Lecture Notes in
Real Algebraic and Analytic Geometry

Proceedings of the RAAG Summer School Lisbon 2003 O-minimal Structures

Editors
M. Edmundo
D. Richardson
A.J. Wilkie

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O-minimal Structures, Lisbon 2003

Proceedings of a Summer School by the European Research and Training Network RAAG

Editors

Mário J. Edmundo
CMAF Universidade de Lisboa
Av. Prof. Gama Pinto 2
1649-003 Lisboa, Portugal

Daniel Richardson
Department of Computer Science
University of Bath
Bath BA2 7AY, England

Alex Wilkie
The Mathematical Institute
University of Oxford
24-29 St. Giles Street
Oxford OX1 3LB, England

PREFACE

This is the proceedings volume of a school organized and financed by the European Community Research and Training Network RAAG (Real Algebraic and Analytic Geometry) HPRN-CT 00271 and co-financed by CMAF/FUL (Portugal). The school took place at CMAF - Universidade de Lisboa, Portugal, during the period June 25-28, 2003. The conference committee was: Mário Edmundo (Oxford and Lisbon), Alex Wilkie (Oxford), Daniel Richardson (Bath) and Fernando Ferreira (Lisbon).

Description of the chapters

- *A brief introduction to o-minimality:* an informal introduction by Charlie Steinhorn to the very basic concepts of o-minimality, to help the reader with no previous knowledge of the subject.

- *Covering definable open sets by open cells:* here Alex Wilkie shows that in an o-minimal expansion $\mathfrak{M}$ of a real closed field, any definable, bounded open set is the union of finitely many open cells.

- *O-minimal (co)homology and applications:* this is a brief survey by Mário Edmundo of o-minimal homology and cohomology theory with emphasis on the applications to o-minimal generalizations of the Jordan-Brouwer separation theorem and to degree theory for definable continuous maps.

- *Type-definability, compact Lie groups and o-minimality:* here Anand Pillay studies type-definable subgroups of small index in definable groups, and the structure on the quotient, in first order structures. Some conjectures are raised in the case where the ambient structure is o-minimal. The gist is that in this o-minimal case, any definable group G should have a smallest type-definable subgroup of bounded index, and that the quotient, when equipped with the logic topology, should be a compact Lie group of the “right” dimension. The author gives positive answers to the conjectures in the special cases when G is 1-dimensional, and when G is definably simple.

- *"Complex-like" analysis in o-minimal structures:* in these notes Y'acov Peterzil and Sergei Starchenko survey the content of three of their recent papers where they treat analogues of basic notions in complex analysis, over an arbitrary algebraically closed field of characteristic zero, in the presence of an o-minimal structure.

- *The elementary theory of elliptic functions I: the formalism and a special case:* in his contribution, Angus Macintyre, considers the elementary theory of all Weierstrass functions, with emphasis on definability and decidability. The work can be seen as a refinement of the work of Bianconi, using ideas of Wilkie and Macintyre to get effective model-completeness. The novelty is the subsequent use of a conjecture of Grothendieck-André to get decidability in many cases.

- *O-minimal expansions of the real field II:* this note by Patrick Speissegger mentions an application of the o-minimality of the Pfaffian closure to the theory of o-minimal structures. It also describes the state of affairs concerning the model completeness conjecture for the Paffian closure and tries to formulate an open question testing the limit of the Pfaffian closure's applicability.

- *On the gradient conjecture for definable functions:* here Adam Parusiński presents the main ideas of the proof of gradient conjecture of R. Thom in the analytic case and discuss which of them can be carried over to the o-minimal case.

- *Algebraic measure, foliations and o-minimal structures:* here J.-Marie Lion presents the main ideas of the proof of the following result. $\mathcal{F}_\lambda$ be a family of codimension p foliations defined on a family M_λ of manifolds and let X_λ be a family of compact subsets of M_λ. Suppose that $\mathcal{F}_\lambda$, M_λ and X_λ are definable in an o-minimal structure and that all leaves of $\mathcal{F}_\lambda$ are closed. Given a definable family Ω_λ of differential p-forms satisfaying $i_Z\Omega_\lambda = 0$ for any vector field Z tangent to $\mathcal{F}_\lambda$, then there exists a constant $A > 0$ such that the integral of $|\Omega_\lambda|$ on any transversal of $\mathcal{F}_\lambda$ intersecting each leaf in at most one point is bounded by A. This result is applied to prove that p-volumes of transverse sections of $\mathcal{F}_\lambda$ are uniformly bounded.

- *Limit sets in o-minimal structures:* in his contribution, Lou van den Dries, shows that taking Hausdorff limits of a definable family in an o-minimal expansion of the real field is a very tame operation: it preserves definability, cannot raise dimension, creates fewer limits than the family has members, and respects Lebesgue measure. He also proves similar results for Gromov-Hausdorff limits and Tychonov limits of definable families of sets, and for pointwise limits of definable families of functions. The first part of these notes is purely geometric, and the second part uses some model theory. An appendix gives Gabrielov's geometric proof that a definable family has few Hausdorff limits.

The Editors
Mário Edmundo
Daniel Richardson
Alex Wilkie

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It is a pleasure to remember this useful and interesting workshop. And such a beautiful city! The workshop would not have happened without the enthusiasm and hard work of Mário Edmundo. Thanks are also due to the participants for the very high quality of their presentations; and for the hospitality of the Universidade de Lisboa; and, of course, for the support of RAAG.

Daniel Richardson

List of contributors

Lou van den Dries, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbanna, IL 61801, USA.

Mário Edmundo, CMAF Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal.

Jean-Marie Lion, IRMAR, Université Rennes 1, 35042 Rennes cedex, France.

Angus Macintyre, Queen Mary, University of London, England.

Adam Parusinski, Département de Mathématiques, Université d'Angers, 2, bd Lavoisier, 49045 Angers cedex 01, France.

Ya'acov Peterzil, Department of Mathematics, University of Haifa, Haifa, Israel.

Anand Pillay, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbanna, IL 61801, USA.

Patrick Speissegger, Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada.

Sergei Starchenko, Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA.

Charles Steinhorn, Department of Mathematics, Vassar College, Poughkeepsie, NY 12604, USA.

Alex J. Wilkie, The Mathematical Institute, University of Oxford, 24-29 St. Giles Street, Oxford OX1 3LB, England.

Contents

A brief introduction to o-minimality

Charles Steinhorn[1]

We presume throughout some familiarity with basic model theory, in particular with the notion of a *definable* set. An excellent reference is [24].

A dense linearly ordered structure

$$\mathcal{M} = (M, <, \ldots)$$

is *o-minimal* (short for ordered-minimal) if every definable set (with parameters) is the union of finitely many points and open *intervals* (a, b), where $a < b$ and $a, b \in M \cup \{\pm\infty\}$.

The "minimal" in o-minimal reflects the fact that the definable subsets in one variable of such a structure $\mathcal{M}$ form the smallest collection possible: they are *exactly* those sets that *must* be definable in the presence of a linear order.

This definition is the ordered analogue of *minimal* structures, those whose definable sets are finite or cofinite, that is, whose definable sets are those that must be definable (in the presence of equality) in every structure. The more familiar *strongly minimal* structures have the property every elementarily equivalent structure is minimal. Not every minimal structure is strongly minimal; see 2.9 below for the surprising situation in the ordered context.

The importance of o-minimality derives from the power that definability and the tools of model theory provide when combined with the wealth of examples of o-minimal structures that are now known. We discuss several examples in the next section. In Sections 2 and 3, we present several of the important model-theoretic consequences of o-minimality. We develop some of the geometric, topological, and algebraic properties enjoyed by o-minimal

[1]Partially supported by NSF grant DMS-0070743.

structures in § 4. In the final section of the paper, we describe how Vapnik-Chervonenkis dimension figures in o-minimal structures and briefly sketch an application in statistical learning theory. References to many of the early and most basic results are omitted for brevity; they may be found in [7]

There is now a vast literature on the subject. Excellent general references include the monograph [7], and the surveys [8] and [22]. O-minimal structures whose underlying order type is that of the real numbers were first discussed in [6]; the general definition and the first systematic model-theoretic treatment was developed in [32] and [18], wherein the name "o-minimality" was introduced. All credits for results not cited explicitly may be found in one of the sources mentioned in this paragraph.

Unless otherwise stated, for the remainder of this paper, all structures $\mathcal{M}$ are linearly ordered by $<$ and all topology is that induced by the order topology on structure.

1 Examples of o-minimal structures

Several "classical" examples of o-minimal structures were known from the beginning, as their definable sets were well-understood long before the development of o-minimality.

1.1 Dense linear orderings without endpoints.

The axioms for all such structures are those of $(\mathbb{Q}, <)$. Quantifier-elimination holds relative to these axioms—that is, every formula is equivalent relative to the axioms to a formula without quantifiers—and o-minimality is an immediate consequence.

1.2 The semilinear sets.

The semilinear sets are are those subsets of $\mathbb{R}^n$, for n varying, that are given by finite boolean combinations of linear equalities and inequalities over $\mathbb{R}$. The semilinear sets are exactly those sets definable by quantifier-free formulas in the real vector space $\mathbb{R}_{\text{lin}} = (\mathbb{R}, <, +, 0, -, \mu_r)_{r \in \mathbb{R}}$, where $-$ is the unary function given by $x \mapsto -x$ and for each $r \in \mathbb{R}$, the unary function μ_r is just scalar multiplication by r. The structure $\mathbb{R}_{\text{lin}}$ admits quantifier elimination—this is essentially the usual linear elimination—so that the definable sets in $\mathbb{R}_{\text{lin}}$ are exactly the semilinear sets. O-minimality is then immediate.

More generally, the same quantifier elimination works for any ordered vector space V over an ordered division ring, whence V is o-minimal also. As a special case, divisible ordered abelian groups are o-minimal.

1.3 Real closed fields.

Let $\overline{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 0, 1, <)$ denote the field of real numbers. In a seminal paper [39], Tarski proved that a uniform quantifier elimination applies to all real closed fields, and so in particular to $\overline{\mathbb{R}}$. So the definable sets in $\overline{\mathbb{R}}$ are exactly the quantifier-free definable sets. These consist of sets in $\mathbb{R}^n$ given by finite boolean combinations of (real) polynomial equalities and inequalities, that is, the semialgebraic sets. O-minimality follows at once.

Tarski's quantifier elimination is effective, thereby providing a decision procedure for real algebraic questions. The algorithm his work provides is highly infeasible, and during the last thirty years or so, these issues have spawned a great deal of work (see [2], e.g.).

In [39], Tarski asks the fundamental question of whether results analogous to those he proves for $\overline{\mathbb{R}}$ can be shown for the field of real numbers with the exponential function adjoined. This question inspired van den Dries' original paper [6], and became a central concern after the initial development of o-minimality. Wilkie [43], in a dramatic breakthrough, solved this problem.

1.4 The real exponential field.

Let $\overline{\mathbb{R}}_{\text{exp}} = (\mathbb{R}, <, +, -, \cdot, \exp, 0, 1, <)$, where $\exp(x) = e^x$. Wilkie [43] showed that every formula is equivalent to an existential formula. This roughly amounts to saying that every definable set is the projection along some coordinate axes of a quantifier-free set (in a higher dimension). In model-theoretic terminology, he showed that the real exponential field is model-complete. O-minimality now follows from earlier work of Khovanskii that quantifier-free sets have just finitely many connected components.

Decidability is another matter. Macintyre and Wilkie [23] reduce the decidability of the theory of the real exponential field to Schanuel's Conjecture, an assertion about transcendence degree that is generally considered to be well beyond current methods.

Finding o-minimal structures that expand $\overline{\mathbb{R}}$ by adjoining mathematically important functions quickly became one of the central foci of research in the subject. We illustrate with just a few examples from the extensive literature.

1.5 The real exponential field with restricted analytic functions.

The class of *restricted analytic functions*, **an**, consists of all functions $g\colon \mathbb{R}^n \to \mathbb{R}$ for which there is an analytic $f\colon U^{\text{open}} \subset \mathbb{R}^n \to \mathbb{R}$ such that $[0,1]^n \subset U$, $g_{|[0,1]^n} = f_{|[0,1]^n}$, and $g(\overline{x}) = 0$ otherwise.

Let $\overline{\mathbb{R}}_{\text{an,exp}} = (\mathbb{R}, <, +, -, \cdot, \exp, g, 0, 1, <)_{g\in\text{an}}$. This structure, studied extensively in [10] building on earlier work going back to Gabrielov [13] in the late 1960's, is o-minimal and behaves well model-theoretically. If the (definable) natural logarithm function log (with the convention that $\log x = 0$ for $x \leq 0$) is adjoined to the language, then the resulting structure $(\overline{\mathbb{R}}_{\text{an,exp}}, \log)$ eliminates quantifiers. Many elementary functions are definable in $\overline{\mathbb{R}}_{\text{an,exp}}$. To illustrate, the sine and cosine functions restricted to any bounded interval (*but not on all of* $\mathbb{R}$) are definable, and hence so is the arctangent function.

1.6 The real field with Pfaffian functions.

We say that a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *Pfaffian* if there are functions $f_1, \ldots, f_k\colon \mathbb{R}^n \to \mathbb{R}$ and polynomials $p_{ij}\colon \mathbb{R}^{n+i} \to \mathbb{R}$ such that

$$\frac{\partial f_i}{\partial x_j}(\overline{x}) = p_{ij}(\overline{x}, f_1(\overline{x}), \ldots, f_i(\overline{x}))$$

for all $i = 1, \ldots, k$, $j = 1, \ldots n$, and $\overline{x} \in \mathbb{R}^n$.

Wilkie [44] proves (by quite different methods than used previously) that the expansion of $\overline{\mathbb{R}}$ by all Pfaffian functions is o-minimal. For a discussion of further work in this direction, see Speissegger's survey [37].

1.7 Quasianalytic Denjoy-Carleman classes.

Here, we refer to the recent paper [33] of Rolin, Speissegger, and Wilkie for definitions. There model-completeness and o-minimality is proved for structures obtained by adjoining functions in certain *Denjoy-Carlemann classes* to the real field $\overline{\mathbb{R}}$. As a consequence, it follows that there are incompatible o-minimal expansions of $\overline{\mathbb{R}}$, and hence no largest o-minimal expansion of $\overline{\mathbb{R}}$.

2 Some basic model theory for o-minimal structures

A central and successful theme in model theory has long been to explore the algebraic consequences of model-theoretic hypotheses like the following early result in the development of o-minimality. The more recent deep theorem of Peterzil and Starchenko, Theorem 3.4, circumscribes the possibilities much more strictly.

Proposition 2.1

1. *Every o-minimal ordered group is a divisible ordered abelian group.*
2. *Let $\mathcal{G} = (G, +, <, \ldots)$ be an o-minimal expansion of an ordered group. The only subgroups of G that are definable in $\mathcal{G}$ are G and $\{0\}$.*
3. *Every o-minimal ordered ring is a real closed field.*

In fact, it is proved in [29] that if $\mathcal{K}$ is an integral domain that is definable in an o-minimal structure, then there is a one-dimensional (see the paragraphs after 2.5 for the definition of dimension) real closed field $\mathcal{R}$ that is a subring of $\mathcal{K}$ such that $\mathcal{K}$ is definable isomorphic to $\mathcal{R}$, $\mathcal{R}(\sqrt{-1}\,)$, or the ring of quaternions over $\mathcal{R}$.

At the root of many, if not most o-minimality arguments is the

Theorem 2.2 (Monotonicity Theorem)
Let $\mathcal{M} = (M, <, \ldots)$ be o-minimal and let $f\colon M \to M$ be definable. Then there are

$$-\infty = a_0 < a_1 < \cdots < a_{k-1} < a_k = \infty$$

in $M \cup \{\pm\infty\}$ (definable with the same parameters used to define f) such that for each $j < k$ either $f_{|(a_j, a_{j+1})}$ is constant or is a strictly monotone bijection of intervals in M.

