}(E_j')$, $\widetilde{\sigma}_{e,-}(E_j) = \widetilde{\sigma}_{e,-}(E_j')$, $j = 1, \ldots, l-1$. Consequently, since $E_j\mathcal{H}$ is a Pontryagin space (see Lemma 1.9 (3)), also $E_j'\mathcal{H}$ is a Pontryagin space and $\kappa_+(E_j\mathcal{H}) < \infty$ ($\kappa_-(E_j\mathcal{H}) < \infty$) if and only if $\kappa_+(E_j'\mathcal{H}) < \infty$ (resp. $\kappa_-(E_j'\mathcal{H}) < \infty$). Since for a selfadjoint operator B in a Pontryagin space with finite negative (positive) index, $\widetilde{\sigma}_{e,+}(B)$ (resp. $\widetilde{\sigma}_{e,-}(B)$) coincides with $\sigma_e(B) \setminus \sigma_{p,norm}(B)$, we have

$$\widetilde{\sigma}_{e,\pm}(A) \cap G_j = \widetilde{\sigma}_{e,\pm}(A \underset{\smile}{+} V) \cap G_j, \quad j = 1, \ldots, l-1. \tag{1.10}$$

3. Assume that $(1 - E_0)\mathcal{H}$ is a Pontryagin space. Then by $E_0' - E_0 \in \mathfrak{S}_\infty$, $(1 - E_0')\mathcal{H}$ is a Pontryagin space as well. Hence $A \underset{\smile}{+} V$ is the direct sum of selfadjoint operators in Pontryagin space with pairwise disjoint spectra such that at most one of the operators is unbounded. Therefore, by a reasoning similar to that in the proof of Lemma 1.9, $A \underset{\smile}{+} V$ is definitizable. Again as above we find

$$\widetilde{\sigma}_{e,\pm}(A) \cap (\overline{\mathbb{R}} \setminus G_0) = \widetilde{\sigma}_{e,\pm}(A \underset{\smile}{+} V) \cap (\overline{\mathbb{R}} \setminus G_0),$$

which along with (1.10) implies (ii). In this case the assertions (iii) and (iv) are evident.
4. Assume now that $(1 - E_0)\mathcal{H}$ is not a Pontryagin space. This implies, in view of

$$(1 - E_0)E(\overline{\mathbb{R}})\mathcal{H} = \operatorname{closp}\{(1 - E_0)E((-n, n))x : n \in \mathbb{N}, x \in \mathcal{H}\}$$

and Lemma 1.9, (1) and (2), that ∞ is an accumulation point of $\sigma(A) \cap (0, \infty)$ and of $\sigma(A) \cap (-\infty, 0)$ and, on the other hand, that either

$$\sup_{\nu \in (-\infty, s_1)} \{\kappa_+(E((\nu, s_1))\mathcal{H})\} < \infty \text{ and } \sup_{\nu \in (s_l, \infty)} \{\kappa_-(E((s_l, \nu))\mathcal{H})\} < \infty. \tag{1.11}$$

or similar relation with κ_+ and κ_- interchanged hold. It follows that ∞ is a critical point. We assume that (1.11) holds. This is no restriction.

Then, as a consequence of Theorem 1.11, $A\dot{+}V|P'\mathcal{H}$, where P' is as in Theorem 1.11, is a definitizable selfadjoint operator. As the non–real spectrum of $A\dot{+}V$ consists only of finitely many points belonging to $\sigma_{p,norm}(A\dot{+}V)$ (Lemma 1.10, Theorem 1.11 and relation (i)), also $A\dot{+}V|(I - E_0')\mathcal{H}$ is a definitizable selfadjoint operator. This implies, as above, that $A\dot{+}V$ is definitizable. Assertions (iii) and (iv) are consequences of Theorem 1.11.

In order to prove (ii) observe that for sufficiently large ν the projections $E((s_l,\nu))$ and $E((s_l,\nu); A\dot{+}V)$ are defined and their ranges are Pontryagin spaces. Therefore, by the same reasoning as in the proof of (1.10),

$$\widetilde{\sigma}_{e,+}(A) \cap (s_l,\nu) = \widetilde{\sigma}_{e,+}(A\dot{+}V) \cap (s_l,\nu). \tag{1.12}$$

Similarly, for sufficiently small ν,

$$\widetilde{\sigma}_{e,-}(A) \cap (\nu,s_1) = \widetilde{\sigma}_{e,-}(A\dot{+}V) \cap (\nu,s_1). \tag{1.13}$$

These relations along with (1.10) imply the assertion (ii) and Theorem 1.12 is proved. □

1.6. Locally Definitizable Operators

By $\mathbb{C}^+$ we denote the open upper half plane and by $\mathbb{C}^-$ the open lower half plane.

Let A_0 be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and let Δ be an open subset of $\overline{\mathbb{R}}$ such that no point of Δ is an accumulation point of the non–real spectrum $\sigma(A_0)\setminus\mathbb{R}$ of A_0. We shall say that A_0 belongs to the class $S^\infty(\Delta)$, if for every closed subset Δ' of Δ there exists $m \geq 1$, $M > 0$ and an open neighbourhood $\mathcal{U}$ of Δ' in $\overline{\mathbb{C}}$ such that

$$\|(A_0-\lambda)^{-1}\| \leq M(|\lambda|+1)^{2m-2}|\mathrm{Im}\,\lambda|^{-m}$$

holds for all $\lambda \in \mathcal{U}\setminus\overline{\mathbb{R}}$.

Definition 1.13. Let Ω be a domain in $\overline{\mathbb{C}}$ which is symmetric with respect to $\mathbb{R}$ such that $\Omega\cap\overline{\mathbb{R}} \neq \emptyset$ and $\Omega\cap\mathbb{C}^+$ and $\Omega\cap\mathbb{C}^-$ are simply connected. Let A_0 be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A_0)\cap(\Omega\setminus\overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A_0, and no point of $\Omega\cap\overline{\mathbb{R}}$ is an accumulation point of the non–real spectrum $\sigma(A_0)\setminus\mathbb{R}$

of A_0. The operator A_0 is called *definitizable over* Ω, if $A_0 \in S^\infty(\Omega \cap \overline{\mathbb{R}})$ and every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ such that both components of $I \setminus \{\lambda\}$ are of definite type (cf. Definition 1.3) with respect to A_0.

We remark that A_0 is definitizable (cf. Section 1.4) if and only if A_0 is definitizable over $\overline{\mathbb{C}}$ ([J6]).

In the following we will introduce for an operator which is definitizable over Ω the notions of critical points and open sets of type π in a way similar to Section 1.4.

Assume that A_0 is definitizable over Ω. A point $t \in \Omega \cap \overline{\mathbb{R}}$ is called a *critical point* of A_0 if there is no open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ of definite type with $t \in \Delta$. The set of critical points of A_0 is denoted by $c(A_0)$. For the properties of the spectral function $E(\cdot\,; A_0)$ of A_0 we refer to [J6] and [J1]. We mention only that for every connected subset δ, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$, whose endpoints are not critical points of A_0, $E(\delta; A_0)$ is defined and this projection is selfadjoint in $(\mathcal{H}, [\cdot, \cdot])$.

Furthermore, by [J6, Theorem 2.13], an open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ is of positive type (negative type) with respect to A_0 if and only if $\kappa_-(E(\delta; A_0)\mathcal{H}) = 0$ (resp. $\kappa_+(E(\delta; A_0)\mathcal{H}) = 0$) for every (in $\overline{\mathbb{R}}$) connected subset δ of Δ with $\overline{\delta} \subset \Delta$.

We shall say that an open subset Δ of $\Omega \cap \overline{\mathbb{R}}$ is of *type* π_+ (*type* π_-) with respect to A_0 if $\kappa_-(E(\delta; A_0)\mathcal{H}) < \infty$ (resp. $\kappa_+(E(\delta; A_0)\mathcal{H}) < \infty$) for every (in $\overline{\mathbb{R}}$) connected subset δ of Δ with $\overline{\delta} \subset \Delta$ such that $E(\delta; A_0)$ is defined. The open set $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is called of *type* π with respect to A_0, if Δ is of type π_+ or of type π_- with respect to A_0.

1.7. Perturbations of Fundamentally Reducible Operators

Let A_0 be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$ with $\rho(A_0) \neq \emptyset$. The operator A_0 is called *fundamentally reducible* if there exists a fundamental symmetry J of $\mathcal{H}$ which commutes with the resolvent of A_0, i.e. $(A_0 - \lambda)^{-1} J = J (A_0 - \lambda)^{-1}$ for some $\lambda \in \rho(A_0)$. In this case A_0 is a selfadjoint operator in the Hilbert space $(\mathcal{H}, [J\cdot, \cdot])$. We set $P_\pm := \frac{1}{2}(I \pm J)$ and $\mathcal{H}_\pm := P_\pm \mathcal{H}$. Then we have $P_\pm \mathcal{D}(A_0) = (A_0 - i)^{-1} \mathcal{H}_\pm \subset \mathcal{D}(A_0)$. Denote by $A_{0,\pm}$ the operators in $\mathcal{H}_\pm$ defined by $A_{0,\pm} x := A_0 x$ for $x \in P_\pm \mathcal{D}(A_0) =: \mathcal{D}(A_{0,\pm})$. It is easy to see that $A_{0,+}$ is a selfadjoint operator in the Hilbert space $(\mathcal{H}_+, [\cdot, \cdot])$ and $A_{0,-}$ is a selfadjoint operator in the Hilbert space $(\mathcal{H}_-, -[\cdot, \cdot])$. Further,

$$\sigma(A_0) = \sigma(A_{0,+}) \cup \sigma(A_{0,-})$$

holds. For a densely defined closed operator C we denote by $\sigma_{e,ess}(C)$ the extended essential spectrum, i.e. $\sigma_{e,ess}(C) = \sigma_{ess}(C)$ (see Section 1.3) if C is bounded and $\sigma_{e,ess}(C) = \sigma_{ess}(C) \cup \{\infty\}$ otherwise. It is easy to see that the following lemma holds. We omit the simple proof.

Lemma 1.14. *Let A_0 be a fundamentally reducible operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$. Then*

(1.14) $\sigma_{++}(A_0) = \sigma(A_{0,+}) \setminus \sigma(A_{0,-})$ *and* $\sigma_{--}(A_0) = \sigma(A_{0,-}) \setminus \sigma(A_{0,+})$

hold and the operator A_0 is definitizable over

$$\overline{\mathbb{C}} \setminus \{\sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})\}.$$

An open connected set Δ of $\overline{\mathbb{R}} \setminus \{\sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})\}$ is of type π_+ (π_-) with respect to A_0 if and only if $\Delta \subset \overline{\mathbb{R}} \setminus \sigma_{e,ess}(A_{0,-})$ ($\Delta \subset \overline{\mathbb{R}} \setminus \sigma_{e,ess}(A_{0,+})$, resp.) holds.

If, in particular, $\sigma_e(A_{0,+}) \cap \sigma_e(A_{0,-})$ consists of at most finitely many points, then A_0 is a definitizable operator in $(\mathcal{H}, [\cdot,\cdot])$ and $c_\infty(A_0) = \sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})$ and $c_s(A_0) = \{\emptyset\}$ hold.

With the help of (1.14) it is easily seen that $\sigma(A_{0,+})$ and $\sigma(A_{0,-})$ do not depend on the choice of the fundamental symmetry J, which commutes with the resolvent of A_0.

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. In the sequel we denote by $\mathfrak{S}_p(\mathcal{H}, \mathcal{K})$, $p \in [1, \infty)$, the linear space of all operators $T \in \mathfrak{S}_\infty(\mathcal{H}, \mathcal{K})$ such that $\sum_{j=1}^{\infty} s_j^p < \infty$, where s_j, $j = 1, 2, \ldots$, are the eigenvalues of $\sqrt{T^*T}$ (taking multiplicities into account). We set $\mathfrak{S}_p := \mathfrak{S}_p(\mathcal{H}, \mathcal{H})$.

In [J1] perturbations of fundamentally reducible definitizable operators are considered. In the following theorem we consider perturbations of fundamentally reducible operators, in particular we do not assume that the unperturbed operator is definitizable. For the case that the unperturbed operator is a bounded fundamentally reducible operator similar results are obtained in [L1] and [LMaM1]. For the convenience of the reader we will give a proof of the following theorem although parts of the proof are similar to [J1, proof of Theorem 3.1].

Theorem 1.15. *Let A_j, $j = 0, 1$, be a selfadjoint operator in $(\mathcal{H}, [\cdot,\cdot])$ with $i \in \rho(A_j)$. Assume, further, that A_0 is fundamentally reducible and that there exists $p \in [1, \infty)$ such that*

$$(A_0 + i)^{-1} - (A_1 + i)^{-1} \in \mathfrak{S}_p.$$

Then the following holds.

(1) *The operator* A_1 *is definitizable over* $\overline{\mathbb{C}} \setminus \{\sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})\}$.

(2) *An open connected subset* Δ *of* $\overline{\mathbb{R}}$, $\Delta \subset \overline{\mathbb{R}} \setminus \{\sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})\}$, *is of type* π_+ (π_-) *with respect to* A_0 *if and only if it is of type* π_+ (π_-, *resp.) with respect to* A_1

Proof. 1. We set $\mathbf{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbf{D}^c := \overline{\mathbb{C}} \setminus \overline{\mathbf{D}}$ and $\mathbf{T} := \{z \in \mathbb{C} : |z| = 1\}$.

Let J be a fundamental symmetry of $\mathcal{H}$ commuting with the resolvent of A_0. By ψ we denote the linear fractional transformation defined by

$$\psi(z) = -\frac{z-i}{z+i}.$$

If U_j, $j = 0, 1$, denotes the Cayley transform $\psi(A_j)$ of A_j, then

$$(I \pm J)U_1(I \pm J) = (I \pm J)U_0(I \pm J) + (I \pm J)(U_1 - U_0)(I \pm J).$$

From this it follows that

$$P_+U_1P_- \in \mathfrak{S}_\infty, \qquad P_-U_1P_+ \in \mathfrak{S}_\infty, \tag{1.15}$$

$$P_+U_1P_+ - P_+U_0P_+ \in \mathfrak{S}_\infty, \qquad P_-U_1P_- - P_-U_0P_- \in \mathfrak{S}_\infty, \tag{1.16}$$

where $P_\pm = \frac{1}{2}(I \pm J)$. This implies $\sigma(U_1) \setminus \mathbf{T} \subset \sigma_{p,norm}(U_1)$ (see e.g. [K2, Theorem 12]). By a well–known result ([K2, Theorem 13], [B, Theorem VIII.3.1]) there exists an U_1–invariant maximal nonnegative (nonpositive) subspace $\mathcal{M}_+$ (resp. $\mathcal{M}_-$) of $\mathcal{H}$ such that $\sigma(U_1|\mathcal{M}_+) \setminus \mathbf{T} = \sigma(U_1) \cap \mathbf{D}$ (resp. $\sigma(U_1|\mathcal{M}_-) \setminus \mathbf{T} = \sigma(U_1) \cap \mathbf{D}^c$) and all root spaces corresponding to eigenvalues of U_1 in $\mathbf{D}$ (resp. in $\mathbf{D}^c$) are contained in $\mathcal{M}_+$ ($\mathcal{M}_-$, resp.).

2. We will prove $\sigma_{ess}(U_1|\mathcal{M}_+) = \sigma_{ess}(U_0|P_+\mathcal{H})$. Let $K_+ \in \mathcal{L}(P_+\mathcal{H}, P_-\mathcal{H})$ be the angular operator (see [K2]) corresponding to $\mathcal{M}_+$, $\mathcal{M}_+ = \{x : x = x_+ + K_+x_+, x_+ \in P_+\mathcal{H}\}$. We consider the following extended angular operator

$$\widetilde{K_+} := \begin{bmatrix} 0 & 0 \\ K_+ & 0 \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = P_+\mathcal{H} + P_-\mathcal{H}$. Let $x_+ \in P_+\mathcal{H}$. Then $(I + K_+)x_+ \in \mathcal{M}_+$ and $(U_1 - \lambda)(I + K_+)x_+ \in \mathcal{M}_+$ for $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
&(U_1 - \lambda)(I + K_+)x_+ = \\
&= (P_+U_1P_+ + P_+U_1P_-K_+ - \lambda P_+)x_+ + \\
&+(P_-U_1P_+ + P_-U_1P_-K_+ - \lambda P_-K_+)x_+ = \\
&= (I + K_+)(P_+U_1P_+ + P_+U_1P_-K_+ - \lambda P_+)x_+.
\end{aligned}$$

The operator $I + K_+$ considered as an operator acting from $P_+\mathcal{H}$ onto $\mathcal{M}_+$ is an isomorphism, therefore it follows

$$\sigma_{ess}(U_1|\mathcal{M}_+) = \sigma_{ess}(P_+U_1P_+ + P_+U_1P_-\widetilde{K_+}|P_+\mathcal{H}).$$

In view of (1.15) and (1.16) we have

$$\begin{aligned}\sigma_{ess}(U_1|\mathcal{M}_+) &= \sigma_{ess}(P_+U_1P_+|P_+\mathcal{H}) = \\ &= \sigma_{ess}(P_+U_1P_+) \setminus \{0\} = \sigma_{ess}(P_+U_0P_+) \setminus \{0\} = \sigma_{ess}(U_0|P_+\mathcal{H}).\end{aligned} \tag{1.17}$$

By a similar reasoning the following holds

$$\sigma_{ess}(U_1|\mathcal{M}_-) = \sigma_{ess}(U_0|P_-\mathcal{H}). \tag{1.18}$$

3. Let Δ be a connected open subset of $\overline{\mathbb{R}}$ such that $\overline{\Delta} \subset \overline{\mathbb{R}} \setminus \{\sigma_{e,ess}(A_{0,+}) \cap \sigma_{e,ess}(A_{0,-})\}$. Let t_1 and t_2 be the endpoints of Δ (in $\overline{\mathbb{R}}$). Assume that t_1 is an accumulation point of $\sigma(A_1) \setminus \mathbb{R}$. Then there exists sequences $(\lambda_n^+) \subset \mathbb{C}^+ \cap \sigma(A_1)$ and $(\lambda_n^-) \subset \mathbb{C}^- \cap \sigma(A_1)$ such that $\lambda_n^\pm \to t_1$ for $n \to \infty$. Therefore

$$\psi(\lambda_n^+) \in \sigma(U_1|\mathcal{M}_+) \text{ and } \psi(\lambda_n^-) \in \sigma(U_1|\mathcal{M}_-), \quad n \in \mathbb{N},$$

and relations (1.17) and (1.18) imply

$$\begin{aligned}&\psi(t_1) \in \sigma_{ess}(U_1|\mathcal{M}_+) \cap \sigma_{ess}(U_1|\mathcal{M}_-) = \\ &= \sigma_{ess}(U_0|P_+\mathcal{H}) \cap \sigma_{ess}(U_0|P_-\mathcal{H}) = \psi(\sigma_{e,ess}(A_{0,+})) \cap \psi(\sigma_{e,ess}(A_{0,-})).\end{aligned}$$

But this is a contradiction to $t_1 \in \overline{\Delta}$, hence t_1 and t_2 are no accumulation points of $\sigma(A_1) \setminus \mathbb{R}$. As a consequence of [J2] assertions (1) and (2) hold and Theorem 1.15 is proved. □

2. A Class of Analytic Operator Functions and Their Linearizations

2.1. Definition of a Class of Analytic Operator Functions T. Jordan Chains of T

Let again $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and $J_{\mathcal{H}}$ a fundamental symmetry of $\mathcal{H}$. Throughout Chapter 2 we assume that A is a definitizable selfadjoint operator in $\mathcal{H}$. For the scale corresponding to A we simply write $\mathcal{H}_s := \mathcal{H}_s(A, J_{\mathcal{H}})$. The case that $(\mathcal{H}, [\cdot,\cdot])$ is a Hilbert space, i.e. $J_{\mathcal{H}} = 1$, and A is an arbitrary selfadjoint operator in this Hilbert space is not excluded.

We first define a class of holomorphic operator functions the values of which may be unbounded operators in $\mathcal{H}$ with domains depending on the variable. Let G be a locally holomorphic operator function with values in $\mathfrak{S}_{\infty}^{(A)}$ on an open set $O \subset \mathbb{C}$ symmetric with respect to the real axis. Moreover, we assume that

$$G(\overline{\lambda}) = G(\lambda)^{+}, \quad \lambda \in O.$$

Then the operator function

$$T(\lambda) := \lambda - (A \dot{+} G(\lambda)), \quad \lambda \in O, \tag{2.1}$$

is defined (see Lemma 1.6). For every $\lambda \in O$, $T(\lambda)$ is a densely defined closed operator in the Krein space $\mathcal{H}$. By definition we have

$$\mathcal{D}(T(\lambda)) = \{x \in \mathcal{H}_{\frac{1}{2}} : \widetilde{T}(\lambda)x \in \mathcal{H}\}, \tag{2.2}$$

where

$$\widetilde{T}(\lambda) := \lambda E - \widetilde{A} - G(\lambda), \quad \lambda \in O, \tag{2.3}$$

and

$$T(\lambda) = \widetilde{T}(\lambda) \,|\, \mathcal{D}(T(\lambda)).$$

The domain of $T(\lambda)$ may depend on λ. From the relation $\widetilde{T}(\lambda)^+ = \widetilde{T}(\overline{\lambda})$, $\lambda \in O$, we easily conclude that

$$T(\lambda)^+ = T(\overline{\lambda}), \quad \lambda \in O.$$

The following simple lemma shows that the function T is holomorphic on O in the sense of [Ka, Theorem VII.1.3].

Lemma 2.1. *For every $\lambda_0 \in O$ there exists a $z_0 \in \mathbb{C}$, $\lambda_0 + z_0 \in \rho(A)$, and a neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0 in O such that the following equivalent assertions are true.*

(i) *For every $\lambda \in \mathcal{U}(\lambda_0)$ the operator $\widetilde{T}(\lambda) - z_0 E$ is an isomorphism.*

(ii) *For every $\lambda \in \mathcal{U}(\lambda_0)$, $T(\lambda) - z_0$ has a bounded inverse $(T(\lambda) - z_0)^{-1} \in \mathcal{L}(\mathcal{H})$.*

If (i) *or* (ii) *holds for some $z_0 \in \mathbb{C}$ and some neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0, then the $\mathcal{L}(\mathcal{H})$-valued function $\lambda \mapsto (T(\lambda) - z_0)^{-1}$ is holomorphic in $\mathcal{U}(\lambda_0)$.*

Proof. The selfadjoint operator $\operatorname{Re}\lambda_0 - A$ is definitizable in $\mathcal{H}$. Then, by Lemma 1.10, for sufficiently large $\eta > 0$, we have

$$-i\eta - i \operatorname{Im}\lambda_0 \in \rho((\operatorname{Re}\lambda_0 - A)\dot{+}(-G(\lambda_0))) \cap \rho(\operatorname{Re}\lambda_0 - A).$$

Hence the assumptions of Lemma 1.7 are fulfilled, and by that lemma

$$(i\eta + \lambda_0)E - \widetilde{A} - G(\lambda_0) = \widetilde{T}(\lambda_0) + i\eta E$$

is an isomorphism. Then there exists a neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0 such that (i) holds with $z_0 = -i\eta$.

That (i) and (ii) are equivalent is a consequence of Lemma 1.7. The holomorphy assertion follows from the holomorphy of $\widetilde{T}$ in O. □

Remark 2.2. To establish Lemma 2.1 we made use of the assumption that A is definitizable. We mention that Lemma 2.1 and all other results of Sections 2.1–2.3 remain valid if A is the orthogonal direct sum of a definitizable selfadjoint operator and a bounded selfadjoint operator. For simplicity we restrict ourselves to the case of a definitizable selfadjoint operator A.