The Monotonicity Theorem suggests that definable sets in *more* than one variable in o-minimal structures might enjoy good topological finiteness properties also. That this is true is a remarkable consequence of o-minimality. To develop these results, we require

Definition 2.3 Let $\mathcal{M}$ be a densely ordered structure. The collection of $\mathcal{M}$*-cells* is the subcollection $\mathcal{C} = \cup_{n=1}^{\infty}\mathcal{C}_n$ of the $\mathcal{M}$-definable subsets of M^n for $n = 1, 2, 3, ...$ defined recursively as follows.

(i) *Cells in* M

The collection of cells $\mathcal{C}_1$ in M consists of all single point sets $\{a\} \subset M$ and all open intervals $(a, b) \subseteq M$, where $a < b$ and $a, b \in M \cup \{\pm\infty\}$.

(ii) *Cells in* M^{n+1}

Assume the collection of cells $\mathcal{C}_n$ in M^n has been defined. The collection $\mathcal{C}_{n+1}$ of cells in M^{n+1} consists of cells of two different kinds:

(a) *Graphs.* Let $C \in \mathcal{C}_n$ and let $f\colon C \subseteq M^n \to M$ be $\mathcal{M}$-definable and continuous. Then

$$\mathrm{graph}(f) = \{(\overline{x}, y) \in M^{n+1} \mid f(\overline{x}) = y\} \subseteq M^{n+1}$$

is a cell;

(b) *Generalized Cylinders.* Let $C \in \mathcal{C}_n$. Let $f, g\colon C \subseteq M^n \to M$ be $\mathcal{M}$-definable and continuous such that $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in C$. Then the cylinder set

$$(f, g)_C = \{(\overline{x}, y) \in M^{n+1} \mid f(\overline{x}) < y < g(\overline{x})\} \subset M^{n+1}$$

is a cell.

Cells in an o-minimal structure $\mathcal{M}$ clearly are $\mathcal{M}$-definable. They also possess good topological and geometric properties, which we now illustrate.

Let $X \subset M^n$ be definable in $\mathcal{M}$. The set X is *definably connected* if there do not exist disjoint *definable* sets $Y_1, Y_2 \subset X$, both open in X, such that $X = Y_1 \cup Y_2$.

Proposition 2.4 *Cells in o-minimal structures are definably connected.*

The next fundamental result shows that cells are the basic building blocks of definable sets in o-minimal structures.

Theorem 2.5 (Cell Decomposition Theorem)
Let $\mathcal{M}$ be o-minimal. Then every definable set $S \subset M^n$ can be partitioned definably into finitely many cells. If $f\colon X \subset M^n \to M$ is definable, then the partition can be taken such that the restriction of f to each cell is continuous.

Definable sets in an o-minimal structure thus consist of finitely many definably connected components. If $\mathcal{M}$ is an o-minimal expansion of $(\mathbb{R}, <)$, then cells are easily seen to be connected and so definable sets may be partitioned into finitely many connected components, generalizing a fact long known in the semialgebraic setting.

If an o-minimal structure is an expansion of a real closed field, then the cell decomposition can be further strengthened: for each definable set $X \subset M^n$ and $k = 1, 2, \ldots$, there is a decomposition of M^n that respects X and for which the data in the decomposition are $\mathcal{C}^k$.

Cells come naturally equipped with a topologically defined dimension. For each cell $C \subseteq M^n$ there is a largest $k \leq n$ and $i_1, \ldots, i_k \in \{1, 2, \ldots, n\}$ such that if $\pi\colon M^n \to M^k$ is the projection mapping given by

$$\pi(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k}),$$

then $\pi(C) \subseteq M^k$ is an open cell in M^k. This value of k is called the *dimension* of C. The dimension of a cell can be shown to correspond to the number of times that a generalized cylinder is formed in a construction of the cell. The dimension of a definable set is defined to be the maximum dimension of a cell in any partition of the set into cells.

Working in a countably saturated o-minimal structure, it is possible to assign an algebraic dimension to a definable set that coincides with its topological dimension. Let $a, \overline{b} = (b_1, \ldots b_p)$ be elements of a structure $\mathcal{M}$ (not necessarily linearly ordered). Then a is said to be *algebraic* over $\overline{b}$ if there is a formula $\varphi(x, \overline{y})$ and $k \in \mathbb{N}$ such that

$$\mathcal{M} \models \varphi(a, \overline{b}) \;\wedge\; \exists^{\leq k} x\, \varphi(x, \overline{b}).$$

Observe that if $\mathcal{M}$ is linearly ordered and some $a \in M$ is algebraic over $\overline{b}$ in $\mathcal{M}$, then a actually is definable over $\overline{b}$ (as the first, second, etc., element satisfying the formula φ as above).

In a variety of model-theoretic contexts, symmetry of algebraic dependence provides the basis for a satisfactory notion of dimension, as does ordinary algebraic dependence in transcendence theory. The Monotonicity Theorem yields this symmetry as an easy consequence.

Corollary 2.6 (Exchange Lemma) *Let $\mathcal{M}$ be o-minimal and $a, c, \overline{b}$ be in M. If a is algebraic over $c, \overline{b}$ but not algebraic over $\overline{b}$ then c is algebraic over $a, \overline{b}$.*

Now suppose that $\mathcal{M}$ is a countably saturated o-minimal structure and $X \subset M^n$ is definable. Let the *algebraic dimension* of X be the greatest $k \leq n$ for which there is some $\overline{a} = (a_1, \ldots, a_n) \in X$ such that there are algebraically independent $a_{i_1}, \ldots, a_{i_k} \subset \{a_1, \ldots, a_n\}$. Then it is not difficult to prove that

Proposition 2.7 *Algebraic dimension coincides with (topological) dimension for definable sets in countably saturated o-minimal structures.*

Among the most unexpected consequences of o-minimality are various "uniform finiteness" results. Informally, these assert that a property holds uniformly over a parametrically definable family of definable sets. To make this more precise, we require some notation. Let $S \subset M^{n+p}$ be a definable set in an o-minimal structure $\mathcal{M}$, and for each $\overline{b} \in M^n$ set

$$S_{\overline{b}} = \{\overline{y} \in M^p \,|\, (\overline{b}, \overline{y}) \in S\}.$$

The family

$$\mathcal{S} = \{S_{\overline{b}} \,|\, \overline{b} \in M^n\}$$

we call a *definable family* of definable sets.

The following theorem is proved together with the Cell Decomposition Theorem as part of a simultaneous induction.

Theorem 2.8 (Uniform Bounds Theorem) *Let $\mathcal{M}$ be o-minimal and let $S \subset M^{n+1}$ be a definable set. Then there is a fixed $K \in \mathbb{N}$ such that if $S_{\overline{b}}$ is finite for all $\overline{b} \in M^n$, then $|S_{\overline{b}}| \leq K$.*

The Uniform Bounds Theorem immediately implies

Corollary 2.9 *If $\mathcal{M}$ is o-minimal then so is every $\mathcal{N} \equiv \mathcal{M}$.*

This fact opens the door to the use of the model-theoretic Compactness Theorem to deduce a variety of uniformity results about a single o-minimal structure, such as an o-minimal expansion of the field of real numbers. We discuss one result in Section 4—see 4.2; others may be found in [7].

3 Some further model theory for o-minimal structures

Here we present some finer purely model-theoretic results that have significant geometric or algebraic consequences.

Definition 3.1 A type $p(\overline{x})$ over a structure $\mathcal{M}$ —i.e., a maximal consistent set of formulas with parameters from $\mathcal{M}$— is *definable* if for every formula $\varphi(\overline{x}, \overline{y})$ we have

$$\{\overline{b} \mid \varphi(\overline{x}, \overline{b}) \in p\} \subset M^{|\overline{y}|}$$

is a definable set in $\mathcal{M}$.

It can be shown that all types over a structure $\mathcal{M}$ are definable if and only if for every $\mathcal{N} \succ \mathcal{M}$ and every $X \subset N^k$ definable in $\mathcal{N}$, the set $X \cap M^k$ is definable in $\mathcal{M}$ (that is, $\mathcal{N}$ is a *conservative extension* of $\mathcal{M}$).

It is easy to find a type over an o-minimal structure that is not definable. For example, let $p(x)$ be the type of π over the structure $(\mathbb{Q}, <)$ determined by the formulas $\{x > q \,|\, q < \pi\} \cup \{x < q \,|\, q > \pi\}$. For a complete theory T, all types over all models of T are definable if and only if T is stable. Thus for definability of types to hold for an o-minimal structure, a further hypothesis is needed.

Let $\mathcal{M} = (M, <, \ldots)$ be a linearly ordered structure. A partition $M = C_1 \cup C_2$ into nonempty subsets such that every element of C_1 is less than every element of C_2 is called a *cut* if C_1 has no greatest element and C_2 has no least element. The structure $\mathcal{M}$ is *Dedekind complete* if there are no cuts in $\mathcal{M}$.

Theorem 3.2 ([25]) *Let $\mathcal{M}$ be a Dedekind complete o-minimal structure. Then all types over an $\mathcal{M}$ are definable.*

In particular, the theorem applies to all o-minimal structures whose underlying order type is $(\mathbb{R}, <)$. Here is a consequence.

Corollary 3.3 *Let $\mathcal{M} = (\mathbb{R}, <, +, \ldots)$ be o-minimal, and let*

$$f_{\overline{a}} \colon B \subset \mathbb{R}^m \to \mathbb{R} \quad \textit{for } \overline{a} \in A \subset \mathbb{R}^m$$

be a definable family $\mathcal{F}$ of functions in $\mathcal{M}$. Then every $g \colon B \to \mathbb{R}$ which is in the closure of $\mathcal{F}$ (with respect to the product topology on $(\mathbb{R} \cup \{\pm\infty\})^B$) is definable.

The definability of types also can be used to prove the definability of the Hausdorff limits of a compact, definable family of sets in an o-minimal expansion of the real field. See [9] or [21] for more about this application.

The final theorem of this section yields a striking classification of the structure that definability confers on an interval around a point in an o-minimal structure. Almost twenty-five years ago, Zilber conjectured that strongly minimal structures divide into three classes—"trivial", "vector space-like", and "field-like"—according to the algebraic structure that could be recovered from purely model-theoretic hypotheses. Although this conjecture in its most general form was refuted by Hrushovski [14], it has offered a robust paradigm that has proved true in certain contexts (e.g., see [15]) and has inspired considerable research activity. The beautiful theorem of Peterzil and Starchenko, below, shows that the so-called Zilber trichotomy holds for o-minimal structures.

We say that an element a in an o-minimal structure is *trivial* if there does not exist an open interval I with $a \in I$ and a definable $F\colon I^2 \to I$ which is strictly monotone in each variable.

Theorem 3.4 (Trichotomy Theorem; [27]) *Let $\mathcal{M}$ be ω-saturated and o-minimal, and let $a \in M$. Then exactly one of the following holds:*

i. a is trivial;

ii. the structure that $\mathcal{M}$ induces on some convex neighborhood of a is an ordered vector space over an ordered division ring;

iii. the structure that $\mathcal{M}$ induces on some open interval about a is an o-minimal expansion of a real closed field.

Hence, an o-minimal structure $\mathcal{M}$ either is trivial, or there is some point satisfying (ii) and no points satisfying (iii), or there is some point satisfying (iii). Note that such a result must be local in character: the ordered sum of a dense linear order with greatest element followed by a real closed field shows that the possibilities can mix for innocent reasons. In the next section, we present an striking result for groups definable in an o-minimal structure whose proof relies crucially on the Trichotomy Theorem.

4 Some geometry, topology and algebra of o-minimal structures

It has long been discussed by researchers in the field—perhaps first made explicit by van den Dries—that o-minimal structures (particularly over the real numbers) provide a rich framework for "tame topology," or *topologie modérée*, as envisioned by Grothendieck (see, e.g., the article by Teissier in [35]). That is, o-minimality does not allow for set theoretic complexity (or, as some would prefer, pathology). The small selection of theorems in this section are mainly intended to illustrate this point of view; much more can be found in [7] or [5].

We begin with two finer topological and geometrical results that are available for o-minimal expansions of real closed fields. These are due to van den Dries; proofs can be found in [7].

Theorem 4.1 (Triangulation Theorem) *Let $\mathcal{R} = (R, +, \cdot, <, \ldots)$ be an o-minimal expansion of a real closed field. Then each definable set $X \subset R^m$ is definably homeomorphic to a semilinear set. More precisely, X is definably homeomorphic to a union of simplices of a finite simplicial complex in R^m.*

It is worth noting that Shiota [36] has proved an o-minimal version of the so-called "Hauptvermutung," namely that homeomorphic compact semilinear subsets of $\mathbb{R}^n$ that are homeomorphic via a function definable in an o-minimal expansion of the real field $\overline{\mathbb{R}}$ are in fact homeomorphic via a semilinear mapping.

The Triangulation Theorem can be used to prove the finiteness of the number of homeomorphism types of a definable family of subsets of R^n in an o-minimal expansion of a real closed field $\mathcal{R}$.

Theorem 4.2 *Let $S \subset R^{m+n}$ be definable in an o-minimal expansion of a real closed field $\mathcal{R}$, so that $\{S_{\overline{a}} \,|\, \overline{a} \in R^m\}$ forms a definable family of subsets of R^n. Then there is a definable partition $\{B_1, \ldots, B_p\}$ of R^m such that for all $\overline{a}_1, \overline{a}_2 \in R^m$, the sets $S_{\overline{a}_1}$ and $S_{\overline{a}_2}$ are (definably) homeomorphic if and only if there is some $j = 1, \ldots, p$ such that $\overline{a}_1, \overline{a}_2 \in B_j$.*

It is possible to extract topological consequences even using only that a structure is an o-minimal expansion of an ordered group. We illustrate with a definable version of "curve selection."

Definition 4.3 Let $\mathcal{M} = (M, <, \ldots)$. A *definable curve* $\mathcal{C}$ in $X \subseteq M^n$ is the image $\mathcal{C} = \sigma((a,b))$ of an $\mathcal{M}$-definable continuous embedding $\sigma\colon (a,b) \subset M \to X$, where $-\infty \leq a < b \leq \infty$.

Theorem 4.4 *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and let $S \subseteq M^n$ be definable. Then for every $\overline{a} \in \operatorname{cl} S \setminus S$, there is a definable curve $\sigma\colon (0,\epsilon) \to S$ such that $\lim_{t \to \epsilon} \sigma(t) = \overline{a}$.*

The Cell Decomposition Theorem provides the basis for a combinatorial version of the Euler characteristic for definable sets in an o-minimal structure.

Definition 4.5 Let $S \subset M^n$ be definable in the o-minimal structure $\mathcal{M}$ and $\mathcal{P}$ be a partition of S into cells. Let

$$n(\mathcal{P}, k) = \#\text{ cells of dimension } k \text{ in } \mathcal{P}$$

and

$$E_{\mathcal{P}}(S) = \sum (-1)^k n(\mathcal{P}, k).$$

By taking a common refinement of two partitions of a definable set into cells, it can be shown that

Proposition 4.6 *If $\mathcal{P}$ and $\mathcal{P}'$ are partitions of S into cells, then $E_{\mathcal{P}}(S) = E_{\mathcal{P}'}(S)$.*

This allows us to define $E(S) = E_{\mathcal{P}}(S)$ for any partition $\mathcal{P}$. The Euler characteristic enjoys many properties that it would be expected to satisfy.

Theorem 4.7 *Let $\mathcal{M}$ be o-minimal.*

1. *Let A and B be disjoint definable subsets of M^n. Then*

$$E(A \cup B) = E(A) + E(B).$$

2. *Let $A \subset M^m$ and $B \subset M^n$ be definable. Then*

$$E(A \times B) = E(A)E(B).$$

3. *Let $f\colon A \subset M^m \to M^n$ be definable and injective. Then $E(A) = E(f(A))$.*

Note that (3), above does not require continuity. Moreover, if $\mathcal{M}$ is an o-minimal expansion of a real closed field, it can be shown that the existence of a bijection between two definable sets in $\mathcal{M}$ is equivalent to their having the same dimension and Euler characteristic.