We define the *resolvent set* $\rho(T)$ of T to be the set of all $\lambda \in O$ such that $0 \in \rho(T(\lambda))$. The *spectrum* of T is by definition

$$\sigma(T) := O \setminus \rho(T).$$

It may happen that $\rho(T)$ is empty. The set $\rho(T)$ coincides with the set of all $\lambda \in O$ such that $\widetilde{T}(\lambda)$ is an isomorphism. If $\rho(T) \neq \emptyset$, the functions $\lambda \mapsto T(\lambda)^{-1}$ and $\lambda \mapsto \widetilde{T}(\lambda)^{-1}$ are locally holomorphic in $\rho(T)$. A point $\lambda \in O$ is called an *eigenvalue* of T if there exists a nonzero $x \in \mathcal{D}(T(\lambda))$ such that $T(\lambda)x = 0$. Then x is called an *eigenelement* of T corresponding to λ. The set of all eigenvalues of T is denoted by $\sigma_p(T)$. If for some $x \in \mathcal{H}_{\frac{1}{2}}$ we have $\widetilde{T}(\lambda)x = 0$ then, by definition of T, $x \in \mathcal{D}(T(\lambda))$ and $T(\lambda)x = 0$. If $\lambda_0 \in \sigma_p(T)$, then $x_0, \dots, x_{m-1} \in \mathcal{H}$ is called a *Jordan chain of* T *at* λ_0 if, for some $z_0 \in \mathbb{C}$, $z_0 \neq 0$, and some neighbourhood $\mathcal{U}(\lambda_0)$ of λ_0 such that statement (i) or (ii) of Lemma 2.1 holds, $x_0, \dots, x_{m-1} \in \mathcal{H}$ is a Jordan chain of the operator function $\lambda \mapsto (T(\lambda) - z_0)^{-1} + z_0^{-1}$ at λ_0, which is holomorphic in $\mathcal{U}(\lambda_0)$. Recall that for an $\mathcal{L}(\mathcal{H})$-valued function S holomorphic in some domain $\mathcal{U} \subset \mathbb{C}$ the vectors $x_0, \dots, x_{m-1} \in \mathcal{H}$, $x_0 \neq 0$, are called a Jordan chain of S at $\lambda_0 \in \sigma_p(S)$ if $\sum_{k=0}^{j} (k!)^{-1} S^{(k)}(\lambda_0) x_{j-k} = 0$, $j = 0, 1, \dots, m-1$, holds.

If z_1 has the same properties as z_0, we find

$$(2.4) \qquad \begin{aligned} &(T(\lambda_0) - z_1)^{-1} + z_1^{-1} = \\ &= z_0 z_1^{-1}(1 - (z_0 - z_1)(T(\lambda_0) - z_1)^{-1})((T(\lambda_0) - z_0)^{-1} + z_0^{-1}). \end{aligned}$$

Hence the above definition does not depend on the choice of z_0.

Lemma 2.3. *The elements* $x_0, \dots, x_{m-1} \in \mathcal{H}$ *are a Jordan chain of* T *at* $\lambda_0 \in \sigma_p(T)$ *if and only if* $x_0, \dots, x_{m-1} \in \mathcal{H}_{\frac{1}{2}}$, $x_0 \neq 0$, *and*

$$(2.5) \qquad \sum_{k=0}^{j} \frac{1}{k!} \widetilde{T}^{(k)}(\lambda_0) x_{j-k} = 0, \quad j = 0, 1, \dots, m-1, \quad \widetilde{T}^{(0)} := \widetilde{T}.$$

Proof. For all λ in a neighbourhood of the point λ_0, we define $Q(\lambda) := (T(\lambda) - z_0)^{-1}$. Let $x_0, \dots, x_{m-1} \in \mathcal{H}$ be a Jordan chain of T at λ_0. Then we have $\sum_{k=0}^{j} \frac{1}{k!} Q^{(k)}(\lambda_0) x_{j-k} = -z_0^{-1} x_j$, $j = 0, \dots, m-1$, and, since $Q(\lambda)$ is a restriction of $\widetilde{Q}(\lambda) := (\widetilde{T}(\lambda) - z_0 E)^{-1} \in \mathcal{L}(\mathcal{H}_{-\frac{1}{2}}, \mathcal{H}_{\frac{1}{2}})$, the same relation holds with Q replaced by $\widetilde{Q}$. Hence $x_0, \dots, x_{m-1} \in \mathcal{H}_{\frac{1}{2}}$.

Let now $x_0, \dots, x_{m-1}$ be arbitrary elements of $\mathcal{H}_{\frac{1}{2}}$ and set $x(\lambda) := x_0 + (\lambda - \lambda_0)x_1 + \ldots + (\lambda - \lambda_0)^{m-1} x_{m-1}$. Evidently, $(T(\lambda) - z_0)^{-1} + z_0^{-1} = Q(\lambda) + z_0^{-1}$ maps $\mathcal{H}_{\frac{1}{2}}$ into itself and its restriction to $\mathcal{H}_{\frac{1}{2}}$ coincides with $z_0^{-1}(\widetilde{T}(\lambda) - z_0 E)^{-1} \widetilde{T}(\lambda)$. Then $\lambda \mapsto (Q(\lambda) + z_0^{-1}) x(\lambda)$ has a zero of order m at λ_0 if and only if $\lambda \mapsto z_0^{-1}(\widetilde{T}(\lambda) - z_0 E)^{-1} \widetilde{T}(\lambda) x(\lambda)$ has a zero of order m

at λ_0 which is equivalent to the fact that $\widetilde{T}(\lambda)x(\lambda)$ has a zero of order m at λ_0. The latter is true if and only if (2.5) holds, which proves Lemma 2.3. □

2.2. Spectral Points of Positive and Negative Type of T

Let T be defined as in (2.1), let $\lambda_0 \in O$ and let z_0, $z_0 \neq 0$, be chosen as in Lemma 2.1. We recall that a point $\lambda_0 \in O$ is said to belong to the *approximative point spectrum* of T, $\lambda_0 \in \sigma_{ap}(T)$, if there exists a sequence $(x_n) \subset \mathcal{D}(T(\lambda_0))$ with $\|x_n\| = 1$, $n = 1, 2, \dots$, and $\|T(\lambda_0)x_n\| \to 0$ if $n \to \infty$.

It is easy to see that we have $\lambda_0 \in \sigma_{ap}(T)$ if and only if $\lambda_0 \in \sigma_{ap}(F_0)$, i.e. $0 \in \sigma_{ap}(F_0(\lambda_0))$, where

$$F_0(\lambda) := -z_0^2((T(\lambda) - z_0)^{-1} + z_0^{-1}) = -z_0 T(\lambda)(T(\lambda) - z_0)^{-1}.$$

Definition 2.4. A point $\lambda_0 \in \sigma(T) \cap \mathbb{R}$ is said to be a spectral point of *positive* (*negative*) *type* of T, and we write $\lambda_0 \in \sigma_{++}(T)$ (resp. $\lambda_0 \in \sigma_{--}(T)$), if $\lambda_0 \in \sigma_{ap}(T)$ and for each sequence $(x_n) \subset \mathcal{H}$ with $\|x_n\| = 1$, and $\|F_0(\lambda_0)x_n\| \to 0$ for $n \to \infty$ we have

$$\liminf_{n\to\infty} \operatorname{Re}[F_0'(\lambda_0)x_n, x_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} \operatorname{Re}[F_0'(\lambda_0)x_n, x_n] < 0).$$

We remark that if λ_0, (x_n) are as in Definition 2.4 then $\operatorname{Im}[F_0'(\lambda_0)x_n, x_n]$ tends to zero for $n \to \infty$.

Definition 2.4 does not depend on the choice of z_0. Indeed, let $z_1 \in \mathbb{C}$, $z_1 \neq 0$, satisfy the same conditions as z_0 (see Lemma 2.1). We set $F_1(\lambda) := -z_1^2((T(\lambda) - z_1)^{-1} + z_1^{-1})$. Relation (2.4) holds and we have

$$F_1'(\lambda)\left(1 - \frac{z_0 - z_1}{z_0 z_1} F_0(\lambda)\right) = \left(1 + \frac{z_0 - z_1}{z_0 z_1} F_1(\lambda)\right) F_0'(\lambda).$$

Let (x_n) and λ_0 be as in Definition 2.4. It follows that

$$\begin{aligned} &[F_1'(\lambda_0)x_n, x_n] - \frac{z_0 - z_1}{z_0 z_1}[F_1'(\lambda_0)F_0(\lambda_0)x_n, x_n] = \\ &= [F_0'(\lambda_0)x_n, x_n] + \frac{z_0 - z_1}{z_0 z_1}[F_0'(\lambda_0)x_n, -\overline{z}_1^2((T(\lambda_0) - \overline{z}_1)^{-1} + \overline{z}_1^{-1})x_n]. \end{aligned}$$

As $\|F_0(\lambda_0)x_n\| \to 0$ and, by (2.4), $\| - \overline{z}_1^2((T(\lambda_0) - \overline{z}_1)^{-1} + \overline{z}_1^{-1})x_n\| \to 0$ the second terms on both sides of the preceding relation converge to zero if $n \to \infty$. This shows that Definition 2.4 does not depend on the choice of z_0.

In the case of operator functions with values in $\mathcal{L}(\mathcal{H})$ this definition of spectral points of positive (negative) type coincides with the usual one (cf. [LMaM2]). This is a consequence of the following Proposition 2.5.

Proposition 2.5. *Let $\lambda_0 \in \sigma_{ap}(T) \cap \mathbb{R}$. The following assertions are equivalent:*

(1) $\lambda_0 \in \sigma_{++}(T)$ $(\lambda_0 \in \sigma_{--}(T))$.

(2) *For each sequence $(x_n) \subset \mathcal{H}_{\frac{1}{2}}$ with $\|x_n\|_{\frac{1}{2}} = 1$ and $\|\widetilde{T}(\lambda_0)x_n\|_{-\frac{1}{2}} \to 0$ for $n \to \infty$ we have*

$$\liminf_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] < 0).$$

(3) *For each sequence $(x_n) \subset \mathcal{D}(T(\lambda_0))$ with $\|x_n\| = 1$ and $\|T(\lambda_0)x_n\| \to 0$ for $n \to \infty$ we have*

$$\liminf_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] < 0).$$

Proof. For $x \in \mathcal{H}_{\frac{1}{2}}$ we have

$$(2.6) \qquad [F_0'(\lambda_0)x, x] = [z_0\widetilde{T}'(\lambda_0)(T(\lambda_0) - z_0)^{-1}x, \overline{z}_0(T(\lambda_0) - \overline{z}_0)^{-1}x].$$

We consider only the case of a spectral point of positive type. For a spectral point of negative type a similar reasoning applies. Assume that (1) holds. Let $(x_n) \subset \mathcal{H}_{\frac{1}{2}}$ be a sequence with $\|x_n\|_{\frac{1}{2}} = 1$ and $\|\widetilde{T}(\lambda_0)x_n\|_{-\frac{1}{2}} \to 0$ for $n \to \infty$. The operator $\widetilde{T}(\lambda_0) - z_0E$ is an isomorphism, therefore it follows $\liminf_{n\to\infty} \|(\widetilde{T}(\lambda_0) - z_0E)x_n\|_{-\frac{1}{2}} > 0$. Hence, we have $\liminf_{n\to\infty} \|x_n\|_{-\frac{1}{2}} > 0$, which implies $\liminf_{n\to\infty} \|x_n\| > 0$. Further, we have

$$(2.7) \qquad \begin{aligned} \|F_0(\lambda_0)x_n\| &\le \|z_0^2((T(\lambda_0) - z_0)^{-1} + z_0^{-1})x_n\|_{\frac{1}{2}} = \\ &= \|z_0(\widetilde{T}(\lambda_0) - z_0E)^{-1}\widetilde{T}(\lambda_0)x_n\|_{\frac{1}{2}} \le \\ &\le |z_0|\|(\widetilde{T}(\lambda_0) - z_0E)^{-1}\|\|\widetilde{T}(\lambda_0)x_n\|_{-\frac{1}{2}}. \end{aligned}$$

Therefore, $\|F_0(\lambda_0)x_n\|$ tends to zero for $n \to \infty$. By the second inequality in (2.7) and a similar inequality for z_0 replaced by $\overline{z}_0$ we find

$$\lim_{n\to\infty} \|z_0(T(\lambda_0) - z_0)^{-1}x_n + x_n\|_{\frac{1}{2}} = \lim_{n\to\infty} \|\overline{z}_0(T(\lambda_0) - \overline{z}_0)^{-1}x_n + x_n\|_{\frac{1}{2}} = 0.$$

Then (1) and (2.6) imply

$$0 < \liminf_{n\to\infty} \operatorname{Re}\,[F_0'(\lambda_0)x_n, x_n] =$$
$$= \liminf_{n\to\infty} \operatorname{Re}\,[\widetilde{T}'(\lambda_0)x_n, x_n] = \liminf_{n\to\infty}\,[\widetilde{T}'(\lambda_0)x_n, x_n],$$

hence (2) holds.

It is easy to see that (2) implies (3). Assume that (3) holds. Let $(x_n) \subset \mathcal{H}$ be a sequence with $\|x_n\| = 1$ and $\|F_0(\lambda_0)x_n\| \to 0$ for $n \to \infty$. We will prove that $\liminf_{n\to\infty} \operatorname{Re}\,[F_0'(\lambda_0)x_n, x_n] > 0$ holds. For this we set $y_n := -z_0(T(\lambda_0) - z_0)^{-1}x_n$. Then

$$\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} |z_0^{-1}|\|F_0(\lambda_0)x_n\| = 0. \tag{2.8}$$

Hence, we have

$$\lim_{n\to\infty} \|y_n\| = 1, \quad \lim_{n\to\infty} \|T(\lambda_0)y_n\| = \lim_{n\to\infty} \|F_0(\lambda_0)x_n\| = 0 \tag{2.9}$$

and, by (3),

$$\liminf_{n\to\infty}\,[\widetilde{T}'(\lambda_0)y_n, y_n] > 0. \tag{2.10}$$

We have

$$\|z_0^{-1}y_n + (T(\lambda_0) - z_0)^{-1}y_n\|_{\frac{1}{2}} =$$
$$= |z_0|^{-1}\|(T(\lambda_0) - z_0)^{-1}(T(\lambda_0) - z_0 + z_0)y_n\|_{\frac{1}{2}} \leq$$
$$\leq |z_0|^{-1}\|(\widetilde{T}(\lambda_0) - z_0E)^{-1}\|\|(T(\lambda_0)y_n\|_{-\frac{1}{2}}$$

and a similar relation with z_0 replaced by $\overline{z}_0$. Then making use of the second relation in (2.9) we find

$$\lim_{n\to\infty} \|z_0^{-1}y_n + (T(\lambda_0) - z_0)^{-1}y_n\|_{\frac{1}{2}} = \lim_{n\to\infty} \|\overline{z}_0^{-1}y_n + (T(\lambda_0) - \overline{z}_0)^{-1}y_n\|_{\frac{1}{2}} = 0.$$

Then by (2.10) and (2.8)

$$0 < \liminf_{n\to\infty} \operatorname{Re}\,[\widetilde{T}'(\lambda_0)z_0(T(\lambda_0) - z_0)^{-1}y_n, \overline{z}_0(T(\lambda_0) - \overline{z}_0)^{-1}y_n] =$$
$$= \liminf_{n\to\infty} \operatorname{Re}\,[F_0'(\lambda_0)y_n, y_n] = \liminf_{n\to\infty} \operatorname{Re}\,[F_0'(\lambda_0)x_n, x_n],$$

and (1) holds. □

Remark 2.6. Every real eigenvalue λ_0 of T belonging to $\sigma_{++}(T) \cup \sigma_{--}(T)$ is semisimple, that is, all Jordan chains of T at λ_0 consist of one element only.

Indeed, assume x_0, x_1 is a Jordan chain of T at λ_0. Then we have, by (2.5),

$$\widetilde{T}(\lambda_0)x_0 = 0, \quad \widetilde{T}'(\lambda_0)x_0 + \widetilde{T}(\lambda_0)x_1 = 0.$$

Hence

$$0 = [\widetilde{T}'(\lambda_0)x_0, x_0] + [\widetilde{T}(\lambda_0)x_1, x_0] = [\widetilde{T}'(\lambda_0)x_0, x_0],$$

which contradicts $\lambda_0 \in \sigma_{++}(T) \cup \sigma_{--}(T)$.

2.3. An Operator Matrix M Connected with T. Relations between the Spectra of T and M.

We assume now, in addition, that the operator function G has a special form. Let $\mathcal{K}$ be one more Krein space. We denote its inner product in the same way as the inner product of $\mathcal{H}$ by $[\cdot,\cdot]$. Let $J_{\mathcal{K}}$ be a fundamental symmetry of $(\mathcal{K}, [\cdot,\cdot])$. In the following D denotes a selfadjoint operator in $\mathcal{K}$ with $\rho(D) \neq \emptyset$. For the scale $\mathcal{K}_s(D, J_{\mathcal{K}})$ we simply write $\mathcal{K}_s$. Let, further, B be a compact operator of $\mathcal{H}_{\frac{1}{2}}$ in $\mathcal{K}_{-\frac{1}{2}}$. Then B^+ is a compact operator of $\mathcal{K}_{\frac{1}{2}}$ in $\mathcal{H}_{-\frac{1}{2}}$:

(b) $$B \in \mathfrak{S}_\infty(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}}), \quad B^+ \in \mathfrak{S}_\infty(\mathcal{K}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}}).$$

In the special case of an operator A with compact resolvent and a bounded operator B from $\mathcal{H}$ to $\mathcal{K}$, the restriction of B to $\mathcal{H}_{\frac{1}{2}}$ regarded as an operator into $\mathcal{K}_{-\frac{1}{2}}$ is compact (Lemma 1.2), i.e. (b) holds.

We set

$$G(\lambda) := -B^+(\widetilde{D} - \lambda E)^{-1}B, \quad \lambda \in \rho(D).$$

This operator function G satisfies the conditions of Section 2.1 and we define T and $\widetilde{T}$ as in (2.1), (2.3), i.e.

(2.11) $$\begin{aligned} T(\lambda) &= \lambda - \{A \buildrel{+}\over{\smile} (-B^+(\widetilde{D} - \lambda E)^{-1}B)\}, \\ \widetilde{T}(\lambda) &= \lambda E - \widetilde{A} + B^+(\widetilde{D} - \lambda E)^{-1}B, \quad \lambda \in \rho(D). \end{aligned}$$

Let $\mathcal{G}$ denote the Krein space $\mathcal{H} \times \mathcal{K}$. We consider on $\mathcal{G}$ the fundamental symmetry $J_{\mathcal{H}} \times J_{\mathcal{K}}$. The operator $\mathbf{L} := A \times D$ is selfadjoint in $\mathcal{G}$ and we have $\rho(\mathbf{L}) \neq \emptyset$. The scale $\mathcal{G}_s := \mathcal{G}_s(\mathbf{L}, J_{\mathcal{H}} \times J_{\mathcal{K}})$ corresponding to $\mathbf{L}$ is $\mathcal{H}_s \times \mathcal{K}_s$. We define an operator $\mathbf{B} \in \mathcal{L}(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}})$ by

(2.12) $$\mathbf{B} := \begin{bmatrix} 0 & B^+ \\ B & 0 \end{bmatrix}$$

with respect to the decompositions $\mathcal{G}_{\frac{1}{2}} = \mathcal{H}_{\frac{1}{2}} \times \mathcal{K}_{\frac{1}{2}}$ and $\mathcal{G}_{-\frac{1}{2}} = \mathcal{H}_{-\frac{1}{2}} \times \mathcal{K}_{-\frac{1}{2}}$. By the assumptions on B, $\mathbf{B} \in \mathfrak{S}_\infty^{(\mathbf{L})}$ (cf. Section 1.2). We set

$$\widetilde{\mathbf{M}} := \widetilde{\mathbf{L}} + \mathbf{B} = \begin{bmatrix} \widetilde{A} & B^+ \\ B & \widetilde{D} \end{bmatrix} \quad \text{and} \quad \mathbf{M} := \mathbf{L} \dot{+} \mathbf{B} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \dot{+} \begin{bmatrix} 0 & B^+ \\ B & 0 \end{bmatrix}$$

with respect to the same decompositions.

Let $\lambda \in \rho(D)$. We apply the Frobenius-Schur factorization to $\widetilde{\mathbf{M}} - \lambda E$:

$$\begin{aligned} \widetilde{\mathbf{M}} - \lambda E &= \begin{bmatrix} 1 & \widetilde{F}(\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\widetilde{T}(\lambda) & 0 \\ 0 & \widetilde{D} - \lambda E \end{bmatrix} \begin{bmatrix} 1 & 0 \\ (\widetilde{F}(\overline{\lambda}))^+ & 1 \end{bmatrix} = \\ &= \mathbf{U}(\lambda) \begin{bmatrix} -\widetilde{T}(\lambda) & 0 \\ 0 & \widetilde{D} - \lambda E \end{bmatrix} \mathbf{U}(\overline{\lambda})^+, \quad \lambda \in \rho(D), \end{aligned} \tag{2.13}$$

where

$$\widetilde{F}(\lambda) := B^+(\widetilde{D} - \lambda E)^{-1}, \quad \mathbf{U}(\lambda) := \begin{bmatrix} 1 & \widetilde{F}(\lambda) \\ 0 & 1 \end{bmatrix}.$$

This relation and Lemma 1.7 imply (2.14) below.

Proposition 2.7. *We have*

$$\rho(T) = \rho(\mathbf{M}) \cap \rho(D). \tag{2.14}$$

Moreover, for $\lambda \in \rho(T)$ *we have*

$$\begin{aligned} &(\widetilde{\mathbf{M}} - \lambda E)^{-1} = \\ &= \begin{bmatrix} -\widetilde{T}(\lambda)^{-1} & \widetilde{T}(\lambda)^{-1}\widetilde{F}(\lambda) \\ \widetilde{F}(\overline{\lambda})^+\widetilde{T}(\lambda)^{-1} & -\widetilde{F}(\overline{\lambda})^+\widetilde{T}(\lambda)^{-1}\widetilde{F}(\lambda) + (\widetilde{D} - \lambda E)^{-1} \end{bmatrix} \end{aligned} \tag{2.15}$$

and

$$\mathcal{D}(\mathbf{M}) = (\widetilde{\mathbf{M}} - \lambda E)^{-1}\mathcal{G}.$$

Denote by I_1 the embedding of $\mathcal{H}$ in $\mathcal{G}$:

$$I_1 : \mathcal{H} \ni x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{G}$$

and by P_1 the orthogonal projection on $\mathcal{H}$ in $\mathcal{G}$ regarded as an operator of $\mathcal{G}$ in $\mathcal{H}$. We have $I_1^+ = P_1$. Then by Proposition 2.7, for $\lambda \in \rho(T) = \rho(\mathbf{M}) \cap \rho(D)$, we have

$$-T(\lambda)^{-1} = P_1(\mathbf{M} - \lambda)^{-1}I_1. \tag{2.16}$$

The following proposition on the point spectra of the operator function T and the operator $\mathbf{M}$ (compare [LMeM, Proposition 1.2], [AL, Lemma 3.2]) can be verified by a simple calculation.

Proposition 2.8. *We have*

$$\sigma_p(T) = \sigma_p(\mathbf{M}) \cap \rho(D).$$

Moreover, the following holds.

(1) *If $x_0, \dots, x_{m-1} \in \mathcal{H}_{\frac{1}{2}}$ is a Jordan chain of T corresponding to a point $\lambda_0 \in \sigma_p(T)$, then the elements*

$$\begin{aligned} \mathbf{x}_0 &= \begin{pmatrix} x_0 \\ -(\widetilde{D} - \lambda_0 E)^{-1} B x_0 \end{pmatrix}, \dots, \\ \mathbf{x}_{m-1} &= \begin{pmatrix} x_{m-1} \\ -\sum_{j=0}^{m-1} (\widetilde{D} - \lambda_0 E)^{-j-1} B x_{m-1-j} \end{pmatrix} \end{aligned} \tag{2.17}$$

form a Jordan chain of $\mathbf{M}$ corresponding to λ_0. Conversely, if $\mathbf{x}_0, \dots, \mathbf{x}_{m-1}$ is a Jordan chain of $\mathbf{M}$ corresponding to a point $\lambda_0 \in \sigma_p(\mathbf{M}) \cap \rho(D)$, then this Jordan chain is of the form (2.17) *and the first components of the vectors in* (2.17) *form a Jordan chain of T corresponding to λ_0.*

(2) *If x_0 and $\mathbf{x}_0$ are as in* (1) *then $[\widetilde{T}'(\lambda_0)x_0, x_0] = [\mathbf{x}_0, \mathbf{x}_0]$ holds.*

The following theorem shows that, for every $\lambda \in \rho(D) \cap \mathbb{R}$, the sign types of λ with respect to T and $\mathbf{M}$ coincide.

Theorem 2.9. *Assume that $\rho(\mathbf{M}) \neq \emptyset$ holds. Then*

$$\sigma_{ap}(T) \cap \mathbb{R} = \sigma_{ap}(\mathbf{M}) \cap \rho(D) \cap \mathbb{R}, \tag{2.18}$$

and we have

$$\sigma_{++}(T) = \sigma_{++}(\mathbf{M}) \cap \rho(D), \quad \sigma_{--}(T) = \sigma_{--}(\mathbf{M}) \cap \rho(D). \tag{2.19}$$

Proof. Let $\lambda_0 \in \sigma_{ap}(T) \cap \mathbb{R}$. We recall that T is only defined on $\rho(D)$, hence $\lambda_0 \in \rho(D)$. According to Proposition 2.7 λ_0 belongs to $\sigma(\mathbf{M}) \cap \mathbb{R}$. The selfadjointness of $\mathbf{M}$ implies $\lambda_0 \in \sigma_{ap}(\mathbf{M})$. Conversely, let $\lambda_0 \in \sigma_{ap}(\mathbf{M}) \cap$

$\rho(D) \cap \mathbb{R}$. Then Proposition 2.7 implies $\lambda_0 \in \sigma(T) \cap \mathbb{R}$. The operator $T(\lambda_0)$ is selfadjoint, therefore λ_0 belongs to $\sigma_{ap}(T) \cap \mathbb{R}$ and (2.18) is proved.

Let $\lambda_0 \in \sigma_{++}(T)$. Relation (2.18) implies $\lambda_0 \in \sigma_{ap}(\mathbf{M}) \cap \rho(D) \cap \mathbb{R}$. Let $(\mathbf{x}_n) = (\binom{x_n}{y_n}) \subset \mathcal{G}_{\frac{1}{2}}$, $x_n \in \mathcal{H}_{\frac{1}{2}}$, $y_n \in \mathcal{K}_{\frac{1}{2}}$, be a sequence with $\|\mathbf{x}_n\|_{\mathcal{G}_{\frac{1}{2}}} = 1$ and $\|(\widetilde{\mathbf{M}} - \lambda_0 E)\mathbf{x}_n\|_{\mathcal{G}_{-\frac{1}{2}}} \to 0$ for $n \to \infty$. From (2.13) and the fact that $\mathbf{U}(\lambda_0) \in \mathcal{L}(\mathcal{G}_{-\frac{1}{2}})$ is invertible we deduce

$$\|(\widetilde{T}(\lambda_0)x_n\|_{\mathcal{H}_{-\frac{1}{2}}} \to 0 \quad \text{and } \|Bx_n + (\widetilde{D} - \lambda_0 E)y_n\|_{\mathcal{K}_{-\frac{1}{2}}} \to 0 \tag{2.20}$$

for $n \to \infty$. We claim that $\liminf_{n\to\infty} \|x_n\|_{\mathcal{H}_{\frac{1}{2}}} > 0$ holds. Suppose that there exists a subsequence (x_{n_k}) of (x_n) which tends to zero in $\mathcal{H}_{\frac{1}{2}}$. Then Bx_{n_k} tends to zero in $\mathcal{K}_{-\frac{1}{2}}$ and relation (2.20) implies $(\widetilde{D} - \lambda_0 E)y_{n_k} \to 0$ for $k \to \infty$ in $\mathcal{K}_{-\frac{1}{2}}$. Since, by Lemma 1.1, $\widetilde{D} - \lambda_0 E$ is an isomorphism we find $\|y_{n_k}\|_{\mathcal{K}_{\frac{1}{2}}} \to 0$ and $\|\mathbf{x}_{n_k}\|_{\mathcal{G}_{\frac{1}{2}}} \to 0$ for $k \to \infty$, a contradiction.

Proposition 2.5 now leads to

$$\begin{aligned} 0 &< \liminf_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] = \\ &= \liminf_{n\to\infty} ([x_n, x_n] + [(\widetilde{D} - \lambda_0 E)^{-1}Bx_n, (\widetilde{D} - \lambda_0 E)^{-1}Bx_n]). \end{aligned}$$

From (2.20) it follows that $(\widetilde{D} - \lambda_0 E)^{-1}Bx_n + y_n$ tends to zero in $\mathcal{K}_{\frac{1}{2}}$, hence

$$0 < \liminf_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n] = \liminf_{n\to\infty} ([x_n, x_n] + [y_n, y_n]) = \liminf_{n\to\infty} [\mathbf{x}_n, \mathbf{x}_n].$$

By Lemma 1.4 λ_0 belongs to $\sigma_{++}(\mathbf{M})$. Similarly, for $\lambda \in \sigma_{--}(T)$.

Conversely, let $\lambda_0 \in \sigma_{++}(\mathbf{M}) \cap \rho(D)$. By (2.18) we have $\lambda_0 \in \sigma_{ap}(T) \cap \mathbb{R}$. Let $(x_n) \subset \mathcal{H}_{\frac{1}{2}}$ be a sequence with $\|x_n\|_{\mathcal{H}_{\frac{1}{2}}} = 1$ and $\|\widetilde{T}(\lambda_0)x_n\|_{\mathcal{H}_{-\frac{1}{2}}} \to 0$ for $n \to \infty$. Set $y_n := -(\widetilde{D} - \lambda_0 E)^{-1}Bx_n$. Then y_n belongs to $\mathcal{K}_{\frac{1}{2}}$ and for $\mathbf{x}_n$, $\mathbf{x}_n := \binom{x_n}{y_n} \in \mathcal{G}_{\frac{1}{2}}$, we have

$$\|\mathbf{x}_n\|_{\mathcal{G}_{\frac{1}{2}}} \geq 1 \quad \text{and } \|(\widetilde{\mathbf{M}} - \lambda_0 E)\mathbf{x}_n\|_{\mathcal{G}_{-\frac{1}{2}}} = \|\widetilde{T}(\lambda_0)x_n\|_{\mathcal{H}_{-\frac{1}{2}}} \to 0.$$

for $n \to \infty$. Lemma 1.4 implies

$$\begin{aligned} 0 &< \liminf_{n\to\infty} [\mathbf{x}_n, \mathbf{x}_n] = \liminf_{n\to\infty} ([x_n, x_n] + [y_n, y_n]) = \\ &= \liminf_{n\to\infty} ([x_n, x_n] + [(\widetilde{D} - \lambda_0 E)^{-1}Bx_n, (\widetilde{D} - \lambda_0 E)^{-1}Bx_n]) = \\ &= \liminf_{n\to\infty} [\widetilde{T}'(\lambda_0)x_n, x_n]. \end{aligned}$$

Hence, by Proposition 2.5, $\lambda_0 \in \sigma_{++}(T)$. Similarly, for $\lambda_0 \in \sigma_{--}(\mathbf{M}) \cap \rho(D)$. □

Observe that Theorem 2.9 implies, in the case $\rho(\mathbf{M}) \neq \emptyset$, that for

$$\lambda_0 \in \sigma_{++}(T) \cap \sigma_p(T) = \sigma_{++}(\mathbf{M}) \cap \sigma_p(\mathbf{M}) \cap \rho(D)$$
$$(\lambda_0 \in \sigma_{--}(T) \cap \sigma_p(T) = \sigma_{--}(\mathbf{M}) \cap \sigma_p(\mathbf{M}) \cap \rho(D))$$

and for each eigenelement x_0 of T corresponding to λ_0 and each eigenvector $\mathbf{x}_0$ of $\mathbf{M}$ corresponding to λ_0 we have $[\widetilde{T}'(\lambda_0)x_0, x_0] > 0$ and $[\mathbf{x}_0, \mathbf{x}_0] > 0$ (resp. $[\widetilde{T}'(\lambda_0)x_0, x_0] < 0$ and $[\mathbf{x}_0, \mathbf{x}_0] < 0$). Further, λ_0 is a semisimple eigenvalue of T and $\mathbf{M}$ (see Remark 2.6).

Now we shortly consider some special assumptions on the operator B and their consequences.

Lemma 2.10. *Assume that the following condition* (β) *is fulfilled.*

(β): *Relation* (b) *holds, and* B *can be extended by continuity to an operator belonging to* $\mathcal{L}(\mathcal{H}, \mathcal{K}_{-\frac{1}{2}})$.

Then, for $\lambda \in \rho(D)$, $\mathcal{D}(T(\lambda)) = \mathcal{D}(A)$ *holds and* $\binom{x}{y}$ *belongs to* $\mathcal{D}(\mathbf{M})$ *if and only if* $x \in \mathcal{D}(A)$ *and, for some* $\lambda \in \rho(T)$, $(\widetilde{D} - \lambda E)^{-1}Bx + y \in \mathcal{D}(D)$ *holds. Furthermore, all Jordan chains of* T *corresponding to points* $\lambda \in \rho(D)$ *belong to* $\mathcal{D}(T(\lambda))$.

Proof. For $\lambda \in \rho(D)$ the range of $B^+(\widetilde{D} - \lambda E)^{-1}B$ is contained in $\mathcal{H}$ and this operator can be extended by continuity to an operator belonging to $\mathcal{L}(\mathcal{H})$, hence $\mathcal{D}(T(\lambda)) = \mathcal{D}(A)$. The description of $\mathcal{D}(\mathbf{M})$ is an easy consequence of (2.15) and the fact that B^+ can be considered as an operator belonging to $\mathcal{L}(\mathcal{K}_{\frac{1}{2}}, \mathcal{H})$. The last assertion follows from (2.5). □

Remark 2.11. The case that $\mathbf{M}$ arises from $\mathbf{L}$ by an $\mathbf{L}$–compact perturbation of the special form (2.12) is contained in our general setting. Indeed, let $\widehat{\mathbf{B}} = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix}$ be a symmetric operator in the Krein space $\mathcal{G}$ and assume that B_1 is a D–compact operator from $\mathcal{K}$ into $\mathcal{H}$, and that B_2 is an A–compact operator from $\mathcal{H}$ into $\mathcal{K}$, in particular, we have $\mathcal{D}(\mathbf{L}) \subset \mathcal{D}(\widehat{\mathbf{B}})$ and $\widehat{\mathbf{B}}$ is $\mathbf{L}$–compact. The restrictions $B_1|\mathcal{K}_1$ and $B_2|\mathcal{H}_1$ belong to $\mathfrak{S}_\infty(\mathcal{K}_1, \mathcal{H})$ and $\mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{K})$ respectively (cf. [Ka, Remark IV.1.12]). By the symmetry of $\widehat{\mathbf{B}}$, $(B_1|\mathcal{K}_1)^+$ is an extension of $B_2|\mathcal{H}_1$.

Let B denote the interpolated operator corresponding to the middle point of the scale between $B_2|\mathcal{H}_1$ and $(B_1|\mathcal{K}_1)^+$. Then B satisfies condition (b).

In this case we have

$$\mathcal{D}(T(\lambda)) = \mathcal{D}(A), \quad T(\lambda) = \lambda - A + (B_1|\mathcal{K}_1)(D-\lambda)^{-1}(B_2|\mathcal{H}_1), \quad \lambda \in \rho(D).$$

Indeed, since by assumption $(B_1|\mathcal{K}_1)(D-\lambda)^{-1}(B_2|\mathcal{H}_1) \in \mathfrak{S}_\infty(\mathcal{H}_1, \mathcal{H})$, $\lambda \in \rho(D)$, the operator $S(\lambda) := \lambda - A + (B_1|\mathcal{K}_1)(D-\lambda)^{-1}(B_2|\mathcal{H}_1)$, $\mathcal{D}(S(\lambda)) = \mathcal{D}(A)$, is closed. Evidently, $T(\lambda)$ is an extension of $S(\lambda)$ or equal to $S(\lambda)$. For fixed $\lambda \in \rho(D)$ there exists an $M > 0$ such that $|\mathrm{Im}\,\mu| \geq M$ implies $\mu \in \rho(J_{\mathcal{H}} S(\lambda))$ (see e.g. [Ka, V.4.3]). This implies $T(\lambda) = S(\lambda)$. Similarly, we find $\mathcal{D}(\mathbf{M}) = \mathcal{D}(A) \times \mathcal{D}(D)$ and

$$\mathbf{M} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} + \begin{bmatrix} 0 & B_1|\mathcal{K}_1 \\ B_2|\mathcal{H}_1 & 0 \end{bmatrix}$$

In this case all Jordan chains of T belong to $\mathcal{D}(T(\lambda)) = \mathcal{D}(A)$.

2.4. The Case of a Definitizable Operator Matrix M

In this section we assume, in addition to the preceding section, that the following condition (ad) is fulfilled.

(ad) A and D are definitizable selfadjoint operators and the following holds.

- (a_1) $c_\infty(A) = \emptyset$ or $c_\infty(A) = \{\infty\}$, (a_2) $\sigma(A) \setminus \mathbb{R} \subset \sigma_{p,norm}(A)$,
- (d_1) $c_\infty(D) = \emptyset$ or $c_\infty(D) = \{\infty\}$, (d_2) $\sigma(D) \setminus \mathbb{R} \subset \sigma_{p,norm}(D)$.
- (ad_1) The intersections $\widetilde{\sigma}_{e,+}(A) \cap \widetilde{\sigma}_{e,-}(D)$ and $\widetilde{\sigma}_{e,-}(A) \cap \widetilde{\sigma}_{e,+}(D)$ are empty or equal to the one point set $\{\infty\}$.
- (ad_2) There exists a real interval of the form (b_+, ∞) which is of positive type with respect to A and D or of negative type with respect to A and D. The same is true for some interval of the form $(-\infty, b_-)$.

Theorem 2.12. *Assume that the definitizable selfadjoint operators A and D and the operator B fulfil conditions* (ad) *and* (b). *Then $\mathbf{L}$ and $\mathbf{M}$ are definitizable selfadjoint operators in $\mathcal{G}$,*

$$\sigma(\mathbf{L}) \setminus \mathbb{R} \subset \sigma_{p,norm}(\mathbf{L}), \quad \sigma(\mathbf{M}) \setminus \mathbb{R} \subset \sigma_{p,norm}(\mathbf{M}), \tag{2.21}$$

and $\mathbf{L}$ and $\mathbf{M}$ have no finite essential critical points. Moreover, the statements (ii) - (iv) *of Theorem* 1.12 *hold with A and $A \dot{+} V$ replaced by $\mathbf{L}$ and $\mathbf{M}$, respectively.*

Proof. If $\mathcal{G}$ is a Pontryagin space, it is easy to see that the theorem is true. Assume that $\mathcal{G}$ is not a Pontryagin space. The first relation of (2.21) is a direct consequence of (a_2) and (d_2). Let E denote the product of the spectral functions of A and D. By $\sigma(\mathbf{L}) \setminus \mathbb{R} \subset \sigma_{p,norm}(\mathbf{L})$, $E(\overline{\mathbb{R}})\mathcal{G}$ is not a Pontryagin space. It follows from (a_1), (d_1) and (ad_1) that $c_\infty(\mathbf{L}) = \emptyset$ or $c_\infty(\mathbf{L}) = \{\infty\}$. Then by ($ad_2$) and by Lemma 1.9 $\mathbf{L}$ is definitizable. The rest of the theorem is a consequence of Theorem 1.12. □

Since by Theorem 2.9 the sign types of spectral points of T and $\mathbf{M}$ coincide and, on the other hand, by Proposition 2.8 the point spectra in $\rho(D)$ of T and $\mathbf{M}$ coincide, the definitizability of $\mathbf{M}$ and the fact that $\mathbf{M}$ has no finite essential critical point have immediate consequences for T.

Corollary 2.13. *Assume that* (ad) *and* (b) *are fulfilled. Then the following holds.*

(i) *All points of $\sigma(T) \cap \mathbb{R}$ with the possible exception of a finite number of points belong to $\sigma_{++}(T) \cup \sigma_{--}(T)$. Every real eigenvalue λ of T belonging to $\sigma_{++}(T) \cup \sigma_{--}(T)$ is semisimple* (*see Remark* 2.6).

The points of $(\sigma(T) \cap \mathbb{R}) \setminus (\sigma_{++}(T) \cup \sigma_{--}(T))$ are eigenvalues of T. If $\lambda \in (\sigma(T) \cap \mathbb{R}) \setminus (\sigma_{++}(T) \cup \sigma_{--}(T))$ the kernel of $T(\lambda)$ has finite defect in the linear space spanned by all Jordan chains of T corresponding to λ.

(ii) *Every point $\lambda \in \sigma(T) \cap \mathbb{R}$ belonging to an open interval of type π_+ (π_-) with respect to A or, equivalently, with respect to $\mathbf{M}$ such that $\lambda \notin \sigma_{++}(T)$ (resp. $\lambda \notin \sigma_{--}(T)$) is an eigenvalue of T. If N_λ is the kernel of $T(\lambda)$, then $\kappa_-(N_\lambda) < \infty$ (resp. $\kappa_+(N_\lambda) < \infty$).*

In the rest of this section we consider the operator function T^{-1} defined in $\rho(T)$ ($\subset \rho(D)$) with values in $\mathcal{L}(\mathcal{H})$, which is connected with the definitizable operator $\mathbf{M}$ by (2.15). The function T^{-1} is meromorphic on $\mathbb{C} \setminus \mathbb{R}$ and $\mathbb{R}$–symmetric

$$(T(\lambda)^{-1})^+ = T(\overline{\lambda})^{-1}, \quad \lambda \in \rho(T). \tag{2.22}$$

We shall consider $T(\lambda)^{-1}$ not only for $\lambda \in \rho(T)$ but also for all points $\lambda_0 \in \overline{\mathbb{C}}$ such that $T(\lambda)^{-1}$ has a unique analytic continuation to some neighbourhood of λ_0. The set of these points λ_0 will be denoted by $\rho_e(T)$. Evidently, $\rho_e(\mathbf{M}) \subset \rho_e(T)$. We set $\sigma_e(T) := \overline{\mathbb{C}} \setminus \rho_e(T)$.

Some aspects of $\mathbb{R}$-symmetric meromorphic operator functions on $\mathbb{C} \setminus \mathbb{R}$ were studied in [J4]. In [J4] the $\mathbb{R}$–symmetry was understood with respect to a Hilbert scalar product; but it can be replaced in all definitions and results of that paper by the $\mathbb{R}$–symmetry with respect to a Krein space inner product (see [J4, Introduction]). In this thesis we consider $\mathbb{R}$–symmetric operator functions with respect to a Krein space adjoint as, for example, (2.22).

We recall that, if $\mathcal{H}$ is a Krein space, an $\mathbb{R}$-symmetric $\mathcal{L}(\mathcal{H})$–valued function N meromorphic on $\mathbb{C} \setminus \mathbb{R}$ is said to belong to the Krein–Langer class $N_\kappa(\mathcal{L}(\mathcal{H}))$ if for arbitrary $n \in \mathbb{N}$, elements $x_1, \ldots, x_n \in \mathcal{H}$ and points $\lambda_1, \ldots, \lambda_n$ of holomorphy of N the $n \times n$ matrix

$$\left(\left[\frac{N(\lambda_i) - N(\overline{\lambda_j})}{\lambda_i - \overline{\lambda_j}} x_i, x_j \right] \right)_{i,j=1}^{n}$$

has at most κ negative eigenvalues and for at least one choice of n, $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_n$ it has exactly κ negative eigenvalues. N_0 is the class of Nevanlinna functions, i.e. the class N_0 coincides with the class of all $\mathbb{R}$–symmetric $\mathcal{L}(\mathcal{H})$–valued functions N holomorphic in $\mathbb{C} \setminus \mathbb{R}$ such that for every λ with $\operatorname{Im} \lambda > 0$ and every $x \in \mathcal{H}$, $\operatorname{Im} [N(\lambda)x, x] \geq 0$ holds.

An $\mathbb{R}$–symmetric $\mathcal{L}(\mathcal{H})$-valued function G meromorphic on $\mathbb{C} \setminus \mathbb{R}$ is called *definitizable* ([J4, Definition 3.1]) if there exists a scalar rational function r, $r(\overline{\lambda}) = \overline{r(\lambda)}$, a Nevanlinna function N and an $\mathcal{L}(\mathcal{H})$–valued meromorphic function n in $\mathbb{C}$ the poles of which are points of holomorphy of G such that

$$r(\lambda)G(\lambda) = N(\lambda) + n(\lambda)$$

for all points $\lambda \in \mathbb{C} \setminus \mathbb{R}$ of holomorphy of rG.

In [J4] there were introduced the so-called intervals of positive and negative type (see also [J6]), and of type π_+ and type π_- with respect to a definitizable operator function. In Theorem 2.14, (ii), below we will use the notations of [J4].

In Theorem 2.14, (iii), and in Section 2.5 we will make use of decompositions of the spectrum of $\mathbf{M}$ similar to those considered in Lemma 1.9 and Theorem 1.12. Assume, additionally, as in Theorem 2.14, (iii), below, that $\mathcal{G}$ is not a Pontryagin space and that $\infty \notin c_s(A) \cup c_s(D)$. Then, by Theorem 2.12 (Theorem 1.12), $\infty \notin c_s(\mathbf{M})$. Let a_j, b_j, s_j, $j = 1, \ldots, l$, $l \geq 1$, be defined as in Lemma 1.9 with A replaced by $\mathbf{M}$, in particular, $s_j \in (a_j, b_j) \cap \rho(\mathbf{M})$, and consider the intervals

$$(-\infty, s_1), (s_1, s_2), \ldots, (s_l, \infty). \tag{2.23}$$

Put $s_0 := -\infty$, $s_{l+1} := \infty$. We define

$$\begin{aligned}\Delta_+ &:= \bigcup\{(s_{j-1}, s_j) : \kappa_-(E((s_{j-1}, s_j); \mathbf{M})\mathcal{G}) < \infty, 1 \le j \le l+1\},\\ \Delta_- &:= \bigcup\{(s_{j-1}, s_j) : \kappa_-(E((s_{j-1}, s_j); \mathbf{M})\mathcal{G}) = \infty, 1 \le j \le l+1\}.\end{aligned} \tag{2.24}$$

Then $\kappa_-(E(\Delta_+; \mathbf{M})\mathcal{G}) < \infty$ and $\kappa_+(E(\Delta_-; \mathbf{M})\mathcal{G}) < \infty$. The system of intervals (2.23) will be denoted by Σ.

Theorem 2.14. *Assume that* (ad) *and* (b) *are fulfilled. Then* $-T^{-1}$ *is a definitizable operator function. Moreover, the following holds.*

(i) *If q is a scalar rational function, $q(\overline{\lambda}) = \overline{q(\lambda)}$, the poles of which are contained in $\rho_e(\mathbf{M})$ such that the form $[q(\mathbf{M})\,\cdot, \cdot]$ on $\mathcal{G}$ has κ negative squares, then $-qT^{-1}$ can be written in the form*

$$-q(\lambda)T(\lambda)^{-1} = N(\lambda) + n(\lambda),$$

where $N \in N_{\kappa'}(\mathcal{L}(\mathcal{H}))$, $0 \le \kappa' \le \kappa$, and n is a meromorphic operator function in $\overline{\mathbb{C}}$ which is holomorphic at all points where q is holomorphic,

(ii) *If an open subset Δ of $\overline{\mathbb{R}}$ (not necessarily contained in $\rho(D) \cap \mathbb{R}$) is of positive type (negative type, type π_+, type π_-) with respect to $\mathbf{M}$, then it is of positive type (resp. negative type, type π_+, type π_-) with respect to $-T^{-1}$* (see [J4, §3.1]).