The o-minimal Euler characteristic hints that more sophisticated tools of algebraic topology might be available in the o-minimal setting. Homology theory for o-minimal structures has been developed by A. Woerhide in his unpublished Ph.D. dissertation [45]. The most difficult of the homology axioms to verify is excision. Woerhide cleverly avoids barycentric subdivision in his proof of this axiom, which is usually employed but is generally unavailable in an o-minimal structure. Edmundo has developed cohomology for o-minimal structures; see [11] and [12] (in this volume). We do not pursue these topics further here.

Proposition 2.1 and the comments following the proposition suggest that the hypothesis of O-minimality has substantial algebraic consequences. To develop this theme further, we require:

Definition 4.8 A *definable manifold* M of dimension m relative to an o-minimal structure $\mathcal{N}$ is a set X equipped with a *definable atlas*, that is a finite family $(U_i, g_i)_{i\in I}$ such that

i. $X = \cup_{i\in I} U_i$

ii. each $g_i\colon U_i \to g_i(U_i)$ is a bijection and $g_i(U_i) \subset N^m$ is definable and open

iii. each $g_i(U_i \cap U_j)$ is definable and open in $g_i(U_i)$

iv. each map
$$g_{ij} = g_j \circ g_i^{-1}\colon g_i(U_i \cap U_j) \to g_j(U_i \cap U_j)$$
is a definable homeomorphism.

In [30], Pillay adapts Hrushovski's proof of Weil's theorem that an algebraic group over an algebraically closed field can be recovered from "birational data" to show that

Theorem 4.9 *Every definable group in an o-minimal structure can be equipped with a (unique) definable manifold structure making the group into a topological group.*

Moreover, if the ambient o-minimal structure is an expansion of a real closed field, the charts and transition maps in the definable manifold can be chosen to be C^k for an arbitrary $k \geq 1$. Note in particular that if the underlying order is $(\mathbb{R}, <)$, then the positive solution to Hilbert's Fifth Problem implies that the group actually is a Lie group. For issues related to this, see Pillay's contribution to this volume [31].

Our discussion now divides into abelian and non-abelian groups. In the non-abelian case, Peterzil, Pillay, and Starchenko [26] prove an o-minimal analogue of Cherlin's longstanding conjecture in the model theory of stable groups, namely that: an infinite simple group of finite Morley rank is definably isomorphic to an algebraic group over an algebraically closed field.

Theorem 4.10 ([26]) *Let $\mathbb{G} = (G, \cdot, e)$ be a an infinite group that is definable in an o-minimal structure $\mathcal{M}$ and that is $\mathbb{G}$-definably simple, i.e., is not abelian and does not contain a $\mathbb{G}$-definable proper non-trivial normal subgroup. Then there is a real closed field $\mathcal{R}$ definable in $\mathcal{M}$ such that G is definably isomorphic to a semialgebraic linear group over $\mathcal{R}$.*

The proof first makes use of the Trichotomy Theorem 3.4 to find the real closed field $\mathcal{R}$. Then enough Lie theory (over an o-minimal expansion of a real closed field) can be developed to establish the conclusion of the theorem.

For abelian groups, the tempting paradigm is the classical fact that every abelian connected real Lie group is Lie isomorphic to a direct sum of copies of the additive group $\mathbb{R}_a$ of real numbers and the group $\mathbb{S}^1$ of complex numbers of unit norm under complex multiplication. Although a definable decomposition of this kind is not in general possible—the first example appears in [38]—the classical model has proved highly suggestive.

To continue, we require an o-minimal analogue of (topological) compactness introduced in [29].

Definition 4.11 Let $\mathcal{M}$ o-minimal. A definable set $X \subset M^n$ is *definably compact* if for every definable $f\colon (a, b) \to X$, both $\lim_{t \to a^+} f(t)$ and $\lim_{t \to b^-} f(t)$ exist in X.

In the classical setting, an abelian connected real Lie group decomposes into the direct sum of a torsion-free group (the copies of $\mathbb{R}_a$) and a compact group (the copies of $\mathbb{S}^1$). This suggests that if in the o-minimal context we replace "compact" by "definably compact" and consider definably compact and torsion-free groups separately, it still might be possible to extract much of the structure found classically even without the availability of a full decomposition.

To this end, observe first that $E((\mathbb{S}^1)^n) = 0$. Strzebonski [38] recognized that the Euler characteristic could be used as a substitute for counting and proved

Theorem 4.12 *If G is a definable group in an o-minimal structure $\mathcal{M}$ such that $E(G) = 0$, then G contains elements of every finite order.*

Hence, to show that a definable, definably compact abelian group G has elements of every finite order, it is good enough to show that $E(G) = 0$. This has now been established by different approaches involving homology and cohomology by Edmundo in [11] and Berarducci and Otero in [4]:

Theorem 4.13 *Let G be a definable, definably compact abelian group in an o-minimal structure $\mathcal{M}$ that is an expansion of a real closed field. Then $E(G) = 0$ and G has elements of all finite orders.*

Moreover, Edmundo goes on to prove in [11] that the number of torsion points of each finite order depends on the (o-minimal topological) dimension of the group exactly as it should in analogy with the classical context.

Lastly, all known examples illustrating the failure of a decomposition involve a definably compact group. This leaves open the possibility that a full decomposition into one-dimensional subgroups might be possible for a definable torsion-free abelian group. In [29] it was shown that an abelian definable group in an o-minimal structure which is not definably compact contains a definable one-dimensional torsion-free subgroup. For further developments see [28].

5 Vapnik-Chervonenkis dimension

In this section we briefly outline how o-minimality figures in an application to machine learning theory. The crux of the matter—and the key to related applications—comes down to a combinatorial property isolated by Vapnik and Chervonenkis that implies strong uniform convergence theorems in probability (see [42]).

Definition 5.1 A collection $\mathcal{C}$ of subsets of a set X *shatters* a finite subset F if $\{F \cap C \mid C \in \mathcal{C}\} = \mathcal{P}(F)$, where $\mathcal{P}(F)$ is the set of all subsets of F. The collection $\mathcal{C}$ is a *VC-class* if there is some $n \in \mathbb{N}$ such that no set F containing n elements is shattered by $\mathcal{C}$, and the least such n is the VC-dimension, $\mathcal{V}(\mathcal{C})$, of $\mathcal{C}$.

Let $\mathcal{C} \cap F = \{C \cap F \mid C \in \mathcal{C}\}$ and for $n = 1, 2, \dots$ let $f_{\mathcal{C}}(n) = \max\{|\mathcal{C} \cap F| \mid F \subset X \text{ and } |F| = n\}$. For a class $\mathcal{C}$ to be a VC-class, there must be some $n \in \mathbb{N}$ for which $f_{\mathcal{C}}(n) < 2^n$. Remarkably, this condition imposes a polynomial bound on $f_{\mathcal{C}}(n)$ for sufficiently large n.

Theorem 5.2 ("Sauer's Lemma" [34]) *Let $p_d(n) = \sum_{i<d} \binom{n}{i}$. Suppose that $f_{\mathcal{C}}(d) < 2^d$ for some d. Then $f_{\mathcal{C}}(n) \leq p_d(n)$ for all n.*

The connection between VC-classes and model theory was drawn by Laskowski [20]. His main result shows that a definable family of definable sets (in any structure) forms a VC-class if and only it does not satisfy the model-theoretic *independence property*. We do not develop this further except to note that it has long been known that definable families of definable sets in a model of a stable theory do not have the independence property, and, as was observed in [32], that this also holds true in the context of o-minimality. Thus we have:

Theorem 5.3 ([20]) *Let $\mathcal{M} = (M, <, \dots)$ be o-minimal and let $S \subset M^{n+k}$ be definable. Then the collection $\mathcal{C} = \{S_{\overline{x}} \mid \overline{x} \in M^n\}$ is a VC-class.*

To relate VC-classes to computational learning theory, we introduce the *probably approximately correct (PAC)* model of machine learning formulated by Valiant in [40]. Several excellent sources expositing PAC learning are available (e.g., see [1] and [17]), so we shall quickly and informally present just a simple version PAC learning.

By an *instance space* X we mean a set that is intended to represent all instances (or objects) in a learner's world. A *concept* c is a subset of X, which we can identify with the characteristic function $c\colon X \to \{0,1\}$. A *concept class* $\mathcal{C}$ is a collection of concepts.

A *learning algorithm* for the concept class $\mathcal{C}$ is a function L which takes as input m-tuples $((x_1, c(x_1)), \ldots, (x_m, c(x_m))$ for $m = 1, 2, \ldots$ and outputs *hypothesis concepts* $h \in \mathcal{C}$ that are consistent with the input. The m-tuples $((x_1, c(x_1)), \ldots, (x_m, c(x_m))$ are intended to represent training data to which the learner is exposed. If X is equipped with a probability distribution, then we can define the *error* of h to be $\mathrm{err}\,(h) = P(h \triangle c)$, where $h \triangle c$ is the symmetric difference of h and c. With this terminology in place, we can state:

Definition 5.4 The learning algorithm L is said to be *PAC* if for every $\epsilon, \delta \in (0,1)$ there is an $m_L(\epsilon, \delta)$ such that for *every* probability distribution P on X and every concept $c \in \mathcal{C}$, we have for all $m \geq m_{L(\epsilon,\delta)}$ that

$$P(\, \{\overline{x} \in X^m \,|\, \mathrm{err}(L((x_i, c(x_i))_{i \leq m}) \leq \epsilon\} \,) \geq 1 - \delta\,.$$

In words, the definition of PAC learning asserts that for a sufficiently large set of training data, the output of the learning algorithm has error no greater than ϵ with probability at least $1 - \delta$.

It is important to observe that PAC learning is independent of both the probability distribution on X and the target concept c. We further say that $\mathcal{C}$ is *PAC learnable* if there is a PAC learning algorithm L for $\mathcal{C}$.

It follows from the results in [41] and [3] that an algorithm that outputs a hypothesis concept h consistent with the sample data is PAC provided that $\mathcal{C}$ is a VC-class. Moreover, for given ϵ and δ, the number of sample points needed is, roughly speaking, proportional to the VC-dimension $\mathcal{V}(\mathcal{C})$. Putting everything together, it follows that a definable family of definable sets $\mathcal{C}$ in an o-minimal structure is PAC learnable.

We close by noting that PAC learning is directly relevant to neural networks whose architecture defines concept classes that consist of definable families of definable sets in o-minimal structures. Substantial work has been done to compute lower and upper bounds for VC-dimension of such neural networks; we refer the reader to [16] and [19].

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Covering definable open sets by open cells

A.J. Wilkie

Abstract

Let $\mathcal{M}=(M,\leq,+,\cdot,\ldots)$ be an o-minimal expansion of a real closed field. I show that any $\mathcal{M}$-definable, bounded open subset of M^n is the union of finitely many open cells.

Perhaps the result mentioned above is well-known. However, I cannot find it in the standard literature (eg. [1]) and as I have been asked on several occasions whether or not it is true (or, rather, been told that it must be true and asked for a suitable reference), it seems worthwhile publishing a proof.

For the moment, let $\mathcal{M}=(M,\leq,\ldots)$ be any o-minimal structure. All sets and functions below are assumed to be $\mathcal{M}$-definable *without* parameters (thereby implying that our constructions are uniform in parameters).

1 Covering definable open sets by open cells

Lemma 1.1 *Let C be a cell in M^n. Then there exists an open cell D in M^n with $C \subseteq D$ and a retraction $H : D \to C$ (i.e. a continuous map such that $H(\overline{x}) = \overline{x}$ for all $\overline{x} \in C$).*

Proof. We proceed by induction on n. For $n = 1$ take $D = C$ and $H = id_C$ if C is an open interval, or $D = (-\infty, \infty)$ and $H(x) = a$ (for all $x \in M$) if $C = \{a\}$. (NB. The function H *is* definable without parameters provided that the cell C is. It is left to the reader to formulate and check the corresponding property for the constructions below.)

Suppose that C is a cell in M^{n+1}.

Case 1: $C = (f, g)_{C'}$ (in the usual notation-see [1]) for some continuous functions $f, g : C' \to M \cup \{-\infty, \infty\}$, where C' is a cell in M^n and where $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in C'$.

By the inductive hypothesis choose an open cell, D' say, in M^n with $C' \subseteq D'$ and a retraction $H' : D' \to C'$. Set $D = (f \circ H', g \circ H')_{D'}$ and define $H : D \to C$ by $H(\overline{x}, y) = (H'(\overline{x}), y)$ (for $\overline{x} \in D'$ and $f \circ H'(\overline{x}) < y < g \circ H'(\overline{x})$).

Case 2: $C = \text{graph}(f)$ for some continuous $f : C' \to M$, where C' is a cell in M^n.

By the inductive hypothesis let D' be an open cell in M^n with $C' \subseteq D'$ and let $H' : D' \to C'$ be a retraction. Set $D = (-\infty, \infty)_{D'}$ and define $H : D \to C$ by $H(\overline{x}, y) = (H'(\overline{x}), f(H'(\overline{x})))$ (for $\overline{x} \in D'$, $y \in M$).
$\square$

I now assume that $\mathcal{M}$ is an expansion of a real closed field. The usual euclidean distance on M^n is denoted $d^{(n)}(\cdot, \cdot)$, where the arguments may be either elements or subsets of M^n.

Lemma 1.2 *Let C be a cell in M^n. Suppose that $f, g : C \to M$ are continuous functions with $f(\overline{x}) < g(\overline{x})$ for all $\overline{x} \in C$ and let U be a bounded, open subset of M^{n+1}. Suppose further that $(f, g)_C \subseteq U$ and that* graph$(f) \subseteq U$ *(respectively* graph$(g) \subseteq U$*). Then there exists an open subset V of M^n and continuous functions $F, G : V \to M$ such that*

(i) $C \subseteq V$;

(ii) $F|_C = f$ and graph$(F) \subseteq U$ *(respectively* graph$(G) \subseteq U$*);*

(iii) $G|_C = g$;

(iv) for all $\overline{x} \in V$, $F(\overline{x}) < G(\overline{x})$;

(v) for all $\overline{x} \in V$ and all $y \in M$ with $F(\overline{x}) \leq y < G(\overline{x})$, (respectively $F(\overline{x}) < y \leq G(\overline{x})$), $(\overline{x}, y) \in U$.

Proof. We prove the unparenthesized statement, the parenthetical one being similar (or, in fact, may be deduced by a suitable inversion).

Apply Lemma 1.1 to obtain an open cell D in M^n, with $C \subseteq D$, and a retraction $H : D \to C$.

Let $V = \{\overline{x} \in D : d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, f \circ H(\overline{x})), U^c)\}$, where $U^c = M^{n+1} \setminus U$. Clearly V is open in M^n and (i) is satisfied since graph$(f) \subseteq U$. Setting $F = f \circ H|_V$ clearly guarantees (ii).

In order to define G we first note that $F(\overline{x}) < g \circ H(\overline{x})$ for all $\overline{x} \in V$. Now fix $\overline{x} \in V$ and define $U_{\overline{x}} = \{y \in M : (\overline{x}, y) \in U\}$. Using o-minimality and the fact that $F(\overline{x}) \in U_{\overline{x}}$, let $y_0 = y_0(\overline{x})$ be the unique element of M satisfying $F(\overline{x}) < y_0 \leq g \circ H(\overline{x})$, $y_0 \notin U_{\overline{x}}$ and $[F(\overline{x}), y_0) \subseteq U_{\overline{x}}$ if $[F(\overline{x}), g \circ H(\overline{x})] \nsubseteq U_{\overline{x}}$, or $y_0 = g \circ H(\overline{x})$ if $[F(\overline{x}), g \circ H(\overline{x})] \subseteq U_{\overline{x}}$. Now notice that the function $y_0 : V \to M$ satisfies the conditions (iii), (iv) and (v) for G ((iii) is satisfied because $(f, g)_C \subseteq U$, by hypothesis, and $f = F|_C$), but it might not be continuous. It will therefore suffice to find a continuous $G : V \to M$ such that for all $\overline{x} \in V$, $F(\overline{x}) < G(\overline{x}) \leq y_0(\overline{x})$ and satisfying $G(\overline{x}) = y_0(\overline{x})$ whenever $\overline{x} \in C$.