(iii) *Assume, in addition, that $\mathcal{G}$ is not a Pontryagin space and that $\infty \notin c_s(A) \cup c_s(D)$, then for every system Σ (see* (2.23) *and* (2.24)*) $-T^{-1}$ can be written in the form*

$$-T(\lambda)^{-1} = S_0(\lambda) + S_1(\lambda) - S_2(\lambda).$$

Here S_0 is a finitely meromorphic operator function in $\overline{\mathbb{C}}$, $S_0(\overline{\lambda}) = S_0(\lambda)^+$, whose (finitely many) poles belong to $\mathbb{C} \setminus \mathbb{R}$. The operator functions S_1 and S_2 belong to the Krein-Langer classes $N_{\kappa_1}(\mathcal{L}(\mathcal{H}))$ and $N_{\kappa_2}(\mathcal{L}(\mathcal{H}))$, respectively, for some nonnegative κ_1, κ_2. The functions S_1 and S_2 are holomorphic in $\mathbb{C} \setminus \mathbb{R}$, S_1 is locally holomorphic on $\overline{\Delta_-}$ and S_2 is locally holomorphic on $\overline{\Delta_+}$ (closures in $\mathbb{R}$). Moreover, it holds $\operatorname{w-lim}_{\eta\to\infty} \eta^{-1} S_j(i\eta) = 0$, $j = 1, 2$.

By the properties mentioned above each of the functions S_0, S_1 and S_2 is uniquely determined up to addition of a selfadjoint operator.

Proof. 1. Let q be as in the theorem. Set $g(z, \lambda) := (q(z) - q(\lambda))(z - \lambda)^{-1}$. Then the operator function $\lambda \mapsto g(\mathbf{M}, \lambda)$ is holomorphic in every point of holomorphy of q. We have, for $\lambda \in \rho(\mathbf{M}) \cap \rho(D)$,

$$-q(\lambda)T(\lambda)^{-1} = P_1 q(\lambda)(\mathbf{M}-\lambda)^{-1}I_1 = P_1 q(\mathbf{M})(\mathbf{M}-\lambda)^{-1}I_1 - P_1 g(\mathbf{M},\lambda)I_1,$$

(see (2.16)). Making use of the above definition of the Krein-Langer classes $N_\kappa(\mathcal{L}(\mathcal{H}))$ we easily verify that the fact that $[q(\mathbf{M})\,\cdot,\cdot]$ has κ negative squares implies that the function $P_1 q(\mathbf{M})(\mathbf{M}-\lambda)^{-1}I_1$ belongs to $N_{\kappa'}(\mathcal{L}(\mathcal{H}))$ for some $\kappa' \leq \kappa$. This proves (i). If we choose q so that $[q(\mathbf{M})x, x] \geq 0$ for all $x \in \mathcal{G}$, then $P_1 q(\mathbf{M})(\mathbf{M}-\lambda)^{-1}I_1$ is a Nevanlinna function. Therefore the function $-T^{-1}$ is definitizable.
2. For every $c > 0$, cA, cB, and cD satisfy the conditions (ad) and (b), and the operator function corresponding to these operators is $cT(c^{-1}\lambda)$. Therefore, it is no restriction to assume that

$$i \in \rho(D) \cap \rho(\mathbf{M}) = \rho(T).$$

If we do this, then

$$\begin{aligned} -T(\lambda)^{-1} = {} & P_1(\mathbf{M}-\lambda)^{-1}I_1 = P_1(\mathbf{M}+i)^{-1}\mathbf{M}(\mathbf{M}-i)^{-1}I_1 + \\ & + P_1(\mathbf{M}+i)^{-1}\{\lambda + (\lambda^2+1)(\mathbf{M}-\lambda)^{-1}\}(\mathbf{M}-i)^{-1}I_1 = \\ = {} & \frac{1}{2}(T(i)^{-1} + (T(i)^{-1})^+) + \\ & + P_1(\mathbf{M}+i)^{-1}\{\lambda + (\lambda^2+1)(\mathbf{M}-\lambda)^{-1}\}(\mathbf{M}-i)^{-1}I_1, \end{aligned} \tag{2.25}$$

and (ii) follows from (2.25) and [J4].
3. Let the assumptions of (iii) be fulfilled. If E_0 is the Riesz-Dunford projection corresponding to the non–real spectrum of the operator $\mathbf{M}$, then $S_0(\lambda) := P_1 E_0(\mathbf{M}-\lambda)^{-1}I_1$ has the required properties. We have $\infty \notin c_s(\mathbf{L})$ and, by Theorem 2.12, $\infty \notin c_s(\mathbf{M})$. If E is the spectral function of $\mathbf{M}$, then the functions $S_1(\lambda) := P_1 E(\Delta_+)(\mathbf{M}-\lambda)^{-1}I_1$ and $S_2(\lambda) := -P_1 E(\Delta_-)(\mathbf{M}-\lambda)^{-1}I_1$ (see 2.24) have the required properties. It is easy to verify the uniqueness statement. □

2.5. Minimality Properties of M and Their Consequences

In this section we assume as in Section 2.4 that conditions (ad) and (b) are fulfilled. In addition, we assume that $\mathcal{G}$ is not a Pontryagin space and that $\infty \notin c_s(A) \cup c_s(D)$. Let, moreover, $i, -i \in \rho(D) \cap \rho(\mathbf{M}) = \rho(T)$. This is no restriction, see part 2 of the proof of Theorem 2.14.

For a bounded operator U in a Krein space we denote the operator $\frac{1}{2}(U + U^+)$ briefly by $\mathrm{Re}^+ U$.

Let F be an $\mathbb{R}$-symmetric meromorphic function in $\mathbb{C} \setminus \mathbb{R}$ with values in $\mathcal{L}(\mathcal{H})$. Assume that i and $-i$ are points of holomorphy of F. If $\widetilde{\mathcal{H}}$ is a Krein space, $\widetilde{M}$ a selfadjoint operator in $\widetilde{\mathcal{H}}$ with $i, -i \in \rho(\widetilde{M})$ whose non–real spectrum consists of poles of the resolvent, if $\Gamma \in \mathcal{L}(\mathcal{H}, \widetilde{\mathcal{H}})$ and

$$(2.26) \quad F(\lambda) = \mathrm{Re}^+ F(i) + \Gamma^+(\lambda + (\lambda^2+1)(\widetilde{M} - \lambda)^{-1})\Gamma, \quad \lambda \in \rho(\widetilde{M}),$$

holds, $\widetilde{M}$ is called a *representing operator* for F. The operator $\mathbf{M}$ considered above is a representing operator for $-T^{-1}$ with $\Gamma = (\mathbf{M} - i)^{-1} I_1$, see (2.25). We recall the definition of minimality and local minimality of representing operators. In the definition we restrict ourselves to the special class of operators $\widetilde{M}$ which will be considered in this section.

Definition 2.15. Assume that (2.26) holds with a definitizable selfadjoint operator $\widetilde{M}$ in $\widetilde{\mathcal{H}}$ and let $\infty \notin c_s(\widetilde{M})$. Then the representing operator $\widetilde{M}$ is called *minimal* (with respect to F) if

$$\operatorname{clsp}\{(\widetilde{M} - z)^{-1}\Gamma x : z \in \rho(\widetilde{M}), x \in \mathcal{H}\} = \widetilde{\mathcal{H}}.$$

If Ω is an open subset of $\overline{\mathbb{C}}$ symmetric with respect to $\mathbb{R}$, $\Omega \cap \mathbb{R} \neq \emptyset$, and $E_{0,\Omega}$ is the Riesz-Dunford projection corresponding to $\widetilde{M}$ and the set $(\sigma(\widetilde{M}) \setminus \mathbb{R}) \cap \Omega$, then $\widetilde{M}$ is called *minimal in* Ω (with respect to F) if

$$\begin{aligned} &\operatorname{clsp}\{(\widetilde{M} - z)^{-1}(E(\Delta;\widetilde{M}) + E_{0,\Omega})\Gamma x : z \in \rho(\widetilde{M}), \Delta \subset \Omega \cap \overline{\mathbb{R}}, x \in \mathcal{H}\} = \\ &= \operatorname{clsp}\{E(\Delta;\widetilde{M})\widetilde{\mathcal{H}} : \Delta \subset \Omega \cap \overline{\mathbb{R}}\} + E_{0,\Omega}\widetilde{\mathcal{H}}. \end{aligned}$$

Here Δ runs through all closed connected subsets of $\Omega \cap \overline{\mathbb{R}}$ for which $E(\Delta;\widetilde{M})$ is defined.

Remark 2.16. Let F and $\widetilde{M}$ be as in Definition 2.15 and let s be a closed subset of $\overline{\mathbb{R}}$ such that s has no more than a finite number of accumulation points and that $s \cap \sigma_p(\widetilde{M}) = \emptyset$. Then the minimality of $\widetilde{M}$ in $\Omega := \overline{\mathbb{C}} \setminus s$ implies the minimality of $\widetilde{M}$. This follows from the fact that there exists a sequence of closed subsets δ_n, $n = 1, 2, \ldots$, of $\overline{\mathbb{R}}$ with $\delta_n \subset \overline{\mathbb{C}} \setminus s$ such that $\overline{\bigcup_{n=1}^{\infty} E(\delta_n)\widetilde{\mathcal{H}}} = E(\overline{\mathbb{R}};\widetilde{M})\widetilde{\mathcal{H}}$.

We recall that if S is an $\mathcal{L}(\mathcal{H})$-valued function belonging to some class $N_\kappa(\mathcal{L}(\mathcal{H}))$ and the weak limit $\mathrm{w-}\lim_{\eta\to\infty} \eta^{-1} S(i\eta)$ is zero, then with the help of the well-known ϵ-method one can construct a Pontryagin space Π_S

with $\kappa_-(\Pi_S) = \kappa$ and a minimal representing operator M_S for S in Π_S ([KL]). For functions $S', S'' \in N_\kappa(\mathcal{L}(\mathcal{H}))$ which differ only by a selfadjoint operator we have $M_{S'} = M_{S''}$. In essence, Π_S is a space of vector-valued functions and M_S is the operator of multiplication by the independent variable in this space. Every minimal representing operator for S is unitarily equivalent to M_S with respect to the indefinite inner products.

In the following theorem we find, with the help of Theorem 2.14, a minimal representing operator for $-T^{-1}$. If $\mathcal{H}' = (\mathcal{H}', [\cdot,\cdot])$ is a Krein space we denote by $-\mathcal{H}'$ the Krein space $(\mathcal{H}', -[\cdot,\cdot])$.

Theorem 2.17. *If Σ and S_0, S_1, S_2 are as in Theorem 2.14, the selfadjoint operator*

$$M_\Sigma := M_{S_0} \times M_{S_1} \times M_{S_2}$$

in $\Pi_\Sigma := \Pi_{S_0} \times \Pi_{S_1} \times (-\Pi_{S_2})$ is definitizable, $c_\infty(M_\Sigma) \subset \{\infty\}$, $\infty \notin c_s(M_\Sigma)$, and M_Σ is a minimal representing operator for $-T^{-1}$. We have $\sigma_e(M_\Sigma) = \sigma_e(T)$ (see Section 2.4 for the definition).

Moreover, every minimal representing operator $\widetilde{M}$ for $-T^{-1}$ for which $\infty \notin c_s(\widetilde{M})$ holds is unitarily equivalent to M_Σ with respect to the Krein space inner products.

Proof. By the definition of the operators M_{S_j} the following minimal representations hold:

$$S_j(\lambda) = \mathrm{Re}^+ S_j(i) + \Gamma_j^+(\lambda + (\lambda^2+1)(M_{S_j} - \lambda)^{-1})\Gamma_j, \quad j = 0,1,2,$$

If we set $\Gamma = (\Gamma_0 \;\; \Gamma_1 \;\; \Gamma_2')^{\mathrm{T}}$, where Γ_2' is the operator Γ_2 considered as an operator belonging to $\mathcal{L}(\mathcal{H}, (-\Pi_{S_2}))$, then

$$(2.27) \qquad -T(\lambda)^{-1} = -\mathrm{Re}^+ T(i)^{-1} + \Gamma^+(\lambda + (\lambda^2+1)(M_\Sigma - \lambda)^{-1})\Gamma.$$

By the definition of S_j a point λ belongs to $\rho_e(T)$ if and only if λ is a point of holomorphy of each of the functions S_j, $j = 0,1,2$. This is equivalent to $\lambda \in \rho_e(M_{S_j})$, $j = 0,1,2$, since the set of all points of holomorphy of an N_κ–function coincides with the extended resolvent set of every minimal representing operator (see [KL, Satz 4.4]). Hence $\sigma_e(M_\Sigma) = \sigma_e(T)$. It is easy to see that $c_\infty(M_\Sigma) = c_\infty((M_{S_0} \times M_{S_1}) \times M_{S_2}) \subset \{\infty\}$. Then it follows from Lemma 1.9 that M_Σ is definitizable. Since ∞ cannot be a singular critical point of a selfadjoint operator in a Pontryagin space, we have $\infty \notin c_s(M_\Sigma)$.

It remains to prove that the representation (2.27) is minimal. Let E_1 and E_2 denote the spectral functions of M_{S_1} and M_{S_2}, respectively. Let $y_j \in \Pi_{S_j}$,

$j = 1, 2$. Then

$$\lim_{m\to\infty} \|E_j([-m,m])y_j - y_j\|_{\Pi_{S_j}} = 0, \quad j = 1, 2.$$

Therefore, in order to prove the minimality of M_Σ, it is sufficient to verify that for every sufficiently large $m > 0$ every vector

$$y' := (y_0', y_1', y_2')^{\mathrm{T}} \in \Pi_{S_0} \times E_1([-m,m])\Pi_{S_1} \times E_2([-m,m])\Pi_{S_2}$$

can be approximated by linear combinations of elements having the form $(M_\Sigma - \lambda)^{-1}\Gamma x$, $x \in \mathcal{H}$. Furthermore, by the minimality of M_{S_0}, M_{S_1} and M_{S_2} it remains to prove that every vector $y'' := (y_0'', y_1'', y_2'')^{\mathrm{T}}$ belonging to $\Pi_{S_0} \times E_1([-m,m])\Pi_{S_1} \times E_2([-m,m])\Pi_{S_2}$ with

$$y_0'' := \sum_{k=1}^{n} (M_{S_0} - \lambda_{0,k})^{-1}\Gamma_0 x_{0,k},$$

$$y_j'' := \sum_{k=1}^{n} (M_{S_j} - \lambda_{j,k})^{-1} E_j([-m,m])\Gamma_j x_{j,k}, \quad j = 1, 2,$$

where $n \in \mathbb{N}$, $x_{j,k} \in \mathcal{H}$, $\lambda_{j,k} \in \rho(M_\Sigma)$, $j = 0, 1, 2$, can be approximated by linear combinations of elements of the form $(M_\Sigma - \lambda)^{-1}\Gamma x$. We have

$$y'' = \sum_{j=0}^{2}\sum_{k=1}^{n} (M_\Sigma - \lambda_{j,k})^{-1} F_j \Gamma x_{j,k},$$

where F_0 is the Riesz-Dunford projection corresponding to the operator M_Σ and $\sigma(M_\Sigma) \setminus \mathbb{R}$,

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_1(\Delta_+ \cap [-m,m]) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_2(\Delta_- \cap [-m,m]) \end{pmatrix}.$$

Since F_1 (F_2) is the spectral projection corresponding to M_Σ and the set $\Delta_+ \cap [-m,m]$ (resp. $\Delta_- \cap [-m,m]$) F_1 and F_2 can be written as strong limits of linear combinations of resolvents of M_Σ. The same holds for F_0. This proves the minimality of M_Σ. The last assertion is a consequence of [J4, Section 2]. □

The representation of $-T(\lambda)^{-1}$ by $\mathbf{M}$,

$$-T(\lambda)^{-1} = -\mathrm{Re}^{+}\, T(i)^{-1} + P_1(\mathbf{M}+i)^{-1}(\lambda + (\lambda^2+1)(\mathbf{M}-\lambda)^{-1})(\mathbf{M}-i)^{-1} I_1,$$

(see (2.25)) may not satisfy the minimality condition. If minimality holds one can always obtain a non-minimal representing operator by replacing the operator D by a direct product of D with another suitably chosen operator.

For minimal $\mathbf{M}$ results from [J4] (cf. Theorem 2.14, (ii)) imply the following proposition.

Proposition 2.18. *Let* $\mathbf{M}$ *be a minimal representing operator for* $-T^{-1}$*. Then* $\mathbf{M}$ *is unitarily equivalent to* M_Σ *with respect to the Krein space inner products. An open subset* Δ *of* $\overline{\mathbb{R}}$ *is of positive type (negative type, type* π_+*, type* π_-*) with respect to* $\mathbf{M}$ *if and only if it is of the same type with respect to* $-T^{-1}$*.*

It turns out that the minimality of $\mathbf{M}$ with respect to $-T^{-1}$ is equivalent to the "minimality of D with respect to T":

Lemma 2.19. $\mathbf{M}$ *is a minimal representing operator for* $-T^{-1}$ *if and only if*

$$\operatorname{closp}\{(\widetilde{D} - zE)^{-1}\mathcal{R}(B) : z \in \rho(D)\} = \mathcal{K}. \tag{2.28}$$

P r o o f. Since $\rho(\mathbf{M})$ is dense in $\mathbb{C}$, in (2.28) $\rho(D)$ can be replaced by $\rho(\mathbf{M}) \cap \rho(D)$. If (2.28) does not hold, then by (2.15) the set

$$\begin{aligned} &\{(\mathbf{M} - \lambda)^{-1} I_1 \mathcal{H} : \lambda \in \rho(\mathbf{M}) \cap \rho(D)\} = \\ &= \left\{ \begin{pmatrix} -\widetilde{T}(\lambda)^{-1}x \\ (\widetilde{D} - \lambda E)^{-1} B \widetilde{T}(\lambda)^{-1} x \end{pmatrix} : x \in \mathcal{H}, \lambda \in \rho(\mathbf{M}) \cap \rho(D) \right\} \end{aligned} \tag{2.29}$$

cannot be total in $\mathcal{G}$, and $\mathbf{M}$ is not minimal.

Assume that (2.28) holds. In view of

$$-i\eta T(i\eta)^{-1} = P_1 i\eta(\mathbf{M} - i\eta)^{-1} I_1, \quad \eta > 0,$$

and the fact that $\infty \notin c_s(\mathbf{M})$ we have

$$\operatorname{s-}\lim_{\eta\to\infty} -i\eta T(i\eta)^{-1} = -1. \tag{2.30}$$

We claim that, for every $x \in \mathcal{H}$,

$$\lim_{\eta\to\infty} \|i\eta(\widetilde{D} - i\eta E)^{-1} B \widetilde{T}(i\eta)^{-1} x\| = 0. \tag{2.31}$$

If (2.31) holds, we have, by (2.29) and (2.30),

$$\mathcal{H} \times \{0\} \subseteq \operatorname{closp}\{(\mathbf{M} - \lambda)^{-1} I_1 x : \lambda \in \rho(\mathbf{M}), x \in \mathcal{H}\},$$

and this relation along with (2.28) implies

$$\operatorname{closp}\{(\mathbf{M}-\lambda)^{-1}I_1 x : \lambda \in \rho(\mathbf{M}), x \in \mathcal{H}\} = \mathcal{G},$$

and $\mathbf{M}$ is minimal.

It remains to prove (2.31). From $\infty \notin c_s(D)$ it follows that, for every $x \in \mathcal{K}_{-\frac{1}{2}}$, we have

$$\lim_{\eta\to\infty} \|(\widetilde{D} - i\eta E)^{-1}x\|_{\mathcal{K}_{\frac{1}{2}}} = 0.$$

In view of $B \in \mathfrak{S}_\infty(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}})$ we obtain

$$\lim_{\eta\to\infty} \|(\widetilde{D} - i\eta E)^{-1}B\|_{\mathcal{L}(\mathcal{H}_{\frac{1}{2}},\mathcal{K}_{\frac{1}{2}})} = 0. \tag{2.32}$$

Similarly,

$$\lim_{\eta\to\infty} \|(i\eta E - \widetilde{A})^{-1}B^+\|_{\mathcal{L}(\mathcal{K}_{\frac{1}{2}},\mathcal{H}_{\frac{1}{2}})} = 0. \tag{2.33}$$

By (2.32) and (2.33), for $x \in \mathcal{H}$ and sufficiently large $\eta > 0$,

$$\eta^{\frac{1}{2}}T(i\eta)^{-1}x = \Big(1 + (i\eta E - \widetilde{A})^{-1}B^+(\widetilde{D} - i\eta E)^{-1}B\Big)^{-1}\eta^{\frac{1}{2}}(i\eta E - \widetilde{A})^{-1}x,$$

and the first inverse on the right hand side of this relation is uniformly bounded for $i\eta$ in some neighbourhood of ∞. From $\infty \notin c_s(A)$ it follows that

$$\lim_{\eta\to\infty} \|\eta^{\frac{1}{2}}(i\eta - A)^{-1}x\|_{\mathcal{H}_{\frac{1}{2}}} = 0, \quad x \in \mathcal{H},$$

and we obtain

$$\lim_{\eta\to\infty} \|\eta^{\frac{1}{2}}T(i\eta)^{-1}x\|_{\mathcal{H}_{\frac{1}{2}}} = 0, \quad x \in \mathcal{H}. \tag{2.34}$$

By $\infty \notin c_s(D)$,

$$\limsup_{\eta\to\infty} \|\eta^{\frac{1}{2}}(\widetilde{D} - i\eta E)^{-1}\|_{\mathcal{L}(\mathcal{K}_{-\frac{1}{2}},\mathcal{K})} < \infty. \tag{2.35}$$

Then (2.34) and (2.35) imply (2.31), and Lemma 2.19 is proved. □

The following theorem shows that we always have local minimality of $\mathbf{M}$ in $\rho_e(D)$ and that the operator M_Σ (see Theorem 2.17) can locally be used as a model for $\mathbf{M}$.

Theorem 2.20. *The representing operator* $\mathbf{M}$ *(for* $-T^{-1}$*) is minimal in* $\rho_e(D)$*. If* Σ *is as in Theorem* 2.17 *and* Δ *is a connected subset of* $\overline{\mathbb{R}} \cap \rho_e(D)$

such that $E(\Delta; M_\Sigma)$ *is defined, then* $E(\Delta; \mathbf{M})$ *is defined and* $\mathbf{M}|E(\Delta;\mathbf{M})\mathcal{G}$ *is unitarily equivalent to* $M_\Sigma|E(\Delta; M_\Sigma)\Pi_\Sigma$.

Proof. Let E_0 be the Riesz-Dunford projection corresponding to $\mathbf{M}$ and $\sigma(\mathbf{M}) \setminus (\mathbb{R} \cup \sigma_e(D))$. In order to prove that $\mathbf{M}$ is minimal in $\rho_e(D)$ it is sufficient to prove that for any bounded closed interval $[a, b]$ contained in $\rho_e(D) \cap \mathbb{R}$ and any $y \in \mathcal{G}$

$$(E([a,b];\mathbf{M}) + E_0)y \tag{2.36}$$

can be approximated by elements of the form

$$\sum_i \alpha_i(\mathbf{M} - \lambda_i)^{-1}(E([a,b];\mathbf{M}) + E_0)\begin{pmatrix} x_1' \\ 0 \end{pmatrix}, \quad x_1' \in \mathcal{H}, \alpha_i \in \mathbb{C}, \lambda_i \in \rho(\mathbf{M}),$$

where the sum is finite.