To this end let $S = \{(\overline{x}, y) \in M^{n+1} : \overline{x} \in V \text{ and } F(\overline{x}) \leq y \leq g \circ H(\overline{x})\}$ and define continuous functions $\theta_1, \theta_2 : S \to M$ by

$$\begin{aligned}
&\theta_1(\overline{x}, y) = 1 - \tfrac{y - F(\overline{x})}{g \circ H(\overline{x}) - F(\overline{x})}, \\
&\theta_2(\overline{x}, y) = \inf\{d^{(n+1)}((\overline{x}, t), U^c) : F(\overline{x}) \leq t \leq y\}.
\end{aligned}$$

Then for each $\overline{x} \in V$, $\theta_1(\overline{x}, \cdot)$ decreases strictly monotonically from 1 to 0 on $[F(\overline{x}), g \circ H(\overline{x})]$ and $\theta_2(\overline{x}, \cdot)$ is positive and decreases (possibly non- strictly) on $[F(\overline{x}), y_0(\overline{x}))$, with initial value $d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$, and is identically zero on $[y_0(\overline{x}), g \circ H(\overline{x})]$ unless, possibly, $y_0(\overline{x}) = g \circ H(\overline{x})$. However, in all cases the product of these functions, $(\theta_1 \cdot \theta_2)(\overline{x}, \cdot)$, certainly does decrease monotonically and strictly from $d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$ to 0 on $[F(\overline{x}), y_0(\overline{x})]$ and is identically zero on $[y_0(\overline{x}), g \circ H(\overline{x})]$. Since all functions involved here are continuous and since $d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$ (by definition of V), it follows from the Intermediate Value Theorem in $\mathcal{M}$ that there is a unique $y_1 = y_1(\overline{x})$ such that $(\overline{x}, y_1) \in S$ and $d^{(n)}(\overline{x}, H(\overline{x})) = (\theta_1 \cdot \theta_2)(\overline{x}, y_1)$. I claim that $F(\overline{x}) < y_1(\overline{x}) \leq y_0(\overline{x})$. For this is clear if $d^{(n)}(\overline{x}, H(\overline{x})) > 0$, and if $d^{(n)}(\overline{x}, H(\overline{x})) = 0$ then $\overline{x} \in C$, and then $(\theta_1 \cdot \theta_2)(\overline{x}, \cdot)$ is non-vanishing on $[F(\overline{x}), g \circ H(\overline{x}))$, forcing $y_1(\overline{x}) = g \circ H(\overline{x}) = y_0(\overline{x}) = g(\overline{x}) > f(\overline{x}) = F(\overline{x})$.

It follows, as discussed above, that the function $G : V \to M$ defined by $G(\overline{x}) = y_1(\overline{x})$ satisfies (iii), (iv) and (v). But because of the uniqueness in the condition determining y_1, we see that

$$\text{graph}(G) = \{(\overline{x}, y) \in V \times M : (\overline{x}, y) \in S, d^{(n)}(\overline{x}, H(\overline{x})) = (\theta_1 \cdot \theta_2)(\overline{x}, y)\},$$

which shows that graph(G) is closed in $V \times M$. But the fact that $F(\overline{x}) < G(\overline{x}) \leq y_0(\overline{x})$ (for all $\overline{x} \in V$)) shows that graph(G) is a subset of the closure of U in M^{n+1} and is therefore bounded (because U is, by hypothesis). It follows that $G : V \to M$ is continuous (see [1], p 103) as required. $\square$

We can now prove the main result of this note.

Theorem 1.3 *Let U be a bounded, open subset of M^n. Then there exists a finite collection of open cells in M^n whose union is U. (Recall our assumption that all sets and functions are $\mathcal{M}$-definable without parameters.)*

Proof. The case $n = 1$ being clear (and, in fact, not requiring U to be bounded), we proceed by induction on n.

Let U be a bounded, open subset of M^{n+1}.

Let $\mathcal{S}$ be a cell decomposition of M^{n+1} compatible with U. I show that each cell $D \in \mathcal{S}$ with $D \subseteq U$ can be covered by finitely many open cells (in M^{n+1}) each of which is contained in U. This is obviously sufficient.

Case 1: $D = (f_1, f_2)_C$ for some cell C in M^n and continuous functions $f_1, f_2 : C \to M$ (with $f_1(\overline{x}) < f_2(\overline{x})$ for all $\overline{x} \in C$).

Let $h_1 = \frac{2f_1+f_2}{3}$ and $h_2 = \frac{f_1+2f_2}{3}$. Then $h_1, h_2 : C \to M$ are continuous functions such that for all $\overline{x} \in C$, $f_1(\overline{x}) < h_1(\overline{x}) < h_2(\overline{x}) < f_2(\overline{x})$. Therefore graph($h_1$) $\subseteq U$ and $(h_1, f_2)_C \subseteq U$, so we may apply Lemma 1.2 (with $f = h_1, g = f_2$) to obtain an open subset V of M^n and continuous functions $F, G : V \to M$ with properties (i)-(v).

Clearly V must be bounded (by (iii)), so by the inductive hypothesis there exists a finite collection, $\mathcal{C}$ say, of open cells in M^n, with $\bigcup \mathcal{C} = \mathcal{V}$. By (iv), (v), for each $A \in \mathcal{C}$, $(F|_A, G|_A)_A$ is an open cell in M^{n+1} contained in U, and by (i), (ii) and (iii) of Lemma 1.2, $(h_1, f_2)_C \subseteq \bigcup\{(F|_A, G|_A)_A : A \in \mathcal{C}\}$.

Similarly, using the paranthetical statement in Lemma 1.2, $(f_1, h_2)_C$ can be covered by finitely many open cells in M^{n+1} each of which is contained in U. The same is therefore true for $(h_1, f_2)_C \cup (f_1, h_2)_C = (f_1, f_2)_C$.

Case 2: $D =$ graph(h) for some continuous function $h : C \to M$ where C is a cell in M^n.

This case follows from Case 1 because, since U is open, there must exist continuous $f_1, f_2 : C \to M$ such that $f_1(\overline{x}) < h(\overline{x}) < f_2(\overline{x})$ (for all $\overline{x} \in C$) and such that $(f_1, h)_C \subseteq U$, $(f_1, h)_C \in \mathcal{S}$, and similarly for $(h, f_2)_C$. Now apply Case 1 to the cell $(f_1, f_2)_C$. $\square$

It is easy to construct an example showing that Theorem 1.3 is false without the assumption that U be bounded. However, I do not know if the assumption that $\mathcal{M}$ is an expansion of a field can be weakened.

References

[1] L. van den Dries, *Tame topology and o-minimal structures*, London Math. Soc. Lecture Note Ser., vol. 248, Cambridge: Cambridge University Press 1998.

O-minimal (co)homology and applications

Mário J. Edmundo[2]

Abstract

Here we give a brief survey of o-minimal homology and cohomology theory with emphasis on the applications to o-minimal generalizations of the Jordan-Brouwer separation theorem and to degree theory for definable continuous maps.

Below we will assume that $\mathcal{N}$ is an o-minimal expansion of an ordered ring $(N, 0, 1, +, \cdot, <)$. Note that, by [15], this ordered ring is necessarily a real closed field. Finally we point out that we do not assume that the first-order theory $\mathrm{Th}(\mathcal{N})$ of $\mathcal{N}$ has a model with the order type of the real numbers.

This contribution is organized in the following way. In Section 1 we present the definition of o-minimal homology and cohomology. These are defined using the Eilenberg-Steenrod axioms adapted to the definable category. The o-minimal Eilenberg-Steenrod axioms imply the Mayer-Vietoris sequence (see Subsection 1.2) and o-minimal analogues of several well knwon results from classical topology (see Subsection 1.3).

The existence of an o-minimal simplicial and singular homology theory was proved by Woerheide in [17] and is discussed in Section 2 where we also include some comments on the o-minimal simplicial and singular cohomology theory together with the corresponding products. For further details on o-minimal (co)homology the reader can see also [11].

In Section 3 we include the first non-trivial applications of o-minimal (co)homology namely, the o-minimal Jordan-Brouwer separation theorem proved by Woerheide ([17]) and its generalizations and the theory of degrees of definable continuous maps.

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1 O-minimal (co)homology

Here we will define the notion of o-minimal homology and o-minimal cohomology. The first notion is from [17].

1.1 O-minimal (co)homology

Let C be either the category whose objects are pairs of definable sets and whose morphisms are continuous definable maps or the full subcategory of pairs of closed and bounded definable sets. So an object of C is a pair (X, A) of definable sets with $A \subseteq X$ and, a morphism of C is a map $f : (X, A) \longrightarrow (Y, B)$ with $f : X \longrightarrow Y$ continuous and definable and $f(A) \subseteq B$.

Below, R is a ring and $G : C \longrightarrow C$ is the functor that sends $(X, A) \in \mathrm{Obj}C$ into $(A, \emptyset) \in \mathrm{Obj}C$ and sends $f : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ into $f_| : (A, \emptyset) \longrightarrow (B, \emptyset) \in \mathrm{Mor}C$.

Definition 1.1 A *homology* (H_*, d_*) *on* C is a sequence $(H_n)_{n\in\mathbb{Z}}$ of covariant functors from C into the category of R-modules together with a sequence $(d_n)_{n\in\mathbb{Z}}$ of natural transformations $d_n : H_n \longrightarrow H_{n-1} \circ G$ such that the following axioms hold.

Homotopy Axiom. If $f, g : (X, A) \longrightarrow (Y, B) \in \mathrm{Mor}C$ and there is a definable homotopy in C between f and g, then

$$H_n(f) = H_n(g) : H_n(X, A) \longrightarrow H_n(Y, B)$$

for all $n \in \mathbb{Z}$.

Exactness Axiom. If $i : (A, \emptyset) \longrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \longrightarrow (X, A)$ are the inclusions in $\mathrm{Mor}C$, then the following sequence is exact.

$$H_n(A, \emptyset) \overset{H_n(i)}{\to} H_n(X, \emptyset) \overset{H_n(j)}{\to} H_n(X, A) \overset{d_n}{\to} H_{n-1}(A, \emptyset)$$

Excision Axiom. For every $(X, A) \in \mathrm{Obj}C$ and every definable open subset U of X such that $\overline{U} \subseteq \mathring{A}$ and $(X - U, A - U) \in \mathrm{Obj}C$, the inclusion $(X - U, A - U) \longrightarrow (X, A)$ induces isomorphisms

$$H_n(X - U, A - U) \longrightarrow H_n(X, A)$$

for all $n \in \mathbb{Z}$.

Dimension Axiom. If X is a one point set, then $H_n(X, \emptyset) = 0$ for all $n \neq 0$. The R-module $H_0(X, \emptyset)$ is called the coefficient R-module.

These axioms are the analogues of the classical Eilenberg-Steenrod axioms for homology functors. We therefore call them the o-minimal Eilenberg-Steenrod axioms.

We will write $X \in \mathrm{Obj}C$ for $(X,\emptyset) \in \mathrm{Obj}C$, $f : X \longrightarrow Y \in \mathrm{Mor}C$ for $f : (X,\emptyset) \longrightarrow (Y,\emptyset) \in \mathrm{Mor}C$ and $H_n(X)$ for $H_n(X,\emptyset)$. Moreover, if M is the coefficient R-module of (H_*, d_*), we will write $H_n(X,A;M)$ instead of just $H_n(X,A)$ and the notation $H_n(X,A)$ will be used for $H_n(X,A;\mathbb{Z})$.

Definition 1.2 A *cohomology* (H^*, d^*) *on* C is a sequence $(H^n)_{n\in\mathbb{Z}}$ of contravariant functors from C into the category of R-modules together with a sequence $(d^n)_{n\in\mathbb{Z}}$ of natural transformations $d^n : H^n \circ G \longrightarrow H^{n+1}$ such that the following axioms hold.

Homotopy Axiom. If $f, g : (X,A) \longrightarrow (Y,B) \in \mathrm{Mor}C$ and there is a definable homotopy in C between f and g, then

$$H^n(f) = H^n(g) : H^n(Y,B) \longrightarrow H^n(X,A)$$

for all $n \in \mathbb{Z}$.

Exactness Axiom. If $i : (A,\emptyset) \longrightarrow (X,\emptyset)$ and $j : (X,\emptyset) \longrightarrow (X,A)$ are the inclusions in $\mathrm{Obj}C$, then the following sequence is exact.

$$H^l(X,A) \overset{H^l(j)}{\to} H^l(X,\emptyset) \overset{H^l(i)}{\to} H^l(A,\emptyset) \overset{d^l}{\to} H^{l+1}(X,A)$$

Excision Axiom. For every $(X,A) \in \mathrm{Obj}C$ and every definable open subset U of X such that $\overline{U} \subseteq \mathring{A}$ and $(X-U, A-U) \in \mathrm{Obj}C$, the inclusion $(X-U, A-U) \longrightarrow (X,A)$ induces isomorphisms

$$H^n(X,A) \longrightarrow H^n(X-U, A-U)$$

for all $n \in \mathbb{Z}$.

Dimension Axiom. If X is a one point set, then $H^n(X,\emptyset) = 0$ for all $n \neq 0$. The R-module $H^0(X,\emptyset)$, is called the coefficient R-module.

For o-minimal cohomology theories we will use conventions similar to those we introduced above for o-minimal homology theories.

1.2 The Mayer-Vietoris sequence

The results we present below are consequence of the axioms for a homology (H_*, d_*) on C with coefficients in M. The proofs are purely algebraic and we refer the reader to the proofs in the classical case.

Proposition 1.3 (Exactness for triples) *If we have inclusions*

$$(A, \emptyset) \xrightarrow{c} (A, B) \xrightarrow{a} (X, B) \xrightarrow{b} (X, A)$$

in C, then there is an exact sequence for all $n \in \mathbb{Z}$

$$\longrightarrow H_n(A, B; M) \xrightarrow{a_*} H_n(X, B; M) \xrightarrow{b_*} H_n(X, A; M) \xrightarrow{c_* \circ d_n}$$

$$\xrightarrow{c_* \circ d_n} H_{n-1}(A, B; M) \longrightarrow .$$

For the proof of Proposition 1.3 see [14] Chapter 4, Section 5.

Definition 1.4 If $(X, A), (X, B) \in \mathrm{Obj}C$, then we call $(X; A, B)$ a *triad in* C. We say that a triad $(X; A, B)$ in C is an *excisive triad in C with respect to* (H_*, d_*) if the inclusion $(A, A \cap B) \longrightarrow (A \cup B, B)$ induces isomorphisms $H_*(A, A \cap B; M) \simeq H_*(A \cup B, B; M)$.

Proposition 1.5 *A triad $(X; A, B)$ in C is an excisive triad if and only if the triad $(X; B, A)$ in C is an excisive triad.*

For the proof of Proposition 1.5 see [16] Lemma 7.13. The next result is in fact equivalent to the excision axiom for (H_*, d_*). For a proof see [9] Theorem 4.3.7.

Proposition 1.6 *If $(X; A, B)$ is a triad in C such that $X = \mathring{A} \cup \mathring{B}$, then $(X; A, B)$ is an excisive triad in C.*

Finally, we present the o-minimal analogue of the classical Mayer-Vietoris theorem. For a proof see [14] Chapter 4, Section 5 (the proof there is for a special case, but the same argument applies to our more general case, see also [8] Chapter III, Section 8). A special case of this result already appears in [17].