Assume first that a and b are no eigenvalues of $\mathbf{M}$. Let $\mathcal{C}(a, b; \epsilon)$, $\epsilon > 0$, be the oriented curve consisting of the straight line connecting $b + i\epsilon$ and $a + i\epsilon$ oriented in the direction of decreasing real part and the straight line connecting $a - i\epsilon$ and $b - i\epsilon$ oriented in the direction of increasing real part. Let G be a smooth bounded domain whose closure is contained in the open upper half plane such that $\overline{G} \subset \rho_e(D)$ and $(\sigma(\mathbf{M}) \setminus \mathbb{R}) \cap \rho_e(D)$ is contained in $G \cup G^*$, $G^* := \{\overline{z} : z \in G\}$. Then the element (2.36) can be written as

$$-\frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\int_{\mathcal{C}(a,b;\epsilon)}(\mathbf{M} - \lambda)^{-1}y\,d\lambda - \frac{1}{2\pi i}\int_{\mathcal{C}}(\mathbf{M} - \lambda)^{-1}y\,d\lambda, \tag{2.37}$$

where $\mathcal{C}$ denotes the boundary of G. If $y = \binom{x_1}{x_2}$, $x_1 \in \mathcal{H}$, $x_2 \in \mathcal{K}$, we have, by Proposition 2.7,

$$\begin{aligned}&(\mathbf{M} - \lambda)^{-1}y = \\ &\begin{pmatrix} -\widetilde{T}(\lambda)^{-1}(x_1 - B^+(D-\lambda)^{-1}x_2) \\ (\widetilde{D} - \lambda E)^{-1}B\widetilde{T}(\lambda)^{-1}(x_1 - B^+(D-\lambda)^{-1}x_2) \end{pmatrix} + \begin{pmatrix} 0 \\ (D-\lambda)^{-1}x_2 \end{pmatrix}.\end{aligned} \tag{2.38}$$

If we substitute (2.38) into (2.37) the last term in (2.38) gives no contribution. Hence approximating the integrals in (2.37) by Riemann sums we see that (2.37) is the limit of elements of the form

$$\sum_i \alpha_i \begin{pmatrix} -\widetilde{T}(\lambda_i)^{-1}(x_1 - B^+(D-\lambda_i)^{-1}x_2) \\ (\widetilde{D} - \lambda_i E)^{-1}B\widetilde{T}(\lambda_i)^{-1}(x_1 - B^+(D-\lambda_i)^{-1}x_2) \end{pmatrix},$$

where the sum is finite and $\alpha_i \in \mathbb{C}$, $\lambda_i \in \rho(\mathbf{M})$. Since every element $x_1 - B^+(D-\lambda_i)^{-1}x_2$ is the limit in $\mathcal{H}_{-\frac{1}{2}}$ of elements of $\mathcal{H}$, we may approximate (2.37) by elements of the form

$$(E([a,b];\mathbf{M}) + E_0) \sum_i \alpha_i (\mathbf{M} - \lambda_i)^{-1} \begin{pmatrix} x_1' \\ 0 \end{pmatrix}.$$

If a or b is an eigenvalue of $\mathbf{M}$, one has to replace a and b in (2.37) by $a - \frac{1}{n}$ and $b + \frac{1}{n}$, respectively. Then (2.36) coincides with the limit of that expression for $n \to \infty$ and the same reasoning applies. This proves the first assertion. The second assertion is a consequence of [J5]. □

Remark 2.21. If $\sigma_e(D)$ has only a finite number of accumulation points and $\sigma(D) \cap \sigma_p(\mathbf{M}) = \emptyset$ then, in view of Remark 2.16, $\mathbf{M}$ is minimal and, hence, the conclusions of Proposition 2.18 hold.

In the following theorem we summarize the relations of sign properties of open subsets of $\overline{\mathbb{R}} \cap \rho_e(D)$ with respect to T, $-T^{-1}$ and $\mathbf{M}$.

Theorem 2.22. *Let Δ be an open connected subset of $\overline{\mathbb{R}} \cap \rho_e(D)$. Then the following assertions are equivalent.*

(i) $\Delta \subseteq \sigma_{++}(T) \cup \rho(T)$ $(\Delta \subseteq \sigma_{--}(T) \cup \rho(T))$.

(ii) Δ *is of positive (resp. negative) type with respect to* $-T^{-1}$.

(iii) Δ *is of positive (resp. negative) type with respect to* $\mathbf{M}$.

Moreover, the following assertions are equivalent.

(ii′) Δ *is of type* π_+ *(type* π_-*) with respect to* $-T^{-1}$.

(iii′) Δ *is of type* π_+ *(resp. type* π_-*) with respect to* $\mathbf{M}$.

(iv′) Δ *is of type* π_+ *(resp. type* π_-*) with respect to* A.

Proof. The equivalence of (i) and (iii) was proved in Theorem 2.9. The equivalences of (ii) and (iii), and of (ii′) and (iii′) follow from Theorem 2.20 and a result of [J5].

Δ is of type π_+ (type π_-) with respect to A if and only if it is of type π_+ (resp. type π_-) with respect to $\mathbf{L}$. By Theorem 2.12 this is equivalent to (iii′). □

Remark 2.23. The equivalence of (ii′) and (iv′) can be viewed as an operator function variant of Theorem 1.12, (ii). Indeed, if, in $T(\lambda)$, we replace the term $-B^+(\widetilde{D} - \lambda E)^{-1}B \in \mathfrak{S}_\infty^{(A)}$ by an operator $V = V^+ \in \mathfrak{S}_\infty^{(A)}$ we get $\lambda - (A \dot{+} V)$. For that case the equivalence mentioned above was proved in 1.12, (ii).

3. A Sturm-Liouville Equation Depending Rationally upon the Eigenvalue Parameter

3.1. The Case of a Bounded Interval

In this section we will apply the results of Chapter 2 to an eigenvalue problem of the form

$$(3.1)\quad (\operatorname{sign} x)^\delta y''(x) + \lambda y(x) + \sum_{j=1}^{n_+} \frac{q_j^+(x)}{u_j^+(x) - \lambda} y(x) + \sum_{j=1}^{n_-} \frac{q_j^-(x)}{u_j^-(x) - \lambda} y(x) = 0,$$

$\delta = 0, 1$, $\lambda \in \mathbb{C}$, on the interval $I := [-1, 1]$ with boundary conditions

$$(3.2)\qquad y(-1) = y(1) = 0,$$

and its connections to an eigenvalue problem for some systems of differential equations. Here the functions $q_j^\pm$, $u_j^\pm$ are real valued measurable functions, $q_j^+(x) \geq 0$, $j = 1, \dots, n_+$, $q_j^-(x) \leq 0$, $j = 1, \dots, n_-$, for almost all $x \in I$ and satisfy the condition (I) below. We do not exclude that one of the sums in (3.1) is missing. In this case we set $n_+ = 0$ or $n_- = 0$, respectively. It is always assumed that $n_+ + n_- \geq 1$. Let in Section 3.1 the following condition be fulfilled:

(I) $\dfrac{q_j^\pm}{1 + |u_j^\pm|} \in L_1(I), \quad j = 1, \dots, n_\pm.$

The functions $q_j^\pm$, $u_j^\pm$ are not assumed to be integrable. To simplify notation we will use the same letter for the functions $q_j^\pm$, $u_j^\pm$ and the operators of multiplication by these functions in $L_2(I)$.

Now we write the left hand side of (3.1) with the help of an operator function of the form (2.11). Let $\mathcal{H} = L_2(I)$ and let $\mathcal{D}$ denote the set of all absolutely continuous functions $y \in \mathcal{H}$ which have an absolutely continuous derivative y' with $(y')' \in \mathcal{H}$, and which satisfy the boundary conditions (3.2). If $\delta = 0$ we set

$$(Ay)(x) := -y''(x), \quad y \in \mathcal{D}. \tag{3.3}$$

A is a uniformly positive selfadjoint operator in $\mathcal{H}$.

For $\delta = 1$ we introduce in $\mathcal{H}$ an indefinite inner product $[\cdot,\cdot]_{\mathcal{H}}$,

$$[f,g]_{\mathcal{H}} := \int_I (\operatorname{sign} x) f(x)\overline{g(x)}dx.$$

Then $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ is a Krein space and the operator of multiplication by $\operatorname{sign} x$ in $\mathcal{H}$, denoted by $J_{\mathcal{H}}$, is a fundamental symmetry of $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$. In this case let A be the positive selfadjoint operator in $(\mathcal{H}, [\cdot,\cdot]_{\mathcal{H}})$ defined by

$$(Ay)(x) := -(\operatorname{sign} x)\, y''(x), \quad y \in \mathcal{D}.$$

We have $c(A) = c_\infty(A) = \{\infty\}$ and $\sigma(A) \subset \mathbb{R}\setminus\{0\}$. It is well known (see e.g. [CL]) that $\infty \notin c_s(A)$ and A is similar to a selfadjoint operator in a Hilbert space.

In both cases $\delta = 0$ and $\delta = 1$ the conditions on A of Chapter 2 are fulfilled. For the scales $\mathcal{H}_s(A)$ ($\delta = 0$, cf. Section 1.1) and $\mathcal{H}_s(A, J_{\mathcal{H}})$ ($\delta = 1$, cf. Section 1.2) we simply write $\mathcal{H}_s$. It is well known that $\mathcal{H}_{\frac{1}{2}}$ coincides (up to equivalent scalar products) with the Sobolev space $H_0^1(I)$ of all absolutely continuous functions f on I with $f(-1) = f(1) = 0$ such that $f' \in L_2(I)$ ([K1, §6, 9^0]).

We introduce the Hilbert space

$$\mathcal{K} := (L_2(I))^{n_+} \times (L_2(I))^{n_-}. \tag{3.4}$$

Then the operator D,

$$D := \operatorname{diag}(u_1^+, \ldots, u_{n_+}^+, u_1^-, \ldots, u_{n_-}^-), \tag{3.5}$$

defined on the set $\mathcal{D}(D) := \mathcal{D}(u_1^+) \times \cdots \times \mathcal{D}(u_{n_-}^-)$ is selfadjoint in $\mathcal{K}$.

In the following we consider on $\mathcal{K}$ the indefinite inner product $[\cdot,\cdot]_{\mathcal{K}}$ defined by

$$\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right]_{\mathcal{K}} := (f_1, g_1)_{(L_2(I))^{n_+}} - (f_2, g_2)_{(L_2(I))^{n_-}} \tag{3.6}$$

for $f_1, g_1 \in (L_2(I))^{n_+}$, $f_2, g_2 \in (L_2(I))^{n_-}$. Then $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$ is a Krein space and $J_{\mathcal{K}} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the corresponding fundamental symmetry. As D commutes with $J_{\mathcal{K}}$, D is a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$. For the scale $\mathcal{K}_s(D, J_{\mathcal{K}})$ we simply write $\mathcal{K}_s$ (cf. Section 1.2).

Let q, u be any of the pairs q_j^+, u_j^+ or q_j^-, u_j^-. Since the embedding of $\mathcal{H}_{\frac{1}{2}} = H_0^1(I)$ into $C(I)$ is compact (see e.g. [A, VI.6.2]) and (I) holds, the operator of multiplication by $|q^{\frac{1}{2}}|(1+|u|)^{-\frac{1}{2}}$ regarded as an operator from $H_0^1(I)$ into $L_2(I)$ is compact. Therefore,

$$B := \left[(q_1^+)^{\frac{1}{2}}, \ldots, (q_{n_+}^+)^{\frac{1}{2}}, |q_1^-|^{\frac{1}{2}}, \ldots, |q_{n_-}^-|^{\frac{1}{2}}\right]^{\mathrm{T}} \in \mathfrak{S}_\infty(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}}).$$

and $B^+ \in \mathfrak{S}_\infty(\mathcal{K}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}})$.

In the Krein space

$$\mathcal{G} := \mathcal{H} \times \mathcal{K} \tag{3.7}$$

we consider the operator $\mathbf{L} := A \times D$ which is selfadjoint in $\mathcal{G}$. $\mathcal{G}_s = \mathcal{H}_s \times \mathcal{K}_s$ is the scale corresponding to $\mathbf{L}$. Then $\mathbf{B}$,

$$\mathbf{B} := \begin{bmatrix} 0 & B^+ \\ B & 0 \end{bmatrix}, \tag{3.8}$$

belongs to $\mathfrak{S}_\infty(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}})$. We define $T, \widetilde{T}, \mathbf{M}, \widetilde{\mathbf{M}}$ as in Section 2.3. Then the expression on the left hand side of (3.1) defined for all those $y \in H_0^1(I)$ (in distribution sense) for which this expression belongs to $L_2(I)$, coincides with $T(\lambda)y$. Then Lemmas 2.1, 2.3, Propositions 2.5, 2.7, 2.8 and Remark 2.6 hold, in particular we have

$$\rho(T) = \rho(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_p(T) = \sigma_p(\mathbf{M}) \cap \rho(D).$$

By Proposition 2.8, for $\lambda \in \rho(D)$, a function $y \in \mathcal{H}_{\frac{1}{2}}$ is a solution of the equation (3.1) if and only if $\mathbf{y} := (y, y_1^+, \ldots, y_{n_+}^+, y_1^-, \ldots, y_{n_-}^-)^{\mathrm{T}}$, for some $y_j^+, y_k^- \in L_2(I)$, $j = 1, \ldots, n_+$, $k = 1, \ldots, n_-$, is a solution of

$$(\mathbf{M} - \lambda)\mathbf{y} = \begin{bmatrix} -(\operatorname{sign} x)^\delta \frac{d^2}{dx^2} - \lambda & (q_1^+)^{\frac{1}{2}} & \cdots & -|q_{n_-}^-|^{\frac{1}{2}} \\ (q_1^+)^{\frac{1}{2}} & u_1^+ - \lambda & & \\ \vdots & & \ddots & \\ |q_{n_-}^-|^{\frac{1}{2}} & & & u_{n-}^- - \lambda \end{bmatrix} \begin{bmatrix} y \\ y_1^+ \\ \vdots \\ y_{n_-}^- \end{bmatrix} = 0.$$

Lemma 3.1. *Assume that condition* (I) *is fulfilled. Then* $\rho(\mathbf{M}) \neq \emptyset$. *Moreover, we have*

$$\sigma_{ess}(\mathbf{M}) = \sigma(\mathbf{M}) \setminus \sigma_{p,norm}(\mathbf{M}) = \sigma(D). \tag{3.9}$$

Proof. The operators A and D are similar to selfadjoint operators in Hilbert space. This implies as in the proof of Lemma 2.19 (see (2.32), (2.33) and the following relation) that for sufficiently large $\eta > 0$ we have $i\eta \in \rho(T)$. Then, by Proposition 2.7, $\rho(\mathbf{M}) \neq \emptyset$.

We have $\sigma_{ess}(\mathbf{M}) = \sigma_{ess}(\mathbf{L})$ (see (1.6)). As A has a compact resolvent, $\sigma_{ess}(\mathbf{L}) = \sigma_{ess}(D)$. Since D is the direct product of the operators of multiplication by the functions $u_j^{\pm}$, we have $\sigma_{ess}(D) = \sigma(D)$. Therefore, $\sigma_{ess}(\mathbf{M}) = \sigma(D)$.

As $\rho(\mathbf{M}) \neq \emptyset$ each $\lambda \in \mathbb{C} \setminus \mathbb{R} \subset \rho(D) = \mathbb{C} \setminus \sigma_{ess}(\mathbf{M})$ belongs to $\rho(\mathbf{M}) \cup \sigma_{p,norm}(\mathbf{M})$. Hence $\sigma(\mathbf{M}) \setminus \sigma_{p,norm}(\mathbf{M})$ and $\sigma_{ess}(\mathbf{M})$ are subsets of the real axis and $\rho(\mathbf{M})$ is dense in $\mathbb{C}$. Therefore each real point t belongs to the Fredholm domain of $\mathbf{M}$ if and only if $t \in \rho(\mathbf{M}) \cup \sigma_{p,norm}(\mathbf{M})$. This proves Lemma 3.1. □

From $\rho(\mathbf{M}) \neq \emptyset$ it follows that Theorem 2.9 holds, in particular we have

$$\sigma_{++}(T) = \sigma_{++}(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_{--}(T) = \sigma_{--}(\mathbf{M}) \cap \rho(D).$$

We mention that if one replaces in (I) $L_1(I)$ by $L_\infty(I)$ then condition (β) in Lemma 2.10 is fulfilled.

More information about the spectrum of $\mathbf{M}$ is given by the following lemma. In the proof we shall rely on the proof of [LMeM, Lemma 1.3]. In the case $n_+ + n_- = 1$ Lemma 3.2 is contained in [LMeM, Lemma 1.3]. Let μ denote the Lebesgue measure on $\mathbb{R}$. Let f be a function defined on I and let $\lambda \in \mathbb{C}$. We set

$$f^{(-1)}(\lambda) := \{\xi \in I : f(\xi) = \lambda\}.$$

Lemma 3.2. *Assume that condition* (I) *is fulfilled and that* $q_j^{\pm}(t) \neq 0$, $1 \leq j \leq n_\pm$, *for almost all* $t \in I$. *Let* λ *be an isolated eigenvalue of* D. *Then the following holds.*

(1) *If, for all functions* v_1, v_2, $v_1 \neq v_2$, $v_1, v_2 \in \{u_j^+, u_k^- : 1 \leq j \leq n_+, 1 \leq k \leq n_-\}$ *we have* $\mu(v_1^{(-1)}(\lambda) \cap v_2^{(-1)}(\lambda)) = 0$, *then* λ *is no eigenvalue of* $\mathbf{M}$, $\lambda \notin \sigma_p(\mathbf{M})$.

(2) *If there exist at least two functions* v_1, v_2, $v_1 \neq v_2$, $v_1, v_2 \in \{u_j^+, u_k^- : 1 \leq j \leq n_+, 1 \leq k \leq n_-\}$ *such that* $\mu(v_1^{(-1)}(\lambda) \cap v_2^{(-1)}(\lambda)) > 0$ *holds, then* $\lambda \in \sigma_p(\mathbf{M})$ *and* λ *has infinite geometric multiplicity. Moreover every eigenvector* h *of* $\mathbf{M}$ *corresponding to* λ *has the form*

$$h = \left(0, h_1^+, \ldots, h_{n_+}^+, h_1^-, \ldots, h_{n_-}^-\right)^{\mathrm{T}} \tag{3.10}$$

where $h_j^+(t) = 0$, $1 \le j \le n_+$, *for almost all* $t \in I \setminus (u_j^+)^{(-1)}(\lambda)$ *and* $h_j^-(t) = 0$, $1 \le j \le n_-$, *for almost all* $t \in I \setminus (u_j^-)^{(-1)}(\lambda)$. *In particular, if* $\lambda \in \rho(u_j^-)$ *for all* $j = 1, \dots, n_-$, *then* h *is a positive vector and if* $\lambda \in \rho(u_j^+)$ *for all* $j = 1, \dots, n_+$, *then* h *is a negative vector in the Krein space* $\mathcal{G}$.

Proof. We set

$$\Omega(\lambda) := \bigcup_{j=1}^{n_+} (u_j^+)^{(-1)}(\lambda) \cup \bigcup_{j=1}^{n_-} (u_j^-)^{(-1)}(\lambda).$$

By assumption the point λ is an isolated eigenvalue of at least one u_j^+ or u_j^-, hence $\Omega(\lambda)$ has positive measure. Let V be the set of those $\xi \in (-1,1)$ for which there is an $\epsilon_\xi > 0$ such that

$$\mu(\Omega(\lambda) \cap (\xi - \epsilon_\xi, \xi + \epsilon_\xi)) = 0.$$

From Lindelöf's covering theorem we infer that $\mu(\Omega(\lambda) \cap V) = 0$. Thus, by changing the values of $u_j^\pm$, $j = 1, \dots, n_\pm$, on the set $\Omega(\lambda) \cap V$, we may assume that $\Omega(\lambda) \cap V = \emptyset$, which implies $\mu(\Omega(\lambda) \cap (\xi - \epsilon, \xi + \epsilon)) > 0$ for all $\xi \in \Omega(\lambda)$ and $\epsilon > 0$. Now we consider an element $h \in \mathcal{D}(\mathbf{M})$,

$$h = \left(f, g_1^+, \dots, g_{n_+}^+, g_1^-, \dots, g_{n_-}^-\right)^{\mathrm{T}},$$

such that $(\mathbf{M} - \lambda)h = 0$ or, equivalently, $(\widetilde{\mathbf{M}} - \lambda E)h = 0$. By the definition of A, B and D the latter relation holds if and only if

$$(3.11)\quad (\widetilde{A} - \lambda)f + \sum_{j=1}^{n_+} (q_j^+)^{\frac{1}{2}} g_j^+ - \sum_{j=1}^{n_-} |q_j^-|^{\frac{1}{2}} g_j^- = 0,$$

$$(3.12)\quad (q_j^+)^{\frac{1}{2}} f + (u_j^+ - \lambda) g_j^+ = 0 \quad \text{for} \quad 1 \le j \le n_+,$$

$$(3.13)\quad |q_j^-|^{\frac{1}{2}} f + (u_j^- - \lambda) g_j^- = 0 \quad \text{for} \quad 1 \le j \le n_-.$$

The multiplication operators in these relations are regarded as the corresponding extended operators considered above. For each j, $1 \le j \le n_\pm$, equations (3.12) and (3.13) imply $f(\xi) = 0$ a.e. on $\Omega(\lambda)$. Thus for every $\xi_0 \in \Omega(\lambda)$ and every $\epsilon > 0$ there are infinitely many points ξ in $\Omega(\lambda) \cap (\xi_0 - \epsilon, \xi_0 + \epsilon)$ for which $f(\xi) = 0$ holds. Equation (3.11), the fact that h belongs to $\mathcal{G}_{\frac{1}{2}} = \mathcal{H}_{\frac{1}{2}} \times \mathcal{K}_{\frac{1}{2}}$ and assumption (I) imply $f'' \in L_1(I)$, hence f' is continuous and $f(\xi_0) = f'(\xi_0) = 0$ holds. By the continuity of f and f' we obtain that $f(\xi) = 0$ and $f'(\xi) = 0$ for all $\xi \in \overline{\Omega(\lambda)}$.

We claim that $f \equiv 0$ on I. Indeed, let I_1 be a component of $(-1,1) \setminus \overline{\Omega(\lambda)}$. Since $\Omega(\lambda)$ has positive measure, at least one boundary point ξ_1 of I_1 belongs to $\overline{\Omega(\lambda)}$. Hence $f(\xi_1) = f'(\xi_1) = 0$, and on the interval I_1 the function f satisfies the differential equation

$$(\operatorname{sign} x)^\delta f'' + \lambda f + \left(\sum_{j=1}^{n_+} \frac{q_j^+}{u_j^+ - \lambda} + \sum_{j=1}^{n_-} \frac{q_j^-}{u_j^- - \lambda} \right) f = 0.$$

By the assumption that λ is an isolated eigenvalue of D, the coefficients of this equation belong to $L_1(I_1)$. Then the uniqueness theorem yields $f \equiv 0$ on I_1, hence $f \equiv 0$ on I. From equation (3.12) we deduce $g_j^+(t) = 0$, $1 \le j \le n_+$, for almost all $t \in I \setminus (u_j^+)^{(-1)}(\lambda)$ and from equation (3.13) we deduce $g_j^-(t) = 0$, $1 \le j \le n_-$, for almost all $t \in I \setminus (u_j^-)^{(-1)}(\lambda)$. This proves that each eigenvector h of $\mathbf{M}$ in λ has the form (3.10). If the assumption of (2) holds, it is easy to find an infinite system of linearly independent eigenvectors corresponding to λ of the form (3.10). The rest of assertion (2) is evident.

Let the assumption of (1) be fulfilled. Suppose that λ is an eigenvalue of $\mathbf{M}$. Then there exists a nonzero eigenvector of the form (3.10). The component h_1^+ of h is zero for almost all $t \in I \setminus (u_1^+)^{(-1)}(\lambda)$. If the component h_2^+ of h were not zero on $(u_1^+)^{(-1)}(\lambda)$ we would have $\mu((u_1^+)^{(-1)}(\lambda) \cap (u_2^+)^{(-1)}(\lambda)) \ne 0$, a contradiction. Hence all components of h different from h_1^+ are zero a.e. on $(u_1^+)^{(-1)}(\lambda)$. Then (3.11) gives $h_1^+(t) = 0$ for almost all $t \in I$. In a similar way we find that all components of h are zero, i.e. $h = 0$. This proves assertion (1). □

Now we impose, in addition, the following assumption.

(II) The set $\bigcup_{j=1}^{n_+} \sigma(u_j^+)$ is bounded from below, the set $\bigcup_{j=1}^{n_-} \sigma(u_j^-)$ is bounded from above and these two sets are disjoint.

Conditions (I) and (II) imply that the operators A, B and D fulfil conditions (ad) and (b) from Chapter 2, hence, by Theorem 2.12, the operators $\mathbf{L}$ and $\mathbf{M}$ are definitizable selfadjoint operators in $\mathcal{G}$ and the statements of Theorem 2.12 and Corollary 2.13 hold. Further, T^{-1} is a definitizable operator function (cf. Theorem 2.14). As mentioned above, we have $\infty \notin c_s(A)$. Since D is fundamentally reducible (cf. Section 1.7), $\infty \notin c_s(D)$ and, therefore, $\infty \notin c_s(\mathbf{L})$. Then Theorem 1.12 gives

$$\infty \notin c_s(\mathbf{M}).$$

Hence, also the third part of Theorem 2.14 holds.

By Theorem 2.20 the operator $\mathbf{M}$ is minimal in $\rho_e(D)$ with respect to $-T^{-1}$ and Theorem 2.22 holds. For the minimality of $\mathbf{M}$ with respect to $-T^{-1}$ (cf. Section 2.5) we obtain the following characterization.

Theorem 3.3. *Assume that the conditions* (I) *and* (II) *are fulfilled. The operator* $\mathbf{M}$ *is minimal with respect to* $-T^{-1}$ *if and only if the following two conditions hold.*

(1) *For* $1 \leq j \leq n_+$, $1 \leq k \leq n_-$ *and almost all* $t \in I$ *we have* $q_j^+(t) \neq 0$ *and* $q_k^-(t) \neq 0$.

(2) *For* $1 \leq j < k \leq n_+$ *we have* $u_j^+(t) \neq u_k^+(t)$ *and for* $1 \leq j < k \leq n_-$ *we have* $u_j^-(t) \neq u_k^-(t)$ *for almost all* $t \in I$.

Proof. 1. Let F be the spectral function of D and let $F(\,\cdot\,; u_j^+)$, $j = 1, \ldots, n_+$, $(F(\,\cdot\,; u_k^-)$, $k = 1, \ldots, n_-)$ be the spectral function of u_j^+ (resp. u_k^-). Assume that (1) and (2) hold. We first show that if for some $\mathbf{y} := (y_1^+, \ldots, y_{n_+}^+, y_1^-, \ldots, y_{n_-}^-)^{\mathrm{T}} \in \mathcal{K}$ we have $B^+F(\Delta)\mathbf{y} = 0$ for all bounded intervals Δ, then $\mathbf{y} = 0$. Observe that by condition (I) $\mathcal{R}(B^+)$ is contained in $L_1(I)$ regarded as a linear subspace of $\mathcal{H}_{-\frac{1}{2}}$.

Let M_n be the measurable set of all $x \in I$ such that for every pair $v_1, v_2 \in \{u_1^+, \ldots, u_{n_+}^+, u_1^-, \ldots, u_{n_-}^-\}$, $v_1 \neq v_2$, the relation $|v_1(x) - v_2(x)| \geq \frac{1}{n}$, $n \in \mathbb{N}$, holds. By assumption $\mu(M_n) \to \mu(I) = 2$ for $n \to \infty$. Let N_n be a measurable subset of I such that $\mu(I \setminus N_n) \leq \frac{1}{n}$ and all functions $u_1^+, \ldots, u_{n_+}^+, u_1^-, \ldots, u_{n_-}^-$ are continuous on N_n. Evidently we have

$$\mu(M_n \cap N_n) \to \mu(I) \quad \text{for } n \to \infty. \tag{3.14}$$

Let $x \in M_n \cap N_n$ and $\Delta_{x,n} := (u_1^+(x) - \frac{1}{2n}, u_1^+(x) + \frac{1}{2n})$. Then there exists an interval $\delta_{x,n} = (x - \eta_{x,n}, x + \eta_{x,n})$ such that

$$\{u_1^+(t) : t \in \delta_{x,n} \cap M_n \cap N_n\} \subset \Delta_{x,n}$$

and, if v is one of the functions $u_2^+, \ldots, u_{n_+}^+, u_1^-, \ldots, u_{n_-}^-$,

$$\{v(t) : t \in \delta_{x,n} \cap M_n \cap N_n\} \cap \Delta_{x,n} = \emptyset.$$

Then $F(\Delta_{x,n}; u_1^+)$ is the operator of multiplication by an indicator function which is equal to one in all points of $\delta_{x,n} \cap M_n \cap N_n$. $F(\Delta_{x,n}; v)$ (v as above) is the operator of multiplication by an indicator function which is equal to zero in all points of $\delta_{x,n} \cap M_n \cap N_n$. Then, since $F(\Delta)$ is a diagonal operator,

$$F(\Delta_{x,n}) = \operatorname{diag}\{F(\Delta_{x,n}; u_1^+), F(\Delta_{x,n}; u_2^+), \ldots, F(\Delta_{x,n}; u_{n_-}^-)\},$$

in view of $B^+F(\Delta_{x,n})\mathbf{y}=0$, we find $q_1^+(x)y_1^+(x)=0$ for all $x\in M_n\cap N_n$. By (3.14) we obtain $q_1^+(x)y_1^+(x)=0$ for almost all $x\in I$. By the assumptions on the functions $q_1^+,\dots,q_{n_+}^+,q_1^-,\dots,q_{n_-}^-$ we find $y_1^+(x)=0$ for almost all $x\in I$, and in a similar way $\mathbf{y}=0$.
2. Assume now that for some $\mathbf{y}\in\mathcal{K}$ we have $B^+(D-z)^{-1}\mathbf{y}=0$ for all non–real z. Let Δ be a bounded interval. As $F(\Delta)\mathbf{y}=(D-z_0)^{-1}F(\Delta)\mathbf{x}$ holds for some $\mathbf{x}\in\mathcal{K}$ and some $z_0\in\rho(D)$ it is easy to see that $F(\Delta)\mathbf{y}$ can be written as the limit in $\mathcal{K}_1$ of sums of the form $\sum_{i=1}^n\alpha_i(D-z_i)^{-1}\mathbf{y}$, $z_i\neq\overline{z}_i$, $\alpha_i\in\mathbb{C}$. We obtain $B^+F(\Delta)\mathbf{y}=0$ for all bounded intervals. By the first part of the proof it follows $\mathbf{y}=0$. Thus we have shown that if conditions (1) and (2) hold then $B^+(D-z)^{-1}\mathbf{y}=0$ for all non–real z implies $\mathbf{y}=0$. It follows from Lemma 2.19 that $\mathbf{M}$ is minimal with respect to $-T^{-1}$.
3. For the converse assume first that there exists a measurable set Δ, $\Delta\subset I$, $\mu(\Delta)>0$, and a j_0, $1\le j_0\le n_+$, such that $q_{j_0}^+(t)=0$ for all $t\in\Delta$. Denote by $y_{j_0}^+$ the function which equals 1 on Δ and 0 elsewhere. Then for $\mathbf{y}:=(0,\dots,0,y_{j_0}^+,0,\dots,0)^{\mathrm{T}}\in\mathcal{K}$, for all $x\in\mathcal{H}_{\frac12}$ and $z\in\rho(D)$, we have

$$[\mathbf{y},(\widetilde{D}-zE)^{-1}Bx]=0,$$

hence, by Lemma 2.19, $\mathbf{M}$ is not minimal with respect to $-T^{-1}$. Now assume that there exists a measurable set Δ, $\Delta\subset I$, $\mu(\Delta)>0$, such that $u_1^+(t)=u_2^+(t)$ for all $t\in\Delta$. Then there is a measurable set Δ_0, $\Delta_0\subset\Delta$, $\mu(\Delta_0)>0$ such that q_1^+ and q_2^+ are bounded functions on Δ_0. Let g be the indicator function of Δ_0. Then for $\mathbf{y}:=((q_2^+)^{\frac12}g,-(q_1^+)^{\frac12}g,0,\dots,0)^{\mathrm{T}}\in\mathcal{K}$, for all $x\in\mathcal{H}$ and $z\in\rho(D)$, we have

$$[\mathbf{y},(\widetilde{D}-zE)^{-1}Bx]=0,$$

hence, by Lemma 2.19, $\mathbf{M}$ is not minimal with respect to $-T^{-1}$. Similarly, for u_1^+,u_2^+ replaced by any other pair of the functions $u_1^+,\dots,u_{n_+}^+$ or of the functions $u_1^-,\dots,u_{n_-}^-$. □

Let the conditions (I) and (II) be fulfilled and assume, moreover, that (1) and (2) in Theorem 3.3 hold. Then, by Lemma 3.2, an isolated eigenvalue of D does not belong to $\sigma_p(\mathbf{M})$. Observe that under the same assumptions the operator $\mathbf{M}$ is unitarily equivalent to the operator M_Σ defined in Theorem 2.17 (with respect to the Krein space inner products) and Proposition 2.18 holds.

Next, we consider the case of simple functions $u_j^\pm$ and assume that the following holds.

(III) The spectrum of $u_j^{\pm}$, $1 \le j \le n_{\pm}$, has no finite accumulation points, and $q_j^{\pm}(x) \neq 0$ for almost all $x \in I$, $1 \le j \le n_{\pm}$.

If, e.g. for each $u_j^{\pm}$, $1 \le j \le n_{\pm}$, there exists a finite decomposition of I into measurable sets such that $u_j^{\pm}$ is constant a.e. on each of these sets, then condition (III) is fulfilled.

Then with the help of Lemma 3.2, which holds now for every $\lambda \in \sigma(D)$, we get the following proposition. For the notion of Riesz basis, see [GK].

Proposition 3.4. *Assume that conditions* (I)-(III) *are fulfilled. Then there exists a Riesz basis of $\mathcal{G}$ consisting of eigenvectors and associated vectors of* $\mathbf{M}$.

Proof. By Lemma 3.1 $\sigma(\mathbf{M})$ is discrete with the possible exception of the points of $\sigma(D)$. Let $\lambda \in \sigma(D) = \sigma_{ess}(\mathbf{M})$. Assume that $\lambda \in \cup_{j=1}^{n_+}\sigma(u_j^+)$. Then, in view of condition (II), $\lambda \notin \cup_{j=1}^{n_-}\sigma(u_j^-)$, and λ is contained in some open interval Δ of type π_+ with respect to $\mathbf{L}$. Then by Theorem 2.12 Δ is of type π_+ with respect to $\mathbf{M}$. Hence, if λ is no eigenvalue of $\mathbf{M}$, then $\lambda \in \sigma_{++}(\mathbf{M})$. If λ is an eigenvalue of $\mathbf{M}$, then, by Lemma 3.2, all eigenvectors of $\mathbf{M}$ corresponding to λ are positive. It is a well known fact that this implies $\lambda \in \sigma_{++}(\mathbf{M})$. A similar reasoning applies for $\cup_{j=1}^{n_+}\sigma(u_j^+)$ and $\cup_{j=1}^{n_-}\sigma(u_j^-)$ interchanged. Therefore, all finite accumulation points of $\sigma(\mathbf{M})$ are no critical points. Moreover, we have $\infty \notin c_s(\mathbf{M})$. Then it is easy to find a Riesz basis of $\mathcal{G}$ with the required properties. □

3.2. The Case of the Semiaxis

In this section we will consider the equation

$$y''(x) + \lambda y(x) + \sum_{j=1}^{n_+} \frac{q_j^+(x)}{u_j^+(x) - \lambda} y(x) + \sum_{j=1}^{n_-} \frac{q_j^-(x)}{u_j^-(x) - \lambda} y(x) = 0 \tag{3.15}$$

on the interval $I := [0, \infty)$ with boundary condition

$$y(0) = 0. \tag{3.16}$$

Again, the functions $q_j^{\pm}$, $u_j^{\pm}$ are real valued measurable functions, $q_j^+(x) \ge 0$, $j = 1, \dots, n_+$, $q_j^-(x) \le 0$, $j = 1, \dots, n_-$, for almost all $x \in I$ and satisfy the condition (I) from Section 3.1. As above, we do not exclude that one of the sums in (3.15) is missing. In this case we set $n_+ = 0$ or $n_- = 0$, respectively. It is always assumed that $n_+ + n_- \geq 1$.

Now we write the left hand side of (3.15) with the help of an operator function of the form (2.11). Let $\mathcal{H} = L_2(I)$ and let A be the usual selfadjoint operator associated with $-\frac{d^2}{dx^2}$ and the boundary condition (3.16). As in Section 3.1 we introduce the spaces $\mathcal{K}$ and $\mathcal{G}$ (see (3.4) and (3.7)), the indefinite inner product $[\cdot,\cdot]_{\mathcal{K}}$ (cf. (3.6)) and the operators D and $\mathbf{B}$ (see (3.5) and (3.8)). Again, D is a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot]_{\mathcal{K}})$. In this case $\mathcal{H}_{\frac{1}{2}}$ coincides (up to equivalent scalar products) with the Sobolev space $H_0^1(I)$ of all locally absolutely continuous functions $f \in L_2(I)$ with $f(0) = 0$ such that $f' \in L_2(I)$.

In the following lemma we use, as before, the same symbol h for a function $h \in L_2(I)$ and the operator of multiplication by h in $L_2(I)$.

Lemma 3.5. *Let $h \in L_2(I)$. Then the operator $h(1 + A^2)^{-\frac{1}{4}}$ is Hilbert–Schmidt. In particular, $B \in \mathfrak{S}_2(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}})$.*

Proof. Denote by $\mathcal{F}$ the Fourier transform acting from $L_2(\mathbb{R})$ onto $L_2(\mathbb{R})$. Further, we denote by I_0 the embedding operator acting from $L_2(\mathbb{R})$ into $L_2(I)$,

$$I_0 f := f|I, \quad f \in L_2(\mathbb{R}).$$

Set $g(\xi) := (1 + |\xi|)^{-1}$. Then for $f \in L_2(\mathbb{R})$ and $x \in I$ we have

$$(hI_0\mathcal{F}^{-1})(gf)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} k(x,\xi) f(\xi) d\xi,$$

where $k(x,\xi) = h(x)e^{ix\xi}(1 + |\xi|)^{-1}$. The function k belongs to $L_2(I \times \mathbb{R})$. Therefore the operator $hI_0\mathcal{F}^{-1}g$ is a Hilbert–Schmidt operator acting from $L_2(\mathbb{R})$ into $L_2(I)$.

The operator $(1 + A^2)^{-\frac{1}{4}}$ is a bounded operator acting from $L_2(I)$ onto $\mathcal{H}_{\frac{1}{2}}$. Then $I_0^*(1 + A^2)^{-\frac{1}{4}}$ maps $L_2(I)$ continuously into $H^1(\mathbb{R})$, i.e. the space of all locally absolutely continuous functions $f \in L_2(\mathbb{R})$ with $f' \in L_2(\mathbb{R})$. Hence, the operator $g^{-1}\mathcal{F}I_0^*(1 + A^2)^{-\frac{1}{4}}$ is a bounded operator from $L_2(I)$ into $L_2(\mathbb{R})$ and, finally, it follows that

$$h(1 + A^2)^{-\frac{1}{4}} = hI_0\mathcal{F}^{-1}gg^{-1}\mathcal{F}I_0^*(1 + A^2)^{-\frac{1}{4}}$$

is a Hilbert–Schmidt operator.

To prove the second assertion it is sufficient to prove that the operator $(1+D^2)^{-\frac{1}{4}}B(1+A^2)^{-\frac{1}{4}}$ belongs to $\mathfrak{S}_2(\mathcal{H},\mathcal{K})$. This is equivalent to

$$\left(\frac{(q_j^{\pm})^2}{1+(u_j^{\pm})^2}\right)^{\frac{1}{4}}(1+A^2)^{-\frac{1}{4}} \in \mathfrak{S}_2 \quad \text{for } 1 \le j \le n_{\pm}.$$

Condition (I) implies $\left(\frac{(q_j^{\pm})^2}{1+(u_j^{\pm})^2}\right)^{\frac{1}{4}} \in L_2(I)$, $1 \le j \le n_{\pm}$, hence, by the first part of Lemma 3.5, $B \in \mathfrak{S}_2(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}})$. □

As a consequence of Lemma 3.5 the operator $\mathbf{B}$ belongs to $\mathfrak{S}_2(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}})$, that is, condition (b) of Section 2.3 is fulfilled.

We define $T, \widetilde{T}, \mathbf{M}, \widetilde{\mathbf{M}}$ as in Section 2.3. Then $T(\lambda)y$ coincides with the left hand side of (3.15) defined for all $y \in H_0^1(I)$ for which it belongs to $L_2(I)$. Again Lemmas 2.1, 2.3, Propositions 2.5, 2.7, 2.8 and Remark 2.6 hold, in particular we have

$$\rho(T) = \rho(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_p(T) = \sigma_p(\mathbf{M}) \cap \rho(D).$$

Since $0 \in \rho(T(i\eta))$ for sufficiently large $\eta > 0$, we have $\rho(\mathbf{M}) \neq \emptyset$ and Theorem 2.9 holds.

Assume, in addition, that the following condition (II′) is fulfilled.

(II′) The set $\bigcup_{j=1}^{n_+} \sigma(u_j^+)$ is bounded from below, the set $\bigcup_{j=1}^{n_-} \sigma(u_j^-)$ is a subset of $(-\infty, 0)$ and these two sets are disjoint.

Conditions (I) and (II′) imply that the operators A and D satisfy condition (ad) from Section 2.4, hence $\mathbf{M}$ is a definitizable selfadjoint operator in $\mathcal{G}$ and the statements of Theorem 2.12 and Corollary 2.13 hold. Therefore, as $\infty \notin c_s(A) \cup c_s(D)$, we have $\infty \notin c_s(\mathbf{M})$ and $\sigma_{ess}(\mathbf{M}) = \sigma_{ess}(A) \cup \sigma_{ess}(D) = [0,\infty) \cup \sigma(D)$ (see (1.6)). Further, T^{-1} is a definitizable operator function and Theorem 2.14 holds. By Theorem 2.20 the operator $\mathbf{M}$ is minimal in $\rho_e(D)$ with respect to $-T^{-1}$ and Theorem 2.22 holds. Lemma 3.2 and Theorem 3.3 remain valid for $I = [0,\infty)$, without difficulty the proofs can be extended to this case.

In the following theorem we recall some results on the absence of positive (nonnegative) eigenvalues for Sturm-Liouville operators (e.g. [Kn1] and [Kn2]). These results and the relation between the spectra of T and $\mathbf{M}$ can be used to exclude critical points of $\mathbf{M}$ on the positive (resp. nonnegative) half-axis. For the convenience of the reader we will give a proof of the first part of the following theorem although it is a special case of [Kn1, Theorem 3.2].

Theorem 3.6. *Let* V *be a real valued function belonging to* $L_1(I)$ *and let* $\mathcal{D}$ *denote the set of all functions* f *in* $H_0^1(I)$ *such that* $-f'' + Vf$ *belongs to* $L_2(I)$*. Then, for* $\lambda > 0$*, the equation*

$$-y'' + (V - \lambda)y = 0 \tag{3.17}$$

has no nontrivial solution $y \in \mathcal{D}$*.*

Assume, in addition, that there exists $x_0 > 0$ *such that* $|V(x)| \le \frac{3}{4x^2}$ *for all* $x \in (x_0, \infty)$*. Then the equation*

$$-y'' + Vy = 0$$

has no nontrivial solution $y \in \mathcal{D}$*.*

Proof. Assume that there exists a nontrivial solution $y \in \mathcal{D}$ of (3.17). Then $\overline{y}$ is another solution of (3.17). Therefore, we assume y to be a real valued solution of (3.17). We define

$$E(x) := (y'(x))^2 + \lambda y(x)^2, \quad x \in I.$$

Then $E \in L_1(I)$ and the uniqueness theorem implies $E(x) > 0$ for $x \in I$. It follows that

$$\left|\frac{E'(x)}{E(x)}\right| \le \frac{|V(x)|}{\sqrt{\lambda}} \quad \text{for almost all } x \in I.$$

Hence, by integrating from 0 to some $t > 0$, we conclude

$$E(t) \ge E(0)e^{-\lambda^{-\frac{1}{2}}\|V\|_{L_1(I)}} > 0.$$

This is a contradiction to $E \in L_1(I)$. The second assertion is a consequence of [Kn2, Corollary 3.3]. □

Proposition 3.7. *Assume that the conditions* (I), (II′), (III) *and condition* (2) *from Theorem* 3.3 *are satisfied. Then*

$$\sigma_p(\mathbf{M}) \cap ((0,\infty) \cup \sigma(D)) = \emptyset$$

and

$$c(\mathbf{M}) \subset ((-\infty, 0) \cap \sigma_{p,norm}(\mathbf{M})) \cup \{0, \infty\}.$$

Assume, in addition, that one of the following conditions is satisfied.

(i) *The point* 0 *belongs to* $\sigma(D)$.

(ii) $0 \in \rho(D)$ *and there exist an* $x_0 > 0$ *such that*

$$\sum_{j=1}^{n_+} \left| \frac{q_j^+(x)}{u_j^+(x)} \right| + \sum_{j=1}^{n_-} \left| \frac{q_j^-(x)}{u_j^-(x)} \right| \leq \frac{3}{4x^2} \quad \textit{for all } x \in (x_0, \infty).$$

Then

$$\sigma_p(\mathbf{M}) \cap ([0,\infty) \cup \sigma(D)) = \emptyset, \quad c(\mathbf{M}) \subset ((-\infty, 0) \cap \sigma_{p,norm}(\mathbf{M})) \cup \{\infty\},$$

and $\mathbf{M}$ *is a spectral operator in the sense of Dunford.*

Proof. By Lemma 3.2 (1) we have $\sigma_p(\mathbf{M}) \cap \sigma(D) = \emptyset$. Let $\lambda_0 \in (0,\infty) \setminus \sigma(D)$. Then $\lambda_0 - T(\lambda_0)$ is the operator defined by the differential expression

$$-\frac{d^2}{dx^2} + \sum_{j=1}^{n_+} \frac{q_j^+(x)}{\lambda_0 - u_j^+(x)} + \sum_{j=1}^{n_-} \frac{q_j^-(x)}{\lambda_0 - u_j^-(x)} \tag{3.18}$$

restricted to all those functions from $H_0^1(I)$ which are mapped by (3.18) into $L_2(I)$. Let $q := q_j^+$, $u := u_j^+$ for some $j \in \{1, \dots n_+\}$ (resp. $q := q_j^-$, $u := u_j^-$ for some $j \in \{1, \dots n_-\}$). Then we have $d := \operatorname{ess\,inf}(|\lambda_0 - u|) > 0$. For $x \in I$ it follows

$$\begin{aligned} \left| \frac{q(x)}{\lambda_0 - u(x)} \right| &\leq \frac{1 + \lambda_0 + |u(x)|}{|\lambda_0 - u(x)|} \frac{|q(x)|}{1 + \lambda_0 + |u(x)|} \leq \\ &\leq (1 + 3\lambda_0) \max\{d^{-1}, \lambda_0^{-1}\} \frac{|q(x)|}{1 + \lambda_0 + |u(x)|}, \end{aligned} \tag{3.19}$$

i.e. $\frac{q}{\lambda_0 - u} \in L_1(I)$. By Theorem 3.6 $\lambda_0 - T(\lambda_0)$ has no positive eigenvalue. Hence $\lambda_0 \notin \sigma_p(\lambda_0 - T(\lambda_0))$, that is $\lambda_0 \notin \sigma_p(T)$ and, by Proposition 2.8, $\lambda_0 \notin \sigma_p(\mathbf{M})$. It follows that $\sigma_p(\mathbf{M}) \cap (0,\infty) = \emptyset$. As $\mathbf{M}$ has no finite essential critical points (cf. Theorem 2.12), each finite critical point t_0 of $\mathbf{M}$ belongs to $\sigma_p(\mathbf{M})$; hence, if $t_0 \neq 0$, we have $t_0 \in (-\infty, 0) \setminus \sigma(D)$. Moreover, by the same reasoning as in the proof of Lemma 3.1, we have $\sigma_{ess}(\mathbf{M}) = \sigma(\mathbf{M}) \setminus \sigma_{p,norm}(\mathbf{M})$ and, by relation (1.6), it follows $t_0 \in (-\infty, 0) \setminus \sigma_{ess}(\mathbf{M}) = (-\infty, 0) \cap \sigma_{p,norm}(\mathbf{M})$.