Proposition 1.7 (Mayer-Vietoris) *Let* $(X; X_1, X_2)$ *and* $(Z; Z_1, Z_2)$ *be excisive triad in* C *such that* $X = X_1 \cup X_2$, $Z = Z_1 \cup Z_2$ *and we have the following commutative diagram of inclusions in* C

$$\begin{array}{ccc} (X_1 \cap X_2, Z_1 \cap Z_2) & \xrightarrow{i_1} & (X_1, Z_1) \\ \downarrow^{i_2} & & \downarrow^{j_1} \\ (X_2, Z_2) & \xrightarrow{j_2} & (X, Z). \end{array}$$

Then there is an exact sequence for all $n \in \mathbb{Z}$

$$\longrightarrow H_{n+1}(X, Z; M) \xrightarrow{d_*} H_n(X_1 \cap X_2, Z_1 \cap Z_2; M) \xrightarrow{(j_{1*}, -j_{2*})} H_n(X_1, Z_1; M) \oplus H_n(X_2, Z_2; M) \xrightarrow{i_{1*}+i_{2*}} H_n(X, Z; M) \xrightarrow{d_*} H_{n-1}(X_1 \cap X_2, Z_1 \cap Z_2; M) \longrightarrow .$$

Note that there are o-minimal cohomology analogues of all the results mentionted above.

1.3 Some applications

We end this section with some applications of the o-minimal homology axioms. The results we now present are the o-minimal analogues of well know results from topology.

Using as in classical case the Mayer-Vietoris sequence we obtain the following result. See [17].

Example 1.8 For $n \in \mathbb{N}$, let $\mathbb{S}^n$ be the unit n-sphere in N^{n+1} and let B_n be open unit ball in N^n. As in the classical case, we have $H_0(\mathbb{S}^0) = \mathbb{Z} \oplus \mathbb{Z}$, $H_k(\mathbb{S}^n) = \mathbb{Z}$ for $n > 0$ and $k = 0, n$, and $H_k(\mathbb{S}^n) = 0$ otherwise.

An easy consequence of Example 1.8 is the following corollary which is immediate once we identify N^k with $\mathbb{S}^k$ minus a point.

Corollary 1.9 *If* $n \neq m$, *then* $\mathbb{S}^n$ *and* $\mathbb{S}^m$, *whence* N^n *and* N^m, *are not definably homeomorphic. Also* $\mathbb{S}^{n-1}$ *is not a definable retraction of the closure* $\overline{B}_n$ *of* B_n.

Corollary 1.9 has the following application. The proof here is the same as that of its classical analogue in [8] Chapter IV, Section 2.

Proposition 1.10 *If $f : \overline{B}_n \longrightarrow N^n$ is a definable continuous map, then either $f(x) = 0$ for some $x \in \overline{B}_n$ or $f(x) = \lambda x$ for some $x \in \mathbb{S}^{n-1}$ and $\lambda > 0$.*

Proof. This follows from the fact that the definable map $\rho : \overline{B}_n \longrightarrow N^n$ given by $\rho(x) = (2|x|-1)-(2-2|x|)f(\frac{x}{|x|})$ for $2|x| \geq 1$ and $\rho(x) = -f(4|x|x)$ for $2|x| \leq 1$, must have a zero on B_n, for otherwise, the definable map $r : \overline{B}_n \longrightarrow \mathbb{S}^{n-1}$ given by $r(x) = \frac{\rho(x)}{|\rho(x)|}$ is a definable retract. $\square$

Proposition 1.10 implies the o-minimal analogue of the Brouwer fixed point theorem when we take $f = g - 1_{\overline{B}_n}$ where $g : \overline{B}_n \longrightarrow N^n$ is a definable continuous map.

Corollary 1.11 *If $g : \overline{B}_n \longrightarrow N^n$ is a definable continuous map, then either $g(x) = x$ for some $x \in \overline{B}_n$ or $g(x) = \lambda x$ for some $x \in \mathbb{S}^{n-1}$ and $\lambda > 1$.*

The Brouwer fixed point theorem for definable C^p-maps with $p \geq 1$ was proved in [1]. Note also that both Proposition 1.10 and Corollary 1.11 can be expressed in first-order logic. Hence, if $\mathrm{Th}(\mathcal{N})$ has a model on the real numbers, then they both follow from their classical analogues. For another o-minimal fixed point theorem see [2] or [3].

Given a definable continuous map $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$, the *degree* of f, denoted $\deg f$, is the integer determining the homomorphism $f_* : H_n(\mathbb{S}^n) \longrightarrow H_n(\mathbb{S}^n)$.

In the semi-algebraic case the results below can be obtained by tranfer from $\mathbb{R}$. This was first pointed out in [7]. In the o-minimal case we can tranfer the classical proofs.

Proposition 1.12 *Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a continuous definable maps. Then the following hold:*

(i) If f has no fixed points, then $\deg f = (-1)^{n+1}$.

(ii) If f has no antipodal points (i.e., $f(x) \neq -x$ for all $x \in \mathbb{S}^n$), then $\deg f = 1$.

In particular, if n is even, then f has a fixed point or an antipodal point.

Proof. For (i) see [9] 4.3.29. For (ii) apply (i) to the definable map $f \circ T$, where $T : \mathbb{S}^n \longrightarrow \mathbb{S}^n$, $T(x) = -x$ is the antipodal map. $\square$

Theorem 1.13 (Borsuk's antipodal) *If $f : \mathbb{S}^n \longrightarrow \mathbb{S}^m$ is a definable continuous map such that $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$, then* $\deg f$ *is odd.*

The proof of Theorem 1.13 is similar to that of its classical analogue [9] 4.3.32. For the proof of the following corollary of Theorem 1.13 we refer the reader to [9] 4.3.2.

Corollary 1.14 *The following results hold.*

(Antipodal subsphere) *If $f : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ is a definable continuous map such that $f(-x) = -f(x)$ for all $x \in \mathbb{S}^n$, then $m \leq n$.*

(Borsuk-Ulam Theorem) *If $f : \mathbb{S}^n \longrightarrow N^n$ is a definable continuous map, then there is a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$.*

(Gift wrap) *No definable subset of N^n can be definably homeomorphic to $\mathbb{S}^n$.*

(First hairy ball) *There is a continuous nowhere zero tangent vector field on $\mathbb{S}^n$ if and only if n is odd.*

(Lusternik-Schnirelmann) *If $\mathbb{S}^n$ is covered by $n+1$ closed definable sets $A_1, \ldots, A_{n+1}$, then one of the A_i contains an antipodal pair of points.*

Note that if $\mathrm{Th}(\mathcal{N})$ has a model on the real numbers, then Corollary 1.14 can be deduced from its classical analogue using first-order logic. Also we point out that degrees can be use to develop the theory of winding numbers is the o-minimal context just like in the classical case (see [12] Chapter 3 and 4).

2 Existence of o-minimal (co)homology

In the semi-algebraic case, Delfs constructed in [4] (a simpler proof appears in [5]) the semi-algebraic sheaf cohomology. In [6] Delfs defines the semi-algebraic Borel-Moore homology. Semi-algebraic simplicial and singular (co)homology were constructed by Delfs and Knebusch in [7] (see also [5]) based on the semi-algebraic sheaf cohomology.

For o-minimal expansions of real closed fields, Woerheide gives a direct construction of the o-minimal simplicial and singular homology with coefficients in $\mathbb{Z}$ in [17]. This construction easily gives, as in the classical case treated in [8], the o-minimal simplicial and singular homology and cohomology with arbitrary constant coefficients.

Woerheide's results are based on the definable triangulation theorem and on the method of acyclic models from homological algebra and are rather complicated due to the fact that, in arbitrary o-minimal expansions of fields, the classical simplicial approximation theorem and the method of repeated barycentric subdivisions and the Lebesgue number property for the standard simpleces Δ^n fail.

We give here a brief description of Woerheide's constructions. Like in the case for simplicial homology over $\mathbb{R}$, the o-minimal simplicial homology groups $H_*(X)$ for X a definably compact definable set are defined using the *simplicial chain complex* $(C_*(K), \partial_*)$ where K be a closed simplicial complex in N^n obtained from a definable triangulation of X. The main complication is defining the induced homomorphisms between the homology groups and verifying that is definition is independent of the definable triangulation.

We now turn to A. Woerheide ([17]) definition of the o-minimal singular homology on the category of pairs of definable sets. In this case the construction is essentially the same as for the standard singular homology, only with the word "definable" added here and there. But the standard proof of the excision axiom fails and the difficulty is avoided by the use of the o-minimal triangulation theorem and the results obtained while constructing the simplicial homology.

Definition 2.1 The *standard n-simplex* Δ^n over N is the convex hull of the standard basis vectors $e_0, \dots, e_n$ in N^{n+1}. Let the standard (-1)-simplex Δ^{-1} be the empty set.

Let X be a definable set. For $n \geq -1$, we define $\widetilde{S}_n(X)$ to be the free abelian group on the set of definable continuous maps $\sigma : \Delta^n \longrightarrow X$. For $n < -1$, we set $\widetilde{S}_n(X) = 0$. Note that $\widetilde{S}_{-1}(X) = \mathbb{Z}$. The elements of $\widetilde{S}_n(X)$ are called the definable n-chains.

For $n > 0$ and $0 \leq i \leq n$ let $\epsilon_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ be the continuous definable map given by $\epsilon_i^n(\sum_{j=0}^{n-1} a_j e_j) = \sum_{j<i} a_j e_j + \sum_{j \geq i}^{n-1} a_j e_{j+1}$. Let $\epsilon_0^0 : \Delta^{-1} \longrightarrow \Delta^0$ be the unique map. We define the boundary homomorphism $\partial_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_{n-1}(X)$ to be the trivial homomorphism for $n < 0$ and for $n \geq 0$, ∂_n is given on basis elements by

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ \epsilon_i^n.$$

One verifies that $\partial^2 = 0$ and so $(\widetilde{S}_*(X), \partial_*)$ is a chain complex, the *augmented o-minimal singular chain complex.*

Given an element $\sigma = \sum_{i=1}^{k} n_i\sigma_i$ of $S_m(X)$ where $\sigma_i : \Delta^m \longrightarrow X$ are definable continuous maps, the *support* $\mathrm{Im}\sigma$ of σ is by definiton the definable subset $\cup\{\sigma_i(\Delta^m) : i = 1, \ldots, k\}$ of X.

Definition 2.2 Suppose that (X, A) is a pair of definable sets. Then the *relative o-minimal singular chain complex* $(S_*(X, A), \partial_*)$ is the quotient chain complex $(\widetilde{S}_*(X)/\widetilde{S}_*(A), \overline{\partial_*})$. We define the o-minimal singular chain complex $(S_*(X), \partial_*)$ to be $(S_*(X, \emptyset), \partial_*)$.

For $f : (X, A) \longrightarrow (Y, B)$ a definable continuous map, we have an induced chain map $f_\sharp : S_*(X, A) \longrightarrow S_*(Y, B)$ given on the basis elements of $\widetilde{S}_*(X)$ by $f_\sharp(\sigma) = f \circ \sigma$. We define $H_n(X, A) = H_n(S_*(X, A))$, $H_n(f) = H_n(f_\sharp)$ and set $\widetilde{H}_n(X) = H_n(\widetilde{S}_*(X))$ and $H_n(X) = H_n(X, \emptyset)$.

Using the methods mentioned at the begining of this subsection, the following result is proved in [17].

Theorem 2.3 *The sequence of functors defined in Definition 2.2 determines a homology (H_*, d_*) for the category of pairs of definable sets with coefficients in $\mathbb{Z}$, called the o-minimal singular homology.*

We make now a few comments comparing the classical proof of the excision axiom and Woerhiede proof of the o-minimal excision axiom.

For $z \in \widetilde{S}_*(X)$ with $z = \sum_{j=1}^{l} a_j\alpha_j$ we have a chain map $z_\sharp : \widetilde{S}_*(\Delta^n) \longrightarrow \widetilde{S}_*(X)$ given by

$$z_\sharp\beta = \sum_{i,j} a_j b_i(\alpha_j \circ \beta_i)$$

where $\beta = \sum_{i=1}^{k} b_i\beta_i$.

Let X be a definable set. The *barycentric subdivision*

$$\mathrm{Sd}_n : \widetilde{S}_n(X) \longrightarrow \widetilde{S}_n(X)$$

is defined as follows: for $n \leq -1$, Sd_n is the trivial homomorphism, Sd_{-1} is the identity and, for $n \geq 0$, we set

$$\mathrm{Sd}_n(z) = z_\sharp(b_n.\mathrm{Sd}_{n-1}\partial 1_{\Delta^n})$$

where b_n is the barycentre of Δ^n. Here we use the cone construction which is defined in the following way. Let $X \subseteq N^m$ be a convex definable set and let $p \in X$. The *cone construction over* p *in* X is a sequence of homomorphisms $z \mapsto p.z$: $\widetilde{S}_*(X) \longrightarrow \widetilde{S}_{*+1}(X)$ defined as follows: For $n < -1$, $p.$ is defined as the trivial homomorphism and for $n \geq -1$ and a basis element σ, we set $p.\sigma(\sum_{i=0}^{n+1} t_i e_i)= p$ if $t_0 = 1$ or $t_0 p + (1-t_0)\sigma(\sum_{i=1}^{n+1} \frac{t_i}{1-t_0} e_i)$ if $t_0 \neq 1$.

In the classical case we apply the Lebesgue number property to the repeated barycentric subdivision operator

$$\mathrm{Sd}^k = (\mathrm{Sd}^k_n)_{n\in\mathbb{Z}} : \widetilde{S}^{top}_*(X) \longrightarrow \widetilde{S}^{top}_*(X)$$

where Sd^k is the composition of Sd with itself k times, to prove the following lemma.

Lemma 2.4 *Suppose that* X *is a topological space and let* U *and* V *be open subsets of* X *such that* $X = U \cup V$. *If* $z \in \widetilde{S}^{top}_n(X)$, *then there is a sufficiently large* $k \in \mathbb{N}$ *such that* $\mathrm{Sd}^k_n(z) \in \widetilde{S}^{top}_n(U) + \widetilde{S}^{top}_n(V)$.

This lemma implies the excision axiom. In the o-minimal case Woerheide replaces Sd^k by the *subdivision operator*

$$\mathrm{Sd}^K_i : \widetilde{S}_i(X) \longrightarrow \widetilde{S}_i(X)$$

where (Φ, K) is a definable triangulation of X. The subdivision operator is defined by

$$\mathrm{Sd}^K_i(z) = (\mathrm{Sd}z)_\sharp(\gamma^n_i)_\sharp(\Phi^{-1})_\sharp \tau_K F_n\langle e_{n-i}, \ldots, e_n\rangle$$

where $F_n : \widetilde{C}_*(E^n) \longrightarrow \widetilde{C}_*(K)$ is the o-minimal simplicial chain map induced by $\Phi : E^n \longrightarrow K$ and $\gamma^n_i : \Delta^n \longrightarrow \Delta^i$ is defined by

$$\gamma^n_i(\sum_{j=0}^{n} a_j e_j) = \sum_{j=0}^{i}(a_{n-i+j} + \frac{\sum_{k=0}^{n-i-1} a_k}{i+1})e_j$$

and E^n is the standard simplicial complex such that $|E^n| = \Delta^n$.

Woerheide proves the following lemma which, as in the classical case, implies the o-minimal excision axiom.

Lemma 2.5 *Suppose that X is a definable set and let U and V be open definable subsets of X such that $X = U \cup V$. If $z \in \widetilde{S}_n(X)$, then there is a definable triangulation (Φ, K) of Δ^n compatible with E^n such that $\mathrm{Sd}_n^K(z) \in \widetilde{S}_n(U) + \widetilde{S}_n(V)$.*

Finally, we remark that Woerheide constructions and standard arguments can be used to prove the existence of the o-minimal simplicial and the o-minimal singular homology and cohomology with arbitrary constant coefficients. Moreover, one can also develop the theory of products for the o-minimal singular (co)homology in the same way as in the classical case treated in [8] Chapter VI and VII.

3 Applications

3.1 Jordan-Brouwer separation theorem

We include in this subsection Woerheide proof of the o-minimal Jordan-Brouwer separation theorem and the o-minimal invariance of domain theorem. The standard proof of these results, using topological singular homology, depends on the compactness of closed and bounded subsets of the reals and therefore fails for arbitrary o-minimal expansions of an ordered field. The proof in the o-minimal case is due to Woerheide (see [17]) and uses the definable trivialization theorem to circumvent this difficulty.

Since Woerheide's thesis [17] hasn't been published we include all the details.

The standard proof of Lemma 3.2 below depends on the compactness of closed and bounded subsets of the reals and therefore fails for arbitrary o-minimal expansions of an ordered field. The proof below, due to Woerheide (see [17]), uses the definable trivialization theorem to circumvent this difficulty.

Lemma 3.1 *Let (Φ, K) be a triangulation of a definable set Z and let $f : Z \longrightarrow Z$ be a definable continuous map such that $f(\Phi^{-1}(s)) \subseteq \Phi^{-1}(s)$ for each $s \in K$. Then f is definably homotopic to the identity 1_Z.*

Proof. Define $F : Z \times [0,1] \longrightarrow Z$ by $F(z,t) = \Phi^{-1}(t\Phi(z) + (1-t)f \circ \Phi(z))$. Then F is a definable homotopy between f and 1_Z. $\square$

Lemma 3.2 *Let $\mathbb{S}^n$ be the unit n-sphere in N^{n+1}. If e is a definable subset of $\mathbb{S}^n$ that is definably homeomorphic to $[-1,1]^r$ for some $r \leq n$, then $\mathbb{S}^n - e$ is acyclic, i.e. $\widetilde{H}_q(\mathbb{S}^n - e) = 0$ for all $q \in \mathbb{Z}$.*

Proof. We shall use induction on r to show that, if $z \in \widetilde{S}_q(\mathbb{S}^n - e)$ is a cycle, then $[z] = 0$, where $[z]$ denotes the reduced homology class of z in $\widetilde{H}_q(\mathbb{S}^n - e)$. The case $r = 0$ is trivial since, $\mathbb{S}^n$ minus a point is definably homeomorphic to N^n. From now on, we assume $r > 0$.

Let $h : [-1,1]^r \longrightarrow e$ be a definable homeomorphism. For $[a,b] \subseteq [-1,1]$, $e_{[a,b]}$ denotes $h([-1,1]^{r-1} \times [a,b])$, $i_{[a,b]} : \mathbb{S}^n - e \longrightarrow \mathbb{S}^n - e_{[a,b]}$ is the inclusion and $[z]_{[a,b]} = i_{[a,b]*}([z])$. If $a = b$, the we replace the subscript $[a,a]$ by a.

Let (Φ, K) be a triangulation of $\mathbb{S}^n$ compatible with e as well as with $\mathrm{Im} z = \cup\{z_i(\Delta^q) : i = 1, \ldots, l\}$ where $z = \sum_{i=1}^{l} n_i z_i$. Let $\pi : \mathbb{S}^n \times [-1,1] \longrightarrow [-1,1]$ be the projection and set $A = \{(x,a) \in \mathbb{S}^n \times [-1,1] : x \in e_{[-1,a]}\}$. Note that, for each $a \in [-1,1]$, $\pi^{-1}(a) \cap A$ is definably homeomorphic to $e_{[-1,a]}$.

By the definable trivialization theorem ([10] Chapter IX, Theorem 1.2), π is piecewise definably trivial with respect to A as well as with respect to all sets of the form $\Phi^{-1}(s)$ with $s \in K$. Thus there are points $-1 = a_0 < a_1 < \cdots < a_k = 1$ such that, for $0 \leq i < k$ and each pair $b_1, b_2 \in (a_i, a_{i+1})$, there is a definable homeomorphism $f : \pi^{-1}(b_1) \longrightarrow \pi^{-1}(b_2)$ such that $f(\pi^{-1}(b_1) \cap A) = \pi^{-1}(b_2) \cap A$ and $f(\Phi^{-1}(s) \times \{b_1\}) = \Phi^{-1}(s) \times \{b_2\}$ for each $s \in K$.

Claim (1): There exists $\epsilon > 0$ in N such that $\epsilon < |a_{i+1} - a_i|$ for $0 \leq i < k$, $[z]_{[-1,-1+\epsilon]} = 0$, $[z]_{[1-\epsilon,1]}$ and $[z]_{[a_i-\epsilon,a_i+\epsilon]}$ for $0 < i < k$.

Proof of Claim (1): We shall show that, for any $a \in [-1,1]$, if $[z]_a = 0$, then there exists an $\epsilon > 0$ such that $[z]_{[b,c]} = 0$, where $[b,c] = [a-\epsilon, a+\epsilon] \cap [-1,1]$. Note that if $0 < \epsilon' < \epsilon$ and $[b',c'] = [a-\epsilon', a+\epsilon'] \cap [-1,1]$, then $[z]_{[b',c']} = i_*([z]_{[b,c]})$, where $i : \mathbb{S}^n - e_{[b,c]} \longrightarrow \mathbb{S}^n - e_{[b',c']}$ is the inclusion. The claim then follows easily.

We consider the case that $-1 < a < 1$. Let $\delta = \min\{|a+1|, |1-a|\}$. By induction, $[z]_a = 0$. Thus there exists $w \in \widetilde{S}_{q+1}(\mathbb{S}^n - e_a)$ such that $z = \partial w$. It suffices to show that there exists an $\epsilon \in (0,\delta)$ such that $\mathrm{Im} w \cap e_{[a-\epsilon,a+\epsilon]} = \emptyset$, for then z is a boundary in $\mathbb{S}^n - e_{[a-\epsilon,a+\epsilon]}$, and hence $[z]_{[a-\epsilon,a+\epsilon]} = 0$.

Suppose on the contrary that, for all $\epsilon \in (0,\delta)$, $\mathrm{Im} w \cap e_{[a-\epsilon,a+\epsilon]} \neq \emptyset$. Then, by definable curve selection ([10] Chapter VI, Corollary 1.5), there is a $\lambda \in (0, \delta)$ and a definable map $\gamma : (0, \lambda) \longrightarrow \mathbb{S}^n$ such that, for each $\epsilon \in (0, \lambda)$, we have $\gamma(\epsilon) \in \mathrm{Im}w \cap e_{[a-\epsilon, a+\epsilon]}$. By [10] Chapter VI, Proposition 1.10, $\mathrm{Im}w$ is closed. Furthermore, $e_{[a-\epsilon,a+\epsilon]}$ is closed for each $\epsilon \in (0, \lambda)$. Thus $\lim_{\epsilon \longrightarrow 0}\gamma(\epsilon) \in \mathrm{Im}w \cap e_a$. But this contradicts $\mathrm{Im}w \subseteq \mathbb{S}^n - e_a$.

Claim (2): If $b \in (a_i, a_{i+1})$ and $[z]_{[-1,b]} = 0$, then $[z]_{[-1,c]} = 0$ for all $c \in (a_i, a_{i+1})$.

Proof of Claim (2): Let $b, c \in (a_i, a_{i+1})$ with $[z]_{[-1,b]} = 0$. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a definable homeomorphism such that $f(e_{[-1,c]} = e_{[-1,b]}$ and $f(\Phi^{-1}(s)) = \Phi^{-1}(s)$ for each $s \in K$. Since(Φ, K) is compatible with e, we have $f(\mathbb{S}^n - e) = \mathbb{S}^n - e$. By Lemma 3.1, the restriction $f_{|\mathbb{S}^n - e}$ is definably homotopic to the identity $1_{\mathbb{S}^n - e}$. Thus $(f_{|\mathbb{S}^n-e})_*[z] = [z]$. Let $g = f_{|\mathbb{S}^n - e_{[-1,c]}}$. Then $g : \mathbb{S}^n - e_{[-1,c]} \longrightarrow \mathbb{S}^n - e_{[-1,b]}$ is a definable homeomorpism such that $g \circ i_{[-1,c]} = i_{[-1,b]} \circ f_{|\mathbb{S}^n-e}$. Therefore, since $f_{|\mathbb{S}^n-e}$ is definably homotopic to $1_{\mathbb{S}^n-e}$, we have $g_*([z]_{[-1,c]} = g_* \circ i_{[-1,c]*}([z]) = i_{[-1,b]*} \circ (f_{|\mathbb{S}^n-e})_*([z]) = i_{[-1,b]*}([z]) = [z]_{[-1,b]} = 0$. Hence, since g is an isomorphism, we have $[z]_{[-1,c]} = 0$.

We are now ready to finish the proof of the lemma. Let ϵ be as in Claim (1). Let $b = a_1 - \epsilon$ and $c = a_1 + \epsilon$ where we assume $k > 1$ for simplicity. Then $[z]_{[b,c]} = 0$, $[z]_{[-1,-1+\epsilon]} = 0$ and $\epsilon \in (0, |a_1 + 1|)$. By Claim (2), $[z]_{[-1,b]} = 0$. Note that $\mathbb{S}^n - e_{[-1,b]}$ and $\mathbb{S}^n - e_{[b,c]}$ are open subsets of $\mathbb{S}^n - e_b$, that $\mathbb{S}^n - e_b = \mathbb{S}^n - e_{[-1,b]} \cup \mathbb{S}^n - e_{[b,c]}$, and that $\mathbb{S}^n - e_{[-1,c]} = \mathbb{S}^n - e_{[-1,b]} \cap \mathbb{S}^n - e_{[b,c]}$. By Mayer-Vietoris, the sequence

$$\rightarrow \widetilde{H}_q(\mathbb{S}^n - e_{[-1,c]}) \stackrel{(i_{1*}, i_{2*})}{\rightarrow} \widetilde{H}_q(\mathbb{S}^n - e_{[-1,b]}) \oplus \widetilde{H}_q(\mathbb{S}^n - e_{[b,c]}) \rightarrow \widetilde{H}_q(\mathbb{S}^n - e_b)$$

is exact, where $i_1 : \mathbb{S}^n - e_{[-1,c]} \longrightarrow \mathbb{S}^n - e_{[-1,b]}$ and $i_2 : \mathbb{S}^n - e_{[-1,c]} \longrightarrow \mathbb{S}^n - e_{[b,c]}$ are the inclusions. By induction, $\widetilde{H}_p(\mathbb{S}^n - e_b) = 0$ for all $p \in \mathbb{Z}$, so that $(i_{1*}, i_{2*}) : \widetilde{H}_q(\mathbb{S}^n - e_{[-1,c]}) \longrightarrow \widetilde{H}_q(\mathbb{S}^n - e_{[-1,b]}) \oplus \widetilde{H}_q(\mathbb{S}^n - e_{[b,c]})$ is an isomorphism. Since $(i_{1*}, i_{2*})([z]_{[-1,c]}) = ([z]_{[-1,b]}, [z]_{[b,c]})$ we get $[z]_{[-1,c]} = [z]_{[-1,a_1+\epsilon]} = 0$.

More generally, the same argument shows that $[z]_{[-1,a_{i-1}+\epsilon]} = 0$ implies $[z]_{[-1,a_i+\epsilon]} = 0$ for $1 \leq i < k$. By induction, we arrive at $[z]_{[-1,1]} = 0$, which establishes the lemma. $\square$

Lemma 3.3 *Let s_r be a definable subset of $\mathbb{S}^n$ which is definably homeomorphic to $\mathbb{S}^r$, where $n > 0$. Then $\widetilde{H}_q(\mathbb{S}^n - s_r) = \mathbb{Z}$ for $q = n - r - 1$ and is zero otherwise.*

Proof. We use induction on r. If $r = 0$, then $\mathbb{S}^n - s_r$ has the same definable homotopy type as $\mathbb{S}^{n-1}$, hence $\widetilde{H}_q(\mathbb{S}^n - s_0) \simeq \widetilde{H}_q(\mathbb{S}^{n-1}) = \mathbb{Z}$ for $q = n - r - 1$ and is zero otherwise. Assume $r > 0$. Let $h : \mathbb{S}^r \longrightarrow s_r$ be a definable homeomorphism. Let E^+ denote the closed northern hemisphere of $\mathbb{S}^r$ and E^- the closed southern hemisphere. The intersection $E^+ \cap E^-$ is definably homeomorphic to $\mathbb{S}^{r-1}$. Let $e' = h(E^+)$, $e'' = h(E^-)$ and $s_{r-1} = h(E^+ \cap E^-)$. Note that e' and e'' are definably homeomorphic to $[-1, 1]^r$, $\mathbb{S}^n - s_r = (\mathbb{S}^n - e') \cap (\mathbb{S}^n - e'')$ and $\mathbb{S}^n - s_{r-1} = (\mathbb{S}^n - e') \cup (\mathbb{S}^n - e'')$. Thus, by the Mayer-Vietoris sequence and Lemma 3.2 we get $\widetilde{H}_q(\mathbb{S}^n - s_r) \simeq \widetilde{H}_{q+1}(\mathbb{S}^n - s_{r-1}) = \mathbb{Z}$ for $q = n - r - 1$ and is zero otherwise. □

The following o-minimal Jordan-Brouwer separation theorem is an immediate consequence of Lemma 3.3.

Theorem 3.4 *Let s_{n-1} be a definable subset of $\mathbb{S}^n$ which is definably homeomorphic to $\mathbb{S}^{n-1}$, where $n > 0$. Then $\mathbb{S}^n - s_{n-1}$ has exactly two definably connected components.*

We finish this section with the o-minimal version of invariance of domain theorem. This follows immediately from Theorem 3.4. For details, see [17] or [8] page 79.

Theorem 3.5 (Invariance of Domain) *If $X \subseteq N^n$ is an open definable subset and $f : X \longrightarrow N^n$is an injective definable continuous map, then $f(X) \subseteq N^n$ is also an open definable subset. In other words every injective definable continuous map $f : X \longrightarrow N^n$ is open.*

Recently a direct proof of the o-minimal invariance of domain, avoiding o-minimal homology, was obtain by Johns [13]. Johns result is actually valid in any o-minimal structure (not necessarily an o-minimal expansion of a real closed field).

Note also that the statment of Theorem 3.5 can be expressed in first-order logic. Hence, if the first-order theory Th($\mathcal{N}$) of $\mathcal{N}$ has a model in the real numbers, e.g., $\mathcal{N}$ is a real closed field, then this result can be obtained using the Tarski-Seidenberg transfer principle.

3.2 The general separation theorem

Here we prove the general separation theorem. As in the semi-algebraic case treated in [7], this will follow by transfering the corresponding classical result.

Definition 3.6 A definable set X of dimension n is a *homology definable manifold* of dimension n, if for every $x \in X$, the ring $H_q(X, X - x)$ is $\mathbb{Z}$ if and only if $q = n$ and is zero otherwise.

By [2] Lemma 4.10, a definable manifold X of dimension n is a homology definable manifold of dimension n. By [2] Lemma 4.11, the notion of definably connected, definably compact homology definable manifold can be transfered to $\mathbb{R}$. Thus, as in [7] Proposition 5.5 we have the following result.

Proposition 3.7 *Let X be a definably connected, definably compact homology definable manifold of dimension n. Then $H_n(X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H_n(X) = \mathbb{Z}$ or $H_n(X) = 0$.*

As in [7] Definition 4, we say that a definably connected, definably compact homology definable manifold X of dimension n is *orientable* if we have $H_n(X) = \mathbb{Z}$. An arbitrary definably compact homology definable manifold X is called orientable if every definably connected component of X is orientable.

Arguing as in the proof of [7] Theorem 5.7 and using the o-minimal triangulation theorem instead of the semi-algebraic triangulation theorem we can prove the o-minimal Alexander-Poincaré duality theorem.

Theorem 3.8 (Alexander-Poincaré Duality Theorem) *Let X be a definably connected, definably compact homology definable manifold of dimension n and let $B \subseteq A$ be closed definable subsets of X. Then for every $q \in \mathbb{Z}$*

$$H^q(A, B; \mathbb{Z}/2\mathbb{Z}) \simeq H_{n-q}(X - B, X - A; \mathbb{Z}/2\mathbb{Z}).$$

If X is orientable, then

$$H^q(A, B) \simeq H_{n-q}(X - B, X - A).$$

From Theorem 3.8 we obtain as in [7] the o-minimal version of the general separation theorem. A version of this result for definably compact definably connected C^p-definable manifolds with $p \geq 3$ was proved in [1].