The additional assumption of the second part of Proposition 3.7 implies that $\lambda_0 - T(\lambda_0)$ has no eigenvalue in $[0,\infty)$ (see Theorem 3.6). Then the second part of Proposition 3.7 follows as above. □

3.2. A Case where the Numerator Coefficient of the Floating Singularity Changes Sign

In this section we consider an eigenvalue problem of the form

$$y''(x) + \lambda y(x) + \frac{q(x)}{u(x) - \lambda} y(x) = 0 \tag{3.20}$$

on the interval $I := [-1, 1]$ with boundary conditions

$$y(-1) = y(1) = 0. \tag{3.21}$$

We now assume that the function q is a real valued piecewise continuous function on the interval I, i.e. there exist finitely many closed intervals $[a_j, b_j]$, $j = 1, \dots, N$, $N > 0$, such that $\cup_{j=1}^{N} [a_j, b_j] = I$, q is continuous on each open interval (a_j, b_j), $j = 1, \dots, N$, and the one–sided limits $\lim_{x \downarrow a_j} q(x)$ and $\lim_{x \uparrow b_j} q(x)$, $j = 1, \dots, N$, exist. We assume that the function u is real valued and measurable. Then the functions q and u satisfy condition (I) from Section 3.1.

We set

$$\Delta_+ := \{x \in I : q(x) > 0\}, \quad \Delta_- := \{x \in I : q(x) < 0\} \tag{3.22}$$

and

$$\mathcal{K} := L_2(\Delta_+) \times L_2(\Delta_-).$$

Moreover, we assume that $\mu(\Delta_+) > 0$ and $\mu(\Delta_-) > 0$ hold. The case $\mu(\Delta_+) = 0$ (resp. $\mu(\Delta_-) = 0$) is contained in the considerations of Section 3.1.

We will write the left hand side of (3.20) with the help of an operator function of the form (2.11). Let $\mathcal{H} = L_2(I)$ and let A be the operator defined by (3.3). In the following we consider the embedding operators I_+ and I_- acting from $\mathcal{H}$ into $L_2(\Delta_+)$ and $L_2(\Delta_-)$, respectively,

$$I_+ f := f|\Delta_+, \quad I_- f := f|\Delta_-, \quad f \in \mathcal{H}.$$

We introduce the following abbreviations

$$\begin{aligned} u_+ &:= u|\Delta_+, \quad u_- := u|\Delta_-, \\ q_+ &:= q|\Delta_+, \quad q_- := q|\Delta_-. \end{aligned}$$

Then the operator D

$$\begin{bmatrix} u_+ & 0 \\ 0 & u_- \end{bmatrix}$$

defined on the set $\mathcal{D}(D) := \mathcal{D}(u_+) \times \mathcal{D}(u_-)$ is selfadjoint in $\mathcal{K}$. We consider on $\mathcal{K}$ the indefinite inner product $[\cdot,\cdot]_\mathcal{K}$ defined by

$$\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right]_\mathcal{K} := (f_1, g_1)_{L_2(\Delta_+)} - (f_2, g_2)_{L_2(\Delta_-)},$$

where $f_1, g_1 \in L_2(\Delta_+)$ and $f_2, g_2 \in L_2(\Delta_-)$. Then $(\mathcal{K}, [\cdot,\cdot]_\mathcal{K})$ is a Krein space and $J_\mathcal{K} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the corresponding fundamental symmetry. D is a selfadjoint operator in $(\mathcal{K}, [\cdot,\cdot]_\mathcal{K})$.

The operator B from $\mathcal{H}$ into $\mathcal{K}$, defined by

$$B := \begin{bmatrix} q_+^{\frac{1}{2}} I_+ \\ |q_-|^{\frac{1}{2}} I_- \end{bmatrix},$$

is a bounded operator. Then condition (b) from Section 2.3 holds. Again, we denote by $\mathcal{H}_s$ and $\mathcal{K}_s$ the scales $\mathcal{H}_s(A)$ and $\mathcal{K}_s(D, J_\mathcal{K})$ (cf. Sections 1.1 and 1.2).

We define the Krein space $\mathcal{G}$, the operator $\mathbf{L}$, the scales $\mathcal{G}_s$ and the operator $\mathbf{B}$ as in Section 3.1 (cf. (3.7) and (3.8)).

We define $T, \widetilde{T}, \mathbf{M}, \widetilde{\mathbf{M}}$ as in Section 2.3. Then, by the boundedness of the operator B, for $\lambda \in \rho(D)$ it follows

$$\mathcal{D}(T(\lambda)) = \mathcal{D}(A) \text{ and } \mathcal{D}(\mathbf{M}) = \mathcal{D}(\mathbf{L}) = \mathcal{D}(A) \times \mathcal{D}(D).$$

The expression on the left hand side of (3.20), defined for all $y \in \mathcal{D}(A)$, coincides with $T(\lambda)y$. Then Lemmas 2.1, 2.3 and Propositions 2.5, 2.7, 2.8 and Remark 2.6 hold, in particular we have

$$\rho(T) = \rho(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_p(T) = \sigma_p(\mathbf{M}) \cap \rho(D).$$

By Proposition 2.8, for $\lambda \in \rho(D)$ a function $y \in \mathcal{D}(A)$ is a solution of equation (3.20) if and only if $\mathbf{y} := (y, y_+, y_-)^\mathrm{T}$, for some $y_+ \in \mathcal{D}(u_+), y_- \in \mathcal{D}(u_-)$ is a solution of

$$(\mathbf{M} - \lambda)\mathbf{y} = \begin{bmatrix} -\frac{d^2}{dx^2} - \lambda & I_+^* q_+^{\frac{1}{2}} & -I_-^* |q_-|^{\frac{1}{2}} \\ q_+^{\frac{1}{2}} I_+ & u_+ - \lambda & 0 \\ |q_-|^{\frac{1}{2}} I_- & 0 & u_- - \lambda \end{bmatrix} \begin{bmatrix} y \\ y_+ \\ y_- \end{bmatrix} = 0.$$

Using the same reasoning as in the proof of Lemma 3.1 it is easily seen that $\rho(\mathbf{M}) \neq \emptyset$ and

$$\sigma_{ess}(\mathbf{M}) = \sigma(\mathbf{M}) \setminus \sigma_{p,norm}(\mathbf{M}) = \sigma(D).$$

hold. It follows that Theorem 2.9 holds, that is, in particular,

$$\sigma_{++}(T) = \sigma_{++}(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_{--}(T) = \sigma_{--}(\mathbf{M}) \cap \rho(D).$$

Theorem 3.8. *The operator* $\mathbf{M}$ *is definitizable over*

$$\overline{\mathbb{C}} \setminus ((\{\infty\} \cup \sigma_e(u_+)) \cap \sigma_e(u_-)) .$$

Let Δ *be an open connected subset of* $\overline{\mathbb{R}} \setminus ((\{\infty\} \cup \sigma_e(u_+)) \cap \sigma_e(u_-))$. *Then the following assertions are equivalent.*

(i) Δ *is of type* π_+ (*type* π_-) *with respect to* $\mathbf{M}$.

(ii) Δ *is of type* π_+ (*resp. type* π_-) *with respect to* $\mathbf{L}$.

(iii) $\Delta \subset \overline{\mathbb{R}} \setminus \sigma_e(u_-)$ (*resp.* $\Delta \subset \mathbb{R} \setminus \sigma(u_+)$).

Proof. In $\mathcal{G} = L_2(I) \times L_2(\Delta_+) \times L_2(\Delta_-)$ we consider the operator

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then J is a fundamental symmetry in the Krein space $\mathcal{G}$ which commutes with the resolvent of $\mathbf{L}$, i.e. $\mathbf{L}$ is fundamentally reducible (cf. Section 1.7). Set $P_\pm := \frac{1}{2}(I \pm J)$ and denote by $\mathbf{L}_\pm$ the operator in $P_\pm\mathcal{G}$ defined by $\mathbf{L}_\pm\mathbf{x} := \mathbf{L}\mathbf{x}$ for $\mathbf{x} \in P_\pm\mathcal{D}(\mathbf{L})$. Then we have

$$\sigma_{e,ess}(\mathbf{L}_+) = \{\infty\} \cup \sigma_e(u_+) \text{ and } \sigma_{e,ess}(\mathbf{L}_-) = \sigma_e(u_-).$$

By Lemma 1.14, $\mathbf{L}$ is definitizable over $\overline{\mathbb{C}} \setminus ((\{\infty\} \cup \sigma_e(u_+)) \cap \sigma_e(u_-))$ and assertions (ii) and (iii) are equivalent.

For $\lambda \in \rho(\mathbf{L}) \cap \rho(\mathbf{M})$ we have $(A - \lambda)^{-1} \in \mathfrak{S}_1$, where $\mathfrak{S}_1$ denotes the trace class (cf. Section 1.7). By Proposition 2.7, λ belongs to the resolvent set of T and it follows

$$T(\lambda)^{-1} = (-I + (A - \lambda)^{-1}B^+(D - \lambda)^{-1}B)^{-1}(A - \lambda)^{-1} \in \mathfrak{S}_1.$$

Relation (2.15) implies

$$(\mathbf{L} - \lambda)^{-1} - (\mathbf{M} - \lambda)^{-1} \in \mathfrak{S}_1.$$

Then, as a consequence of Theorem 1.15, (i) and (ii) are equivalent. □

The following lemma can be proved by a reasoning similar to the proof of Lemma 3.2.

Lemma 3.9. *Assume that $q(t) \neq 0$ for almost all $t \in I$. Let λ be an isolated eigenvalue of D. Then the following holds.*

(1) *If $\mu(u_+^{(-1)}(\lambda) \cap u_-^{(-1)}(\lambda)) = 0$, then $\lambda \notin \sigma_p(\mathbf{M})$.*

(2) *If $\mu(u_+^{(-1)}(\lambda) \cap u_-^{(-1)}(\lambda)) > 0$, then $\lambda \in \sigma_p(\mathbf{M})$ and λ has infinite geometric multiplicity.*

Assume, in addition, that the following condition (A) is fulfilled.

(A) The functions q and u belong to $C^1(I)$, $u'(x) > 0$ for all $x \in I$ and the function q has at most finitely many zeros in the interval I.

From condition (A) it follows that the set $\sigma(u_+) \cap \sigma(u_-)$ consists of at most finitely many points, i.e. $\sigma(u_+) \cap \sigma(u_-) = \{u(\xi_1), \dots, u(\xi_n)\}$ for some $\xi_1, \dots, \xi_n \in I$, $n \in \mathbb{N}$, and each ξ_j, $j = 1, \dots, n$, is a zero of the function q. We mention that $u(\xi)$, $\xi \in I$, belongs to $\sigma(u_+) \cap \sigma(u_-)$ if and only if $\xi \in (-1, 1)$, $q(\xi) = 0$ and q changes its sign in ξ. Then by Theorem 3.8, $\mathbf{M}$ is definitizable over $\overline{\mathbb{C}} \setminus \{u(\xi_1), \dots, u(\xi_n)\}$. This will be improved by the following theorem.

Theorem 3.10. *Assume that condition* (A) *is fulfilled. Then the operator $\mathbf{M}$ is definitizable and we have $c_\infty(\mathbf{M}) = \{u(\xi_1), \dots, u(\xi_n)\}$.*

Proof. Let $\xi \in \{\xi_1, \dots, \xi_n\}$. We will show that $\mathbf{M}$ is definitizable over an open neighbourhood of ξ and that $u(\xi) \in c_\infty(\mathbf{M})$.

1. Condition (A) implies

$$0 < M := \max_{x \in I} \left| \frac{q(x)}{u(x) - u(\xi)} \right| \leq \max_{x \in I} \left| \frac{q'(x)}{u'(x)} \right| < \infty.$$

Denote by P the orthogonal projection in $\mathcal{H}$ onto the linear span of the eigenvectors of A corresponding to the eigenvalues λ with $\lambda < M + u(1)$. We define for $\lambda \in \rho(D)$

$$A_0 := (I - P)A + (M + u(1))P, \quad T_0(\lambda) := \lambda - A_0 + B^+(D - \lambda)^{-1}B,$$

$$\mathbf{L}_0 := \begin{bmatrix} A_0 & 0 \\ 0 & D \end{bmatrix} \text{ and } \mathbf{M}_0 := \begin{bmatrix} A_0 & B^+ \\ B & D \end{bmatrix} \tag{3.23}$$

with $\mathcal{D}(A_0) = \mathcal{D}(T_0(\lambda)) = \mathcal{D}(A)$ and $\mathcal{D}(\mathbf{L}_0) = \mathcal{D}(\mathbf{M}_0) = \mathcal{D}(A) \times \mathcal{K}$. Then $A_0 \geq M + u(1)$ and $A - A_0$ is of finite rank. Moreover, $u(\xi) \notin \sigma_p(\mathbf{L}_0) \cup \sigma_p(D)$ and the operators $(\mathbf{L}_0 - u(\xi))^{-1}$ and $(D - u(\xi))^{-1}$ are selfadjoint operators in

$\mathcal{G}$ and $\mathcal{K}$, respectively. We denote by $\mathcal{K}'_s$ and $\mathcal{G}'_s$ the scales $\mathcal{K}_s((D-u(\xi))^{-1})$ and $\mathcal{G}_s((\mathbf{L}_0-u(\xi))^{-1})$, respectively (see Section 1.1). The real number $u(\xi)$ belongs to $\rho(A_0)$, therefore we have $\mathcal{G}'_s = \mathcal{H}\times\mathcal{K}'_s$.

The operator $(D-u(\xi))^{-1}$ $((\mathbf{L}_0-u(\xi))^{-1})$ can be extended to a continuous linear operator $((D-u(\xi))^{\sim})^{-1}$ (resp. $((\mathbf{L}_0-u(\xi))^{\sim})^{-1}$) acting from $\mathcal{K}'_{\frac{1}{2}}$ into $\mathcal{K}'_{-\frac{1}{2}}$ (resp. from $\mathcal{G}'_{\frac{1}{2}}$ into $\mathcal{G}'_{-\frac{1}{2}}$). Then, by condition (A), the operator B is a bounded operator from $\mathcal{H}$ into $\mathcal{K}'_{\frac{1}{2}}$ and the operator $B^+((D-u(\xi))^{\sim})^{-1}B$ is a bounded operator in $\mathcal{H}$, $\|B^+((D-u(\xi))^{\sim})^{-1}B\|\le M$, therefore we can define

$$T_0(u(\xi)) := u(\xi)-A_0+B^+((D-u(\xi))^{\sim})^{-1}B$$

with $\mathcal{D}(T_0(u(\xi)))=\mathcal{D}(A)$. It follows from $A_0\ge M+u(1)$ that $T_0(u(\xi)) << 0$, hence $T_0(u(\xi))$ is boundedly invertible.

2. As a consequence of

$$T_0(u(\xi))^{-1} = -(I-(A_0-u(\xi))^{-1}B^+((D-u(\xi))^{\sim})^{-1}B)^{-1}(A_0-u(\xi))^{-1}\in\mathfrak{S}_1,$$

it follows that the operator $V :=$

$$\begin{bmatrix} -T_0(u(\xi))^{-1}-(A_0-u(\xi))^{-1} & T_0(u(\xi))^{-1}B^+\widetilde{R}(u(\xi),D) \\ \widetilde{R}(u(\xi),D)BT_0(u(\xi))^{-1} & -\widetilde{R}(u(\xi),D)BT_0(u(\xi))^{-1}B^+\widetilde{R}(u(\xi),D) \end{bmatrix},$$

where $\widetilde{R}(u(\xi),D)=((D-u(\xi))^{\sim})^{-1}$, belongs to $\mathfrak{S}_1(\mathcal{G}'_{\frac{1}{2}},\mathcal{G}'_{-\frac{1}{2}})$.

3. We claim that $u(\xi)\notin\sigma_p(\mathbf{M}_0)$. Assume that there exists $\mathbf{y}\in\mathcal{D}(\mathbf{M}_0)$, $\mathbf{y}\ne 0$, $\mathbf{y}=(y,y_+,y_-)^{\mathrm{T}}$ with $y\in\mathcal{D}(A)$, $y_+\in L_2(\Delta_+)$ and $y_-\in L_2(\Delta_-)$ such that $(\mathbf{M}_0-u(\xi))\mathbf{y}=0$ holds, i.e.

$$(A_0-u(\xi))y+I_+^*q_+^{\frac{1}{2}}y_+-I_-^*|q_-|^{\frac{1}{2}}y_- = 0, \tag{3.24}$$

$$q_+^{\frac{1}{2}}I_+y+(u_+-u(\xi))y_+ = 0, \tag{3.25}$$

$$|q_-|^{\frac{1}{2}}I_-y+(u_--u(\xi))y_- = 0. \tag{3.26}$$

Then by condition (A) it follows $y\ne 0$. Moreover, by equation (3.25), we have $\frac{q_+}{u_+-u(\xi)}I_+y=-q_+^{\frac{1}{2}}y_+$ and, by equation (3.26), $\frac{|q_-|}{u_--u(\xi)}I_-y=-|q_-|^{\frac{1}{2}}y_-$. Then equation (3.24) implies $T_0(u(\xi))y=0$, which contradicts the invertibility of $T_0(u(\xi))$.

4. The operator $(\mathbf{M}_0-u(\xi))^{-1}$ is a selfadjoint operator in $\mathcal{G}$ and we have $\mathcal{D}((\mathbf{M}_0-u(\xi))^{-1})=\mathcal{R}((\mathbf{M}_0-u(\xi)))\subset\mathcal{G}'_{\frac{1}{2}}$. Let $\mathbf{y}=(\mathbf{M}_0-u(\xi))\mathbf{x}$ for some $\mathbf{x}\in\mathcal{D}(\mathbf{M}_0)$. A straightforward calculation gives

$$(((\mathbf{L}_0-u(\xi))^{\sim})^{-1}+V)\mathbf{y}=\mathbf{x},$$

that is,

$$(\mathbf{M}_0 - u(\xi))^{-1} \subset (\mathbf{L}_0 - u(\xi))^{-1} \dot{+} V.$$

The operators $(\mathbf{M}_0 - u(\xi))^{-1}$ and $(\mathbf{L}_0 - u(\xi))^{-1} \dot{+} V$ are selfadjoint (cf. Lemma 1.6), hence, we have

$$(\mathbf{M}_0 - u(\xi))^{-1} = (\mathbf{L}_0 - u(\xi))^{-1} \dot{+} V.$$

Condition (A) and Lemma 1.14 imply that the operator $(\mathbf{L}_0 - u(\xi))^{-1}$ is definitizable. As a consequence of [J3, Theorem 3.10] there exist real points t_1, t_2, $t_1 < t_2$, such that $(\mathbf{M}_0 - u(\xi))^{-1}$ is definitizable over a neighbourhood of $(t_2, \infty) \cup \{\infty\} \cup (-\infty, t_1)$. Hence, there exists an open interval δ with $u(\xi) \in \delta$ such that $\mathbf{M}_0$ is definitizable over a neighbourhood of δ.

The reasoning above holds for each ξ_j, $j = 1, \ldots, n$, hence, the operator $\mathbf{M}_0$ is definitizable over $\overline{\mathbb{C}}$, i.e. $\mathbf{M}_0$ is a definitizable operator (cf. [J6]).

5. We will show that $c_\infty(\mathbf{M}_0) = \{u(\xi_1), \ldots, u(\xi_n)\}$. Condition (A) and Lemma 1.14 imply that ∞ is a regular critical point of $(\mathbf{L}_0 - u(\xi))^{-1}$ belonging to $c_\infty((\mathbf{L}_0 - u(\xi))^{-1})$. Then, as a consequence of [J3], ∞ is a regular critical point of $(\mathbf{M}_0 - u(\xi))^{-1}$ belonging to $c_\infty((\mathbf{M}_0 - u(\xi))^{-1})$. Using this argument for each ξ_j, $j = 1, \ldots, n$, we conclude that each $u(\xi_j)$, $j = 1, \ldots, n$, is a regular critical point of $\mathbf{M}_0$ and we have $\{u(x_1), \ldots, u(\xi_n)\} \subset c_\infty(\mathbf{M}_0)$. Theorem 3.8 implies

$$\{u(\xi_1), \ldots, u(\xi_n)\} = c_\infty(\mathbf{M}_0).$$

6. From the definition of $\mathbf{M}_0$ it follows that $\mathbf{M} - \mathbf{M}_0$ is of finite rank. Then, by [JL1], the operator $\mathbf{M}$ is definitizable and $c_\infty(\mathbf{M}) = c_\infty(\mathbf{M}_0) = \{u(\xi_1), \ldots, u(\xi_n)\}$ holds, which completes the proof. □

Corollary 3.11. *Assume that condition* (A) *is satisfied and that*

$$u(1) + \max_{x \in I} \left| \frac{q'(x)}{u'(x)} \right| \leq \frac{\pi^2}{4}.$$

Then the operator $\mathbf{M}$ *is definitizable,* $c_\infty(\mathbf{M}) = \{u(\xi_1), \ldots, u(\xi_n)\}$ *and each* $u(\xi_j)$*,* $j = 1, \ldots, n$*, is a regular critical point of* $\mathbf{M}$ *with* $u(\xi_j) \notin \sigma_p(\mathbf{M})$*.*

Proof. By Theorem 3.10 it remains to show $\{u(\xi_1), \ldots, u(\xi_n)\} \cap c_s(\mathbf{M}) = \emptyset$. This follows from the fact that $\sigma(A) \subset [\frac{\pi^2}{4}, \infty)$, hence $\mathbf{M} = \mathbf{M}_0$ (where $\mathbf{M}_0$ is defined as in (3.23)). □

Remark 3.12. Assume that $q, u \in C^1(I)$, $u'(x) > 0$ for all $x \in I$ and that q has infinitely many zeros. Denote by Ξ the set of all $\xi \in I$ with

$u(\xi) \in \sigma(u_+) \cap \sigma(u_-)$ such that $u(\xi)$ is not an isolated point in $\sigma(u_+) \cap \sigma(u_-)$. Then, by a reasoning similar to the proof of Theorem 3.10, the operator $\mathbf{M}$ is definitizable over $\overline{\mathbb{C}} \setminus \Xi$.