Corollary 3.9 (General separation theorem) *If X is a definably compact, homology definable manifold of dimension $m-1$, contained in N^m and having k definably connected components, then the complement of X has $k+1$ definably connected components.*

Proof. Regarding N^m as $\mathbb{S}^m$ minus a point, then by Theorem 3.8, the exactness axiom and the fact that N^m is acyclic, we have the following isomorphism $\widetilde{H}^q(X;\mathbb{Z}/2\mathbb{Z}) \simeq \widetilde{H}_{m-q-1}(\mathbb{S}^m - X;\mathbb{Z}/2\mathbb{Z})$. By a special case of Theorem 3.8, we have $H^q(X;\mathbb{Z}/2\mathbb{Z}) \simeq H_{m-q-1}(X;\mathbb{Z}/2\mathbb{Z})$. Putting $q = m-1$ the result follows. $\square$

If in Corollary 3.9 we take X definably homeomorphic to $\mathbb{S}^{m-1}$ and we identify N^m with $\mathbb{S}^m$ minus a point in $\mathbb{S}^m - X$, then we recover Theorem 3.4.

We end this subsection with a generalization of Theorem 3.8.

Definition 3.10 We say that a definable set $X \subseteq N^m$ is a *definable homology ∂-manifold of dimension n* if X has a definable subset $\mathring{X}$ which is a definable homology manifold of dimension n such that $\partial X = X - \mathring{X}$ is a definable homology manifold of dimension $n-1$.

We say that a definable homology ∂-manifold X of dimension n is *orientable* if there is a homology class $\zeta \in H_n(X, \partial X)$ such that its image under the homomorphism induced by inclusion is a generator of $H_n(\mathring{X}, \mathring{X} - x)$ for all $x \in \mathring{X}$.

As definable manifolds are definable homology manifolds ([2] Lemma 4.10), a definable ∂-manifold of dimension n is a definable homology ∂-manifold. Since, by [2] Lemma 4.11, the notion of definably compact definable homology manifold can be tranfered to $\mathbb{R}$, if X is a definably compact, orientable definable homology ∂-manifold X of dimension n and $\zeta \in H_n(X, \partial X)$ is the corresponding homology class, then the homology class $\partial\zeta \in H_{n-1}(\partial X)$ is such that its image under the homomorphism induced by inclusion is a generator of $H_{n-1}(\partial X, \partial X - x)$ for all $x \in \partial X$. See [14].

Thus, the following can by transfered from the topological case over the reals. In the semi-algebraic case, this result was announced in [7].

Theorem 3.11 (Lefschetz Duality Theorem) *Suppose that X is a definably compact, orientable, definable homology ∂-manifold of dimension n.*

Then the diagram

$$\begin{array}{ccccccc}
\to H^{q-1}(X) & \to & H^{q-1}(\partial X) & \stackrel{\delta}{\to} & H^q(X,\partial X) & \to & H^q(X) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to H_{n-q+1}(X,\partial X) & \stackrel{\partial}{\to} & H_{n-q}(\partial X) & \to & H_{n-q}(X) & \to & H_{n-q}(X,\partial X) \to
\end{array}$$

is sign-commutative and the vertical arrows are isomorphisms.

3.3 Degrees

We end with the application of Theorem 3.8 to the theory of degrees. This generalizes the special case presented in Subsection 1.3.

Suppose that X is an orientable definably compact definable manifold of dimension n and let K be a definably compact definable subset of X. If K is definably connected, then by Theorem 3.8,

$$\mathbb{Z} \simeq H^0(K) \simeq H_n(U, U - K) \simeq H_n(X, X - K),$$

where U is the definably connected component of X which contains K. Here the second isomorphism is given by the excision axiom. We denote the element in $H_n(X, X-K)$ which under these isomorphisms maps to 1 in $H^0(K)$ by $\zeta_{X,K}$. If K is not definably connected and $K_1, \ldots, K_l$ are the definably connected components of K, then by the o-minimal Mayer-Vietoris theorem, $H_n(X, X-K) \simeq \oplus_{i=1}^{l} H_n(X, X-K_i)$ and we set $\zeta_{X,K} = \oplus_{i=1}^{l} \zeta_{X,K_i}$.

The element $\zeta_{X,K}$ is called the *fundamental class of X around K* and if there is no risk of confusion is denoted by ζ_K. We call the fundamental class of X around X the *fundamental class of X*. By naturality of the isomorphisms of Theorem 3.8 for inclusions of closed definable subsets, we have the following observation.

Proposition 3.12 *Suppose that X is an orientable definably compact definable manifold of dimension n and let K be a nonempty definably compact definable subset of X. Then the fundamental class $\zeta_{X,K}$ of X around K is characterised by the fact that, for all $x \in K$, the homomorphism $H_n(X, X-K) \longrightarrow H_n(X, X-x)$ induced by the inclusion $(X, X-K) \longrightarrow (X, X-x)$ sends $\zeta_{X,K}$ into $\zeta_{X,x}$. Furthermore, if K is definably connected, then $\zeta_{X,K}$ is a generator of $H_n(X, X-K)$.*

Margarita Otero has pointed out another proof of this proposition based on the orientation theory from [2] and using the transfer method.

With the previous results available, the treatment of the theory of degrees in our context is exactly like that in classical case (see [8] Chapter VIII, Section 4).

Below, X, Y and Z will be definably compact, orientable definable manifolds of dimension n. Note that here Proposition 3.12 will play a crucial role starting with the following definition.

Definition 3.13 Let $f : X \longrightarrow Y$ be a continuous definable map and $K \subseteq Y$ a definably compact, definably connected, nonempty definable subset. The *degree of f over K* is the integer defined by $f_*(\zeta_{X,f^{-1}(K)}) = (\deg_K f)\zeta_{Y,K}$.

In the next result, the proofs of (1)-(5) are exactly the same as the proofs of their classical analogues. But for the proof of (5) one uses the fact that a definably connected definable set is definably path connected ([10]).

Proposition 3.14 *Let $f : X \longrightarrow Y$ and $K \subseteq Y$ be as in Definition 3.13. Then we have:*

(1) If $f^{-1}(K) \subseteq K'$ and $K' \subseteq X$ is definably compact, then $f_*(\zeta_{X,K'}) = (\deg_K f)\zeta_{Y,K}$.

(2) If L is a definably compact definable subset of K, then $f_*(\zeta_{X,f^{-1}(L)}) = (\deg_K f)\zeta_{Y,K}$. In particular, $\deg_L f = \deg_K f$.

(3) If X is a finite union of open definable subsets $X_1, \ldots, X_r$ such that the sets $K_i = f^{-1}(K) \cap X_i$ are mutually disjoint, then $\deg_K f = \sum_{i=1}^r \deg_K f_{|X_i}$.

(4) If $g : Z \longrightarrow X$ is a continuous definable map, then we have $\deg_K(f \circ g) = (\deg_{f^{-1}(K)} g)\deg_K f$.

(5) If Y is definably connected, then $\deg_K f$ is independent from K and is denote by $\deg f$.

Remark 3.15 Let $f : X \longrightarrow Y$ and $K \subseteq Y$ be as in Definition 3.13. Then the following hold:
(i) if $f^{-1}(K) = \emptyset$, then $\deg_K f = 0$;

(ii) if f is the inclusion of X onto an open definable subset of Y and $K \subseteq X$, then $\deg_K f = 1$;

(iii) if f is a definable homeomorphism of X onto a definable open subset of Y and $K \subseteq f(X)$, then $\deg_K f \in \{-1, 1\}$.

In Remark 3.15 (i) and (ii) are immeadiate from the definition and (iii) follows from Proposition 3.14 (4).

Remark 3.16 Let $f : X \longrightarrow Y$ be a continuous definable map between orientable definable manifolds of dimension n. Let $y \in Y$ and suppose that $f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then there are open definable subsets $V_1, \ldots, V_k$ of X such that $x_i \in V_i$ and $x_j \notin V_i$ for all $i \neq j$. By Proposition 3.14 (3), we have $\deg_y f = \sum_{i=1}^{k} \deg_y f_{|V_i}$. Hence, $\deg_y f$ equals the number of points in $f^{-1}(y)$ counted with their "multiplicity". By the excision axiom, the multiplicity $\deg_y f_{|V_i}$ of x_i can be determined in any definable open neighbourhood of x_i in X.

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Type-definability, compact Lie groups and o-minimality

Anand Pillay [3]

Abstract

We study type-definable subgroups of small index in definable groups, and the structure on the quotient, in first order structures. We raise some conjectures in the case where the ambient structure is o-minimal. The gist is that in this o-minimal case, any definable group G should have a smallest type-definable subgroup of bounded index, and that the quotient, when equipped with the logic topology, should be a compact Lie group of the "right" dimension. I give positive answers to the conjectures in the special cases when G is 1-dimensional, and when G is definably simple.

1 Introduction

Definable groups in o-minimal structures have been studied for several years, as part of the "general theory" of o-minimality. It is, and was, natural here to work in a saturated model M of an o-minimal theory T, rather than restrict one's attention to models with order type of the reals. (In fact an arbitrary o-minimal theory may not even have models whose order type is that of the reals). In any case, the general thrust of earlier work was that a definable group G in M should "resemble" a real Lie group. Examples of the successes were: (i) G can be definably equipped with a "definable manifold" structure over M with respect to which the group operation is continuous [8], (ii) If G is definably simple, then there is a real-closed field R definable in M and a definable isomorphism between G and a semialgebraically simple semialgebraic subgroup of some $GL(n, R)$ [4], (iii) if G is commutative, definably

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compact, and definably connected, with $\dim(G) = n$, then for each k, the k-torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$ [2].

In this paper we investigate the possibility of recovering a suitable compact Lie group from a definable group G (in a saturated o-minimal structure) by quotienting G by a *type-definable* subgroup of bounded index and endowing the quotient with the logic topology (which will be explained in the next section). A type-definable set in a saturated structure is the intersection of a (small) collection of definable sets. Type-definable sets and groups play an important role in stable and simple theories. In fact in the stable case, any type-definable subgroup of a definable group is an intersection of definable subgroups. Likewise in the supersimple case. This of course fails in the o-minimal case. The natural examples of type-definable sets/subgroups in the o-minimal context are "infinitesimal neighbourhoods". Not much attention seems to have been paid to such type-definable subgroups and the corresponding quotient structures, other than in contexts such as real-closed rings, and weakly o-minimal structures. In any case, we expect the meaning of type-definability in the o-minimal context to be "orthogonal" to its significance in stable/simple theories: if G is a definably connected, definable group in a stable theory, then G has no proper type-definable subgroup of bounded index, but if G is a definably connected definable group in an o-minimal structure, we expect G to have a smallest type-definable subgroup G^{00} of bounded index, and all the "nontrivial topology" of G (definable homology, Betti numbers,..) to be contained in the quotient G/G^{00}.

Let us now state the main conjectures. Our notation, in particular the logic topology, will be explained in detail in Section 2.

Conjecture 1.1 *Let T be an o-minimal theory, M a saturated model of T, and G a definably connected definable group in M, defined over $\emptyset$ say. Then*
(i) G has a smallest type-definable subgroup of bounded index, G^{00} .
(ii) G/G^{00} is a compact connected Lie group, when equipped with the logic topology.
(iii) If moreover G is definably compact, then the dimension of G/G^{00} (as a Lie group) is equal to the o-minimal dimension of G.
(iv) If G is commutative then G^{00} is divisible and torsion-free (namely a $\mathbb{Q}$-vector space).

A motivating example is where T is the theory RCF of real closed fields, R

a saturated model, and G a definably connected definably compact semialgebraic group (such as $SO(3,R)$) which is defined over the reals $\mathbb{R}$. So G can be identified with ${}^*G(\mathbb{R})$, the nonstandard version of the compact Lie group $G(\mathbb{R})$. We then have at our disposal the standard part map $st : G \to G(\mathbb{R})$. We can then take G^{00} to be $ker(st)$ (the infinitesimal subgroup of G), and G/G^{00} with the logic topology identifies with the compact Lie group $G(\mathbb{R})$. (This will be considered in more detail in Section 3.)

However in general T may not have a model whose order-type is that of the reals, and even if it did, there may be groups definable in a saturated model M which are not definably isomorphic to groups defined over $\mathbb{R}$. (This even occurs in the semialgebraic situation.) If we tried to take the group of infinitesimals with respect to the parameters over which G is defined, the quotient may have large cardinality (that of $\overline{M}$). So the thrust of Conjecture 1.1 is to recover the "correct" or "intrinsic" infinitesimals and corresponding standard part map, in an abstract context.

In Section 2 we will give precise definitions and state some results valid for arbitrary theories T. In particular we will point out that [G^{00} exists and G/G^{00}] is a compact Lie group if and only if G has the DCC on type-definable subgroups of bounded index.

In Section 3 we turn to o-minimal structures and verify Conjecture 1.1 in some special cases, where G is 1-dimensional, and where G is definably simple.

Some version of Conjecture 1.1 was stated during the problem session of the Ravello meeting in 2002. I would like to thank Alessandro Berarducci and Margarita Otero for their interest in these problems, and for several stimulating conversations, which encouraged me to write this paper.

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2 Type-definable bounded equivalence relations and the logic topology

In this section we record some observations about bounded type-definable equivalence relations and type-definable subgroups of bounded index, in the context of an arbitrary complete first order theory.

T will denote an arbitrary complete theory, unless otherwise stated. Let us fix a κ-saturated model M of T, where $\kappa > |T|$ is very big, say inaccessible. By "small" or "bounded" we mean of cardinality $< \kappa$. Definability is always meant in the sense of this ambient structure M. A type-definable subset of M^n is by definition a set defined by the conjunction of strictly less than κ-many formulas. Let X be a type-definable set and E a type-definable equivalence relation on X. We say that E is *bounded* if $|X/E| < \kappa$.

If M' is an elementary extension of M and X is a type-definable set in M then $X(M')$ denotes the set defined in M' by the same collection of formulas defining X. If E is a type-definable equivalence relation (not necessarily bounded) on X, then we have a canonical injection $i : X/E \to X(M')/E(M')$, given by $i(a/E) = a/E(M')$, With this notation, we have

Fact 2.1 *E is bounded iff for any elementary extension M' of M, $i : X/E \to X(M')/E(M')$ is a bijection.*

So, a type-definable equivalence relation on a type-definable set is bounded if the set of classes does not change when passing to an elementary extension. A special case is when the number of classes is finite, and in this case a compactness argument shows that E is definable (namely is the restriction to $X \times X$ of a definable equivalence relation).

Remark 2.2 *Let X be a type-definable set. Then a type-definable bounded equivalence relation on X is the same thing as a partition of X into a small set of type-definable sets. That is, on the one hand, given bounded E, the E-classes give a partition of X into a bounded number of type-definable sets, and conversely, given any partition $X = \cup X_i$ of X into a bounded number of type-definable sets, there is a type-definable equivalence relation on X whose classes are precisely the X_i.*

Proof. Suppose we are given the partition $\{X_i : i < \lambda\}$ of X into type-definable sets, where $\lambda < \kappa$. For each $i < \lambda$, let $\{Y_{i,j} : j < \lambda_i\}$ be a (small) family of definable sets such that $X_i = X \cap \cap_j Y_{i,j}$. Let S be the set of finite sets $\{(i_1, j_i), .., (i_n, j_n)\}$ such that $X \subseteq Y_{i_1,j_1} \cup ... \cup Y_{i_n,j_n}$. For each $s \in S$, let $Z_s = \{(x, y) \in X \times X : \bigvee_{(i,j)\in s}(x \in Y_{i,j} \wedge y \in Y_{i,j})\}$. Then a compactness argument shows that $\cap_{s\in S} Z_s$ is an equivalence relation on X whose classes are precisely the X_i. □

We now recall the "logic topology" on X/E from [3].

Definition 2.3 Let X be a type-definable set and E a bounded type-definable equivalence relation on X. Let $\mu : X \to X/E$ be the canonical surjection. Define $Z \subset X/E$ to be closed if $\mu^{-1}(Z) \subset X$ is type-definable in M.