3.4. Sign Changes of the Numerator Coefficient in the Semiaxis Case

In this section we consider equation (3.20) on the interval $I := [0, \infty)$ with the boundary condition

$$y(0) = 0.$$

Again we assume that the function u is real valued and measurable and that the function q is a real valued piecewise continuous function on the interval I, i.e. for each interval $[0, l]$, $l > 0$, there exist finitely many closed intervals $[a_j, b_j]$, $j = 1, \ldots, N$, $N > 0$, such that $\cup_{j=1}^{N} [a_j, b_j] = [0, l]$, q is continuous on each open interval (a_j, b_j), $j = 1, \ldots, N$, and the one–sided limits $\lim_{x \downarrow a_j} q(x)$ and $\lim_{x \uparrow b_j} q(x)$, $j = 1, \ldots, N$, exist. We assume that the functions q and u satisfy condition (I) from Section 3.1 and that $\mu(\Delta_+) > 0$ and $\mu(\Delta_-) > 0$ hold, where Δ_+ and Δ_- are defined as in Section 3.3. The case $\mu(\Delta_+) = 0$ (resp. $\mu(\Delta_-) = 0$) is contained in the considerations in Section 3.2.

We define $\mathcal{H}$, A as in Section 3.2 and $I_\pm$, $u_\pm$, $q_\pm$, $(\mathcal{K}, [\cdot, \cdot]_\mathcal{K})$, D and the scales $\mathcal{H}_s$, $\mathcal{K}_s$ as in Section 3.3. Further, we define the Krein space $\mathcal{G}$, the operator $\mathbf{L}$ and the scales $\mathcal{G}_s$ as in Section 3.3. The operator B,

$$B := \begin{bmatrix} q_+^{\frac{1}{2}} I_+ \\ |q_-|^{\frac{1}{2}} I_- \end{bmatrix},$$

is a bounded operator from $\mathcal{H}_{\frac{1}{2}}$ into $\mathcal{K}_{-\frac{1}{2}}$. It follows from Lemma 3.5 and condition (I) that the operator $I_\pm^* (1 + u_\pm^2)^{-\frac{1}{4}} |q_\pm|^{\frac{1}{2}} I_\pm (1 + A^2)^{-\frac{1}{4}}$ is an element of $\mathfrak{S}_2$, hence $B \in \mathfrak{S}_2(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}_{-\frac{1}{2}})$. Therefore the operator $\mathbf{B}$, where $\mathbf{B}$ is defined as in (3.8), belongs to $\mathfrak{S}_2(\mathcal{G}_{\frac{1}{2}}, \mathcal{G}_{-\frac{1}{2}})$, that is condition (b) of Section 2.3 is fulfilled. We define $T, \widetilde{T}, \mathbf{M}, \widetilde{\mathbf{M}}$ as in Section 2.3.

Then $T(\lambda)y$ coincides with the expression on the left hand side of (3.20) defined for all $y \in H_0^1(I)$ for which it belongs to $L_2(I)$. Then Lemmas 2.1, 2.3 and Propositions 2.5, 2.7, 2.8 and Remark 2.6 hold, in particular we have

$$\rho(T) = \rho(\mathbf{M}) \cap \rho(D) \text{ and } \sigma_p(T) = \sigma_p(\mathbf{M}) \cap \rho(D).$$

Since $0 \in \rho(T(i\eta))$ for sufficiently large $\eta > 0$ we have $\rho(\mathbf{M}) \neq \emptyset$ and Theorem 2.9 holds.

For all $\lambda \in \rho(\mathbf{M}) \cap \rho(\mathbf{L})$ we have (cf. [J3, Lemma 2.3])

$$(\mathbf{L} - \lambda)^{-1} - (\mathbf{M} - \lambda)^{-1} \in \mathfrak{S}_2.$$

The operator $\mathbf{L}$ is fundamentally reducible (cf. proof of Theorem 3.8) and, as a consequence of Lemma 1.14 and Theorem 1.15, the following theorem holds.

Theorem 3.13. *The operator* $\mathbf{M}$ *is definitizable over*

$$\overline{\mathbb{C}} \setminus (([0,\infty] \cup \sigma_e(u_+)) \cap \sigma_e(u_-)).$$

Let Δ *be an open connected subset of* $\overline{\mathbb{R}} \setminus (([0,\infty] \cup \sigma_e(u_+)) \cap \sigma_e(u_-))$. *Then the following assertions are equivalent.*

(i) Δ *is of type* π_+ *(type* π_-*) with respect to* $\mathbf{M}$.

(ii) Δ *is of type* π_+ *(resp. type* π_-*) with respect to* $\mathbf{L}$.

(iii) $\Delta \subset \overline{\mathbb{R}} \setminus \sigma_e(u_-)$ *(resp.* $\Delta \subset \mathbb{R} \setminus ([0,\infty) \cup \sigma(u_+)))$.

Lemma 3.9 remains valid for $I = [0,\infty)$, without difficulty the proof can be extended to this case.

Proposition 3.14. *Let* $\lambda_0 \in (0,\infty) \setminus \sigma(D)$. *Then* $\lambda_0 \notin \sigma_p(\mathbf{M})$, *i.e.*

$$\sigma_p(\mathbf{M}) \subset (-\infty, 0] \cup \sigma(D).$$

Assume, in addition, that $0 \in \rho(D)$ *and that there exist an* $x_0 > 0$ *such that* $|q(x)u(x)^{-1}| \leq \frac{3}{4x^2}$ *for all* $x \in (x_0, \infty)$ *hold. Then*

$$\sigma_p(\mathbf{M}) \subset (-\infty, 0) \cup \sigma(D).$$

Proof. Let $\lambda_0 \in (0,\infty) \setminus \sigma(D)$. Then $\lambda_0 - T(\lambda_0)$ is the operator defined by the differential expression

$$-\frac{d^2}{dx^2} + \frac{q(x)}{\lambda_0 - u(x)} \tag{3.27}$$

restricted to all those functions from $H_0^1(I)$ which are mapped by (3.27) into $L_2(I)$. The function $\frac{q}{\lambda_0 - u}$ belongs to $L_1(I)$ (cf. (3.19)). By Theorem 3.6 $\lambda_0 - T(\lambda_0)$ has no positive eigenvalue. Hence $\lambda_0 \notin \sigma_p(T)$ and, by Proposition 2.8, $\lambda_0 \notin \sigma_p(\mathbf{M})$.

The additional assumption of the second part of Proposition 3.14 implies that $0 \notin \sigma_p(T)$ (see Theorem 3.6) and, by Proposition 2.8, $0 \notin \sigma_p(\mathbf{M})$. □

Assume, in addition, that the following condition (A′) is fulfilled.

(A′) The functions q and u belong to $C^1(I)$, $u'(x) > 0$ for all $x \in I$ and the function q has at most finitely many zeros in the interval I. Moreover, q and u are bounded functions on the interval I and there exists a real number $\gamma < 0$ such that $u(x) \leq \gamma$ for all $x \in I$.

From conditions (I) and (A′) it follows that q belongs to $L_1(I)$ and that the operator B is a bounded operator from $\mathcal{H}$ into $\mathcal{K}$. Then, for $\lambda \in \rho(D)$, we have

$$\mathcal{D}(T(\lambda)) = \mathcal{D}(A) \text{ and } \mathcal{D}(\mathbf{M}) = \mathcal{D}(\mathbf{L}) = \mathcal{D}(A) \times \mathcal{K}.$$

Moreover, the set $\sigma(u_+) \cap \sigma(u_-)$ consists of at most finitely many points, i.e. $\sigma(u_+) \cap \sigma(u_-) = \{u(\xi_1), \ldots, u(\xi_n)\}$ for some $\xi_1, \ldots, \xi_n \in I$, $n \in \mathbb{N}$, and each ξ_j, $j = 1, \ldots, n$, is a zero of the function q. From Theorem 3.13 it follows that $\mathbf{M}$ is definitizable over

$$\overline{\mathbb{C}} \setminus \{u(\xi_1), \ldots, u(\xi_n)\}.$$

This will improved by the following theorem.

Theorem 3.15. *Assume that condition* (A′) *is fulfilled. Then the operator* $\mathbf{M}$ *is definitizable and we have* $c_\infty(\mathbf{M}) = \{u(\xi_1), \ldots, u(\xi_n)\}$. *Furthermore, we have*

$$\sigma_p(\mathbf{M}) \cap (0, \infty) = \emptyset \text{ and } c(\mathbf{M}) \setminus c_\infty(\mathbf{M}) \subset \sigma_p(\mathbf{M}) \cap (-\infty, 0].$$

Assume, in addition, that there exist $x_0 > 0$ *such that* $\left|\frac{q(x)}{u(x)}\right| \leq \frac{3}{4x^2}$ *for all* $x \in (x_0, \infty)$. *Then*

$$\sigma_p(\mathbf{M}) \cap [0, \infty) = \emptyset \text{ and } c(\mathbf{M}) \setminus c_\infty(\mathbf{M}) \subset \sigma_p(\mathbf{M}) \cap (-\infty, 0).$$

Proof. 1. Set $I_1 := [0, \xi_n + 1]$. Let $\mathcal{D}_1$ denote the set of all absolutely continuous functions $y \in L_2(I_1)$ which have an absolutely continuous derivative y' with $(y')' \in L_2(I_1)$, and which satisfy $y(0) = y(\xi_n + 1) = 0$. For $y \in \mathcal{D}_1$ we define

$$A_1 y := -y''.$$

We set

$$\Delta_{+,1} := \{x \in I_1 : q(x) > 0\}, \quad \Delta_{-,1} := \{x \in I_1 : q(x) < 0\}$$

and we introduce the following abbreviations

$$u_{+,1} := u|\Delta_{+,1}, \quad u_{-,1} := u|\Delta_{-,1},$$
$$q_{+,1} := q|\Delta_{+,1}, \quad q_{-,1} := q|\Delta_{-,1}.$$

We denote by $I_{+,1}$ ($I_{-,1}$) the embedding operator acting from $L_2(I_1)$ into $L_2(\Delta_{+,1})$ ($L_2(\Delta_{-,1})$, respectively), i.e.

$$I_{+,1}f := f|\Delta_{+,1}, \quad I_{-,1}f := f|\Delta_{-,1}, \quad f \in L_2(I_1).$$

On $\mathcal{G}_1 := L_2(I_1) \times L_2(\Delta_{+,1}) \times L_2(\Delta_{-,1})$ we consider the indefinite inner product $[\cdot,\cdot]_{\mathcal{G}_1}$ defined by

$$\left[\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}\right]_{\mathcal{G}_1} := (f_1, g_1)_{L_2(I_1)} + (f_2, g_2)_{L_2(\Delta_{+,1})} - (f_3, g_3)_{L_2(\Delta_{-,1})},$$

where $f_1, g_1 \in L_2(I_1)$, $f_2, g_2 \in L_2(\Delta_{+,1})$ and $f_3, g_3 \in L_2(\Delta_{-,1})$. Then $(\mathcal{G}_1, [\cdot,\cdot]_{\mathcal{G}_1})$ is a Krein space.

We define an operator $\mathbf{M}_1$ with domain $\mathcal{D}(\mathbf{M}_1) := \mathcal{D}_1 \times L_2(\Delta_{+,1}) \times L_2(\Delta_{-,1})$ by

$$\mathbf{M}_1\mathbf{y} = \begin{bmatrix} A_1 & I_{+,1}^* q_{+,1}^{\frac{1}{2}} & -I_{-,1}^* |q_{-,1}|^{\frac{1}{2}} \\ q_{+,1}^{\frac{1}{2}} I_{+,1} & u_{+,1} & 0 \\ |q_{-,1}|^{\frac{1}{2}} I_{-,1} & 0 & u_{-,1} \end{bmatrix} \mathbf{y}, \quad \mathbf{y} \in \mathcal{D}(\mathbf{M}_1).$$

If we replace the interval I in Section 3.3 by I_1 and the boundary condition (3.21) by $y(0) = y(\xi_n + 1) = 0$ then we can repeat the reasoning from Section 3.3 for the operator $\mathbf{M}_1$. Hence, Theorem 3.8 and Theorem 3.10 are valid for $\mathbf{M}_1$, i.e. $\mathbf{M}_1$ is a definitizable operator with $c_\infty(\mathbf{M}_1) = \{u(\xi_1), \ldots, u(\xi_n)\}$.

2. Assume that $q(x) < 0$ for all $x \in [\xi_n + 1, \infty)$. We set $I_2 := [\xi_n + 1, \infty)$ and let $\mathcal{D}_2$ denote the set of all locally absolutely continuous functions $y \in L_2(I_2)$ which have a locally absolutely continuous derivative y' with $(y')' \in L_2(I_2)$, and which satisfy $y(\xi_n + 1) = 0$. For $y \in \mathcal{D}_2$ we define

$$A_2 y := -y''.$$

We consider on $\mathcal{G}_2 := L_2(I_2) \times L_2(I_2)$ the indefinite inner product $[\cdot,\cdot]_{\mathcal{G}_2}$ defined by

$$\left[\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\right]_{\mathcal{G}_2} := (f_1, g_1)_{L_2(I_2)} - (f_2, g_2)_{L_2(I_2)},$$

where $f_1, g_1, f_2, g_2 \in L_2(I_2)$. Then $(\mathcal{G}_2, [\cdot,\cdot]_{\mathcal{G}_2})$ is a Krein space. We define an operator $\mathbf{M}_2$ with domain $\mathcal{D}(\mathbf{M}_2) := \mathcal{D}_2 \times L_2(I_2)$ by

$$\mathbf{M}_2\mathbf{y} = \begin{bmatrix} A_2 & -|q|I_2|^{\frac{1}{2}} \\ |q|I_2|^{\frac{1}{2}} & u|I_2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y} \in \mathcal{D}(\mathbf{M}_2).$$

If we replace the interval I in Lemma 3.5 by I_2 then it is easy to see that $|q|I_2|^{\frac{1}{2}}(1+A_2^2)^{-\frac{1}{4}}$ is a compact operator in $L_2(I_2)$, i.e. $|q|I_2|^{\frac{1}{2}}$ satisfies condition (b) from Section 2.3. Thus, by Theorem 2.12, it follows that $\mathbf{M}_2$ is a definitizable operator with $c_\infty(\mathbf{M}_2)=\emptyset$.

From $q(x)<0$ for $x\in I_2$ it follows that we have $\Delta_+=\Delta_{+,1}$ and $\Delta_-=\Delta_{-,1}\cup I_2$ (see (3.22)). Therefore, the operator $\mathbf{U}$ defined by

$$\mathbf{U}(f_1,f_2,f_3)^{\mathrm{T}} := (f_1|I_1, f_2, f_3|\Delta_{-,1}, f_1|I_2, f_3|I_2)^{\mathrm{T}},$$

for $f_1\in L_2(I)$, $f_2\in L_2(\Delta_+)$, $f_3\in L_2(\Delta_-)$ is a unitary operator mapping $\mathcal{G}$ onto $\mathcal{G}_1\times\mathcal{G}_2$. Denote by $\mathcal{D}_0$ the set of all locally absolutely continuous functions $y\in L_2(I)$ which have a locally absolutely continuous derivative y' with $(y')'\in L_2(I)$, and which satisfy $y(0)=y(\xi_n+1)=0$. Then $\mathcal{D}_0\times L_2(\Delta_+)\times L_2(\Delta_-)$ is a subset of $\mathcal{D}(\mathbf{M})$ and for $\mathbf{y}\in\mathcal{D}_0\times L_2(\Delta_+)\times L_2(\Delta_-)$ it follows

$$\mathbf{M}\mathbf{y} = \mathbf{U}^{-1}\begin{bmatrix}\mathbf{M}_1 & 0\\ 0 & \mathbf{M}_2\end{bmatrix}\mathbf{U}\mathbf{y}. \tag{3.28}$$

3. We choose an element $(h,0,0)^{\mathrm{T}}$ from $\mathcal{D}(\mathbf{M})$ such that $h(\xi_n+1)=1$ holds. Then it follows

$$\mathcal{D}(\mathbf{M}) = (\mathcal{D}_0\times L_2(\Delta_+)\times L_2(\Delta_-)) \dot{+} \operatorname{sp}\{(h,0,0)^{\mathrm{T}}\}.$$

Let $\mathbf{y}\in\mathcal{G}$ and $\lambda\in\rho(\mathbf{M})\cap\rho(\mathbf{M}_1)\cap\rho(\mathbf{M}_2)$. Then we have for some $\mathbf{x}\in\mathcal{D}_0\times L_2(\Delta_+)\times L_2(\Delta_-)$ and some $\alpha\in\mathbb{C}$

$$\mathbf{y} = (\mathbf{M}-\lambda)\mathbf{x} + \alpha(\mathbf{M}-\lambda)(h,0,0)^{\mathrm{T}}.$$

Therefore, by relation (3.28), we have

$$(\mathbf{M}-\lambda)^{-1}\mathbf{y} - \mathbf{U}^{-1}\begin{bmatrix}\mathbf{M}_1-\lambda & 0\\ 0 & \mathbf{M}_2-\lambda\end{bmatrix}^{-1}\mathbf{U}\mathbf{y} =$$

$$= \left((\mathbf{M}-\lambda)^{-1} - \mathbf{U}^{-1}\begin{bmatrix}\mathbf{M}_1-\lambda & 0\\ 0 & \mathbf{M}_2-\lambda\end{bmatrix}^{-1}\mathbf{U}\right)\alpha(\mathbf{M}-\lambda)\begin{pmatrix}h\\0\\0\end{pmatrix},$$

i.e. the difference of the resolvents of $\mathbf{M}$ and $\mathbf{U}^{-1}(\mathbf{M}_1\times\mathbf{M}_2)\mathbf{U}$ has a one–dimensional range. Moreover, $\sigma_{ess}(\mathbf{M}_1)\cap\sigma_{ess}(\mathbf{M}_2)=\{u(\xi_n+1)\}$ holds and there exists an open interval $\delta\subset\mathbb{R}$, $u(\xi_n+1)\in\delta$, such that δ is of type π_- with respect to $\mathbf{M}_1$ (cf. Theorem 3.8) and of the same type with respect to $\mathbf{M}_2$ (cf. Theorem 2.12). In addition, there exists an open connected set $\delta'\subset\overline{\mathbb{R}}$, $\infty\in\delta'$, such that δ' is of positive type with respect to $\mathbf{M}_1$ (cf.

Theorem 3.8) and of positive type with respect to $\mathbf{M}_2$ (cf. Theorem 2.12). Thus the operator $\mathbf{M}_1 \times \mathbf{M}_2$ is a definitizable operator in $\mathcal{G}_1 \times \mathcal{G}_2$ and, by [JL1, Theorem 1], $\mathbf{M}$ is a definitizable operator with

$$c_\infty(\mathbf{M}) = c_\infty(\mathbf{M}_1 \times \mathbf{M}_2) = \{u(\xi_1), \dots, u(\xi_n)\}.$$

Assume that $q(x) > 0$ for all $x \in I_2$. Then a similar reasoning applies.

4. As each finite critical point of $\mathbf{M}$ belongs to $\sigma_p(\mathbf{M}) \cup c_\infty(\mathbf{M})$, the remaining assertions of Theorem 3.15 follows from Proposition 3.14. □

Theorem 3.16. *Assume that condition* (A′) *is fulfilled. For some* $\xi_j \in \{\xi_1, \dots, \xi_n\}$, $1 \le j \le n$, *assume that there exists* $\alpha < 0$ *with*

$$u(\xi_j) + \frac{q(x)}{u(x) - u(\xi_j)} \le \alpha \tag{3.29}$$

for all $x \in I$. *Then* $u(\xi_j)$ *is a regular critical point of* $\mathbf{M}$ *with* $u(\xi_j) \notin \sigma_p(\mathbf{M})$.

Proof. Condition (A′) implies $\sigma_p(\mathbf{L}) = \emptyset$. We denote by $\mathcal{K}'_s$ and $\mathcal{G}'_s$ the scales $\mathcal{K}_s((D-u(\xi_j))^{-1})$ and $\mathcal{G}_s((\mathbf{L}-u(\xi_j))^{-1})$, respectively (see Section 1.1).

The operator $(D-u(\xi_j))^{-1}$ $((\mathbf{L}-u(\xi_j))^{-1})$ can be extended to a continuous linear operator $((D-u(\xi_j))^\sim)^{-1}$ (resp. $((\mathbf{L}-u(\xi_j))^\sim)^{-1}$) acting from $\mathcal{K}'_{\frac{1}{2}}$ into $\mathcal{K}'_{-\frac{1}{2}}$ (resp. from $\mathcal{G}'_{\frac{1}{2}}$ into $\mathcal{G}'_{-\frac{1}{2}}$). Then the operator B is a bounded operator from $\mathcal{H}$ into $\mathcal{K}'_{\frac{1}{2}}$ and the operator $B^+((D-u(\xi_j))^\sim)^{-1}B$ is a bounded operator in $\mathcal{H}$. Therefore we can define

$$T(u(\xi_j)) := u(\xi_j) - A + B^+((D - u(\xi_j))^\sim)^{-1}B$$

with $\mathcal{D}(T(u(\xi_j))) = \mathcal{D}(A)$. Relation (3.29) implies $T(u(\xi_j)) \le \alpha$, hence $T(u(\xi_j))$ is boundedly invertible. We set

$$K := (A - u(\xi_j))^{-1}B^+((D - u(\xi_j))^\sim)^{-1}B.$$

Then $I - K = -(A - u(\xi_j))^{-1}T(u(\xi_j))$ is boundedly invertible and we have

$$\begin{aligned} &-T(u(\xi_j))^{-1} - (A - u(\xi_j))^{-1} = \\ &= ((A - u(\xi_j))(I - K))^{-1} - (A - u(\xi_j))^{-1} = \\ &= ((I - K)^{-1} - I)(A - u(\xi_j))^{-1} = (I - K)^{-1}K(A - u(\xi_j))^{-1}. \end{aligned} \tag{3.30}$$

Moreover, condition (A′) and Lemma 3.5 imply

$$I_\pm^* \left(q_\pm^2 + \frac{q_\pm^2}{(u_\pm - u(\xi_j))^2} \right)^{\frac{1}{4}} I_\pm(1 + A^2)^{-\frac{1}{4}} \in \mathfrak{S}_2,$$

hence B belongs to $\mathfrak{S}_2(\mathcal{H}_{\frac{1}{2}}, \mathcal{K}'_{\frac{1}{2}})$. Then, by (3.30), it follows that the difference $-T(u(\xi_j))^{-1} - (A - u(\xi_j))^{-1}$ belongs to $\mathfrak{S}_2$. By a reasoning similar to the proof of Theorem 3.10 one can show that $u(\xi_j) \notin \sigma_p(\mathbf{M})$ and that there exists an operator $V \in \mathfrak{S}_2(\mathcal{G}'_{\frac{1}{2}}, \mathcal{G}'_{-\frac{1}{2}})$ such that

$$(\mathbf{M} - u(\xi_j))^{-1} = (\mathbf{L} - u(\xi_j))^{-1} \dot{+} V.$$

holds. Condition (A′) and Lemma 1.14 imply that the operator $(\mathbf{L} - u(\xi_j))^{-1}$ is definitizable and that the point ∞ is a regular critical point of $(\mathbf{L} - u(\xi_j))^{-1}$. As a consequence of [J3, Theorem 3.10] the point ∞ is a regular critical point of $(\mathbf{M} - u(\xi_j))^{-1}$ and Theorem 3.16 is proved. □

Remark 3.17. Assume that condition (A′) is fulfilled and that

$$\sup_{x \in I} u(x) + \sup_{x \in I} \frac{q'(x)}{u'(x)} \leq 0.$$

Then for each ξ_j, $1 \leq j \leq n$, relation (3.29) is fulfilled, hence each $u(\xi_j)$, $1 \leq j \leq n$, is a regular critical point of $\mathbf{M}$ with $u(\xi_j) \notin \sigma_p(\mathbf{M})$.

Symbols

$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers
μ	the Lebesgue measure

For a linear operator A we denote by

$\rho(A)$	the resolvent set,
$\sigma(A)$	the spectrum,
$\sigma_p(A)$	the point spectrum,
$\mathcal{R}(A)$	the range,
$\mathrm{def}(A)$	the codimension of $\mathcal{R}(A)$,
$\mathrm{nul}(A)$	the dimension of the kernel of A.

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