Remark 2.4 *(i) $Z \subset X/E$ is closed iff there is a type-definable subset Y of X such that $Z = \mu(Y)$.*
(ii) $U \subset X/E$ is "open" (complement of a closed) iff there is some $Y \subset X$ defined by a possibly infinite but small disjunction of formulas, such that $U = \{a/E : \text{the } E\text{-class of } a \text{ is contained in } Y\}$.

Proof. Suppose $Z = \mu(Y)$ where $Y \subset X$ is type-definable. Then $\mu^{-1}(Z) = \{x \in X : \exists y \in Y(E(x, y)\}$ is type-definable. □

The following was pointed out in [3] but we sketch the proof again for the convenience of the reader.

Lemma 2.5 *With the set-up and notion of closed in Definition 2.3, X/E is a compact Hausdorff topological space.*

Proof. Firstly, a finite union of closed sets is clearly closed (as a finite union of type-definable sets is type-definable). Moreover, as X/E is of small size, the intersection of an arbitrary family of closed subsets of X/E is the intersection of a small subfamily, and so its preimage is type-definable in M. (Remember that by definition a type-definable set in M is something defined by the conjunction of a small number of formulas). Thus this notion of closed set makes X/E into a topological space.

Compactness of X/E follows from the compactness theorem of first order logic: Suppose $\{Z_i : i \in I\}$ is a family of closed subsets of X/E with the finite intersection property. As remarked above we may assume I is small. Then $\{\mu^{-1}(Z_i) : i \in I\}$ is a small family of type-definable subsets of X, such that every intersection of finitely many members of the family is nonempty. Compactness implies that $\cap_{i\in I}\mu^{-1}(Z_i) \neq \emptyset$, and thus $\cap_i Z_i$ is nonempty.

Hausdorffness: We may assume that E is defined by a set $\{\phi_i(x, y) : i \in I\}$ of formulas each of which is symmetric and reflexive, and such that for each $i \in I$ there is $j \in I$, such that $\models \phi_j(x, y) \wedge \phi_j(y, z) \to \phi_i(x, z)$.

Let $a, b \in X$ with $\neg E(a, b)$. So there is a formula $\phi_i(x, y)$ such that $\models \neg\phi_i(a, b)$. Pick $\phi_j(x, y)$ as above. Let U_1, U_2 be the open neighbourhoods of a/E and b/E respectively, in X/E, given by the formulas $\phi_j(a, y)$ and $\phi_j(b, y)$, as in Remark 2.4 (ii). Then $U_1 \cap U_2 = \emptyset$. $\square$

Remark 2.6 *Note that the logic topology on X/E is not any kind of quotient topology, as we are not starting with a topology on X. If we try to define the closed subsets of X to be the type-definable ones, then this will not define a topology on X, as the intersection of an arbitrary family of type-definable sets is not necessarily type-definable (by a small set of formulas).*

Suppose that X, Y are type-definable sets and E, E' are type-definable bounded equivalence relations on X, Y respectively. Then $X/E \times Y/E'$ identifies (set-theoretically) with $(X \times Y)/(E \times E')$. It is then rather easy to see that the logic topology on $(X \times Y)/(E \times E')$ corresponds to the product topology on $X/E \times Y/E'$.

Let us now discuss type-definable groups. By a type-definable group $(G, \cdot)$ we mean a type-definable set G together with a type-definable subset of $G \times G \times G$ which is the graph of a group operation $\cdot$. Compactness yields that $\cdot$ is *definable*, namely there is a definable function whose restriction to $G \times G$ is precisely $\cdot$. If the underlying set G is definable we speak of a definable group. In so far as our o-minimal applications are concerned, the type-definable groups we will be looking at will be type-definable subgroups of definable groups. But there is no harm in developing the theory in some greater generality. In any case, by the previous paragraph, we have:

Lemma 2.7 *Let $(G, \cdot)$ be a type-definable group, and H a type-definable normal subgroup of G of bounded index. Then, under the logic topology, G/H is a compact, Hausdorff, topological group.*

Let us bring in the notion of *definable connectedness* for (type-)definable groups There is a lot of room for ambiguity and confusion, as subsequently we will look at o-minimal structures where there is a notion of definable connectedness coming from the underlying order topology. So we will try to be careful.

First, by a *relatively definable* subset of a type-definable set X, we mean something of the form $X \cap Y$ where Y is definable.

Definition 2.8 Let G be a type-definable group. We will say that G is *definably connected*, if G has no relatively definable proper subgroup of finite index. We will say that G is *type-definably connected* if G has no proper type-definable subgroup of bounded index.

Of course, if X is a topological space then we can and will talk about connectedness of X.

Remark 2.9 *Let G be a type-definable group.*
(i) Suppose H is a type-definable subgroup of G of finite index. Then H is relatively definable.
(ii) Any relatively definable subgroup H of G of finite index contains a normal relatively definable subgroup of finite index, defined over the same parameters as G and H.
(iii) Any type-definable subgroup H of G of bounded index in G contains a normal type-definable subgroup of bounded index, type-definable over the same parameters as G and H.
(iv) For any small set A of parameters over which G is type-definable, there is a (unique) smallest type-definable over A subgroup of G of bounded index, which is moreover normal in G. We call this group G^{00}_A.

Proof. (i) is by compactness.
(ii) and (iii). Let $K = \cap_{g\in G} H^g$. Then G/K acts faithfully on G/H, so K has bounded index in G (and finite index if H does). Thus K is type-definable, and by its description it is invariant under any automorphisms fixing any para,eters over which G and H are type-definable.
(iv) The intersection of all A-type-definable subgroups of bounded index in G is also type-definable over A and of bounded index. By (iii) it is also normal in G. □

Lemma 2.10 *Let G be a type-definable, definably connected group. Let H be a type-definable normal subgroup of G of bounded index. Then G/H (with the logic topology) is connected.*

Proof. G/H is a compact topological group. If it is not connected (as a topological space) then it has a open (and so closed) subgroup of finite index.

The preimage under μ is then a type-definable subgroup of G of finite index, which by Remark 2.9 (i) contradicts the definable connectedness of G. $\square$

One of the main issues that will concern us (especially in the o-minimal case) is whether and when G_A^{00} does not depend on the choice of A, namely when G has a smallest type-definable subgroup of bounded index. If it does, we will call this subgroup G^{00}, and note that G^{00} is then type-definably connected. We may also call G^{00} the *type-definably connected component* of G.

We will relate these issues to compact Lie groups. By a Lie group, we mean a real analytic manifold with a real analytic group structure. Second countability is also usually assumed, but note that this follows if we assume the group to be compact. We will say that the topological group G is a Lie group if it can be equipped with the structure of a Lie group, which induces the original topology. Any topological group has at most one structure of a Lie group (any map between Lie groups which is both a group isomorphism and a homeomorphism is an isomorphism of Lie groups).

We will make use of two facts about compact groups. From now on we take compactness to include Hausdorffness.

Fact 2.11 *(i) Any connected compact group is the inverse limit of a directed system of connected compact Lie groups,*
(ii) Any compact Lie group has the DCC on closed subgroups. (That is there is no infinite descending chain of closed subgroups of G.)

Proposition 2.12 *Let G be a type-definable, definably connected group. Then the following are equivalent:*
(i) G has the DCC on type-definable subgroups of bounded index,
(ii) G^{00} exists, and G/G^{00} is a (connected) compact Lie group (under the logic topology).
(iii) for any type-definable normal subgroup H of G of bounded index, G/H is a (connected) compact Lie group (under the logic topology),

Proof. (i) implies (ii): Assume (i). Then clearly G^{00} exists. Let $\mu : G \to G/G^{00}$ be the canonical surjective homomorphism. By Lemmas 2.7 and 2.9 and Fact 2.11(i), G/G^{00} is the inverse limit of a directed system $(G_i)_{i \in I}$ of connected compact Lie groups. Let $\nu_i : G/G^{00} \to G_i$ be the corresponding

surjection. Let $H_i = ker(\mu \circ \nu_i)$. Then H_i is a type-definable subgroup of G which contains G^{00} and so has bounded index. Now $\cap_{i \in I} H_i$ is clearly equal to G^{00}. On the other hand by (i), there is a finite subset J of I such that $\cap_{i \in I} H_i = \cap_{i \in J} H_i$. As $(G_i)_i$ is a directed system, $\cap_{i \in J} H_i = H_{i_0}$ for some $i_0 \in I$. It follows that $\nu_{i_0} : G/G^{00} \to G_{i_0}$ is an isomorphism. Hence G/G^{00} is a compact Lie group.
(ii) implies (iii). Assume (ii). Then for every type-definable normal bounded index subgroup H of G, G/H (with its logic topology) is the image of G/G^{00} under a continuous homomorphic surjection. As G/G^{00} is a compact Lie group, so is G/H.
(iii) implies (i): Assume (iii) and suppose for a contradiction that there is an infinite descending chain $G = G_0 > G_1 > G_2 > \ldots$ of type-definable subgroups of bounded index in G. By Remark 2.9(iii) we may assume the G_i are normal in G. Let $H = \cap_{i \in \omega} G_i$. Then H is type-definable, normal and of bounded index in G. By (iii), G/H is a compact Lie group when equipped with the logic topology. Let $\mu : G \to G/H$ be the canonical surjection. Let $H_i = \mu(G_i)$. So the H_i/H form a descending chain of closed subgroups of G/H, contradicting Fact 2.11(ii). □

3 The o-minimal case

Here we specialise to the case where $M = (M, <, \ldots)$ is a saturated model of an o-minimal theory, and the ordering $<$ is dense with no first or last element. We assume familiarity with the basics of o-minimality ([10]). Recall that the ordering on M gives a topology and that every Cartesian power M^n of M is equipped with the product topology. However, M being saturated, this topology is very disconnected. But there is good behaviour when we only consider the category of definable sets. $\dim(-)$ denotes the o-minimal dimension of a definable set.

Let us first recall the notion of a "definable manifold over M". Such a thing is a definable set X (in M), together with a covering $X = U_1 \cup U_2 .. \cup U_r$ by definable sets, and for each i some definable bijection f_i of U_i with some open definable subset V_i of M^n, such that for each $i < j$, $f_i(U_i \cap U_j)$ and $f_j(U_i \cap U_j)$ are open (definable) subsets of V_i and V_j respectively, and moreover $f_j \circ f_i^{-1}$ is a homeomorphism between V_i and V_j. We say that X has dimension

n as a definable manifold over M and note that this coincides with $\dim(X)$ as a definable set.

One can of course make the same definition over any model M_0 of T. If M_0 has underlying order that of the reals, then a definable manifold over M_0 will have the structure of a manifold in the usual sense. But working over our saturated model M, a definable manifold over M could not be a real manifold. In any case, a definable manifold over M has an induced topology, which we call the t-topology: $Y \subset X$ is open iff each $f_i^{-1}(Y \cap U_i)$ is open in M^n. We say that X is t-definably connected if X is not the disjoint union of two nonempty definable t-open subsets. Note that any definable open subset of some M^n, in particular M^n itself, has a canonical structure of a definable manifold over M.

The following is from [8].

Fact 3.1 *Let G be a definable group in M. Then*
(i) G can be given the structure of a definable manifold over M such that with respect to the t-topology both multiplication and inversion are continuous. We call G with such a definable manifold structure, a definable group manifold over M.
(ii) Any definable homomorphism between definable group manifolds is continuous (so the definable group manifold structure on G is unique).
(iii) Any definable subgroup of a definable group manifold is t-closed.
(iv) G has the DCC on definable subgroups, in particular has a smallest definable subgroup of finite index, which we call G^0.
(v) G is definably connected (in the sense of Definition 2.8), that is $G = G^0$, iff G is t-definably connected under some (any) definable group manifold (over M) structure on G.

From now on, any definable group G will be considered as equipped with its definable group manifold structure, and we will speak of t-open, etc. By virtue of (v) above, definable connectedness in the model-theoretic sense and t-sense coincide.

Lemma 3.2 *Let G be a definable group, and H a type-definable subgroup of bounded index. Then H is t-open.*

Proof. It is enough to show that H has t-interior. Let H be the intersection of the small family $\{X_i : i \in I\}$ of definable subsets of G. We may assume that this family is closed under finite intersection. Note that for each i, finitely many translates of X_i cover G (otherwise, by compactness, for any $\lambda < \kappa$, no set of λ translates of X_i can cover G, so H can not be of bounded index in G).

Suppose now for a contradiction that H has no t-interior. Then for any $a \in G$ and t-open neighbourhood U of a, U is not contained in H. As we can quantify over sufficiently small definable t-open neighbouhoods of a, it follows that for each $a \in G$ there is $i \in I$ such that no t-open neighbourhood of a is contained in X_i. By compactness, it follows that some X_i has no t-interior in G. Thus $\dim(X_i) < \dim(G)$, but then finitely many translates of X_i could not cover G, a contradiction. □

Remark 3.3 *Thus, with notation as in Lemma 3.2, if G/H is equipped with the t-quotient topology, then it is discrete. So the t-quotient topology on G/H could not agree with the logic topology unless G/H is finite.*

A very basic question concerning type-definable groups in o-minimal M is: Suppose H is a type-definable subgroup of the definable group G. Is H definably connected-by-finite? Namely does H have a smallest relatively definable subgroup of finite index?

In the case where H is also of bounded index in G, the above question is, by virtue of Proposition 2.12, a special case of Conjecture 1.1 (i) and (ii).

We now recall the notion of definable compactness from [7].

Definition 3.4 Let X be a definable manifold over M. X is said to be definably compact if whenever $a < b$ are in M and f is a definable continuous function from $[a, b)$ into X then $\lim_{x \to b} f(x)$ exists in X.

In the case where the t-topology on X agrees with the induced topology on X from the ambient space M^k, then definable compactness of X is equivalent to X being closed and bounded in M^k.

We now confirm Conjecture 1.1 in some special cases.

Proposition 3.5 *Conjecture 1.1 holds when* $\dim(G) = 1$.

Proof. Remember that we are assuming G to be definably connected. By [8], G is commutative and we use additive notation. We make heavy use of the description of the possible definable group manifold structures on G given in [9]. There are two cases:

Case I. G, with its t-topology, is not definably compact.

In this case there is a definable total ordering on G (which we call $<$ by abuse of terminology) such that $(G, +, <)$ is an ordered divisible (torsion-free) abelian group, and the order topology on G agrees with the t-topology. Moreover every definable subset of G is a finite union of intervals and points. Note that $<$ is dense without endpoints.

We will prove that $G = G^{00}$, namely that G has NO proper type-definable subgroups of bounded index, which trivially yields Conjecture 1.1 for G.

Let H be a type-definable subgroup of G of bounded index. So H is the intersection of a small family $\{X_i : i \in I\}$ of definable subsets of G which we may assume to be open and symmetric ($X_i = -X_i$).

Claim (1): H is $<$-unbounded. Namely for all $a \in G$ with $a > 0$, there is $x \in H$ with $x > a$.

Proof of Claim (1): If not, let $a \in G$ be positive such that H is contained in the interval $(-a, a)$. Let c_j for $j \in J$ be (positive) representatives of the cosets of H in G. As J is small, by compactness there is $d \in G$ such that $d > c_j$ for all j. But then $c + a$ is in a new coset of H, contradiction.

Claim (2): There is $a > 0$ in G such that $[a, \infty)$ is contained in H.

Proof of Claim (2): By Claim 1, for each i, X_i (being a finite union of open intervals) contains some unbounded interval $[a_i, \infty)$. By compactness (as I is small), H contains some $[a, \infty)$.

It follows from Claim 2, that $H = G$. (Let $b > 0$ be in G. Then $a+b \in H$, so $b = (a + b) - a$ is in H.)

We have shown that in Case 1, $G = G^{00}$.

Case II. G is definably compact.

The content of [9] is that G with its t-topology "resembles" the circle group $\mathbb{S}^1$. We will use the torsion elements of G to define G^{00} as the "right" group of infinitesimals, and then show that G/G^{00} with the logic topology IS $\mathbb{S}^1$.

Here are the details. First we describe what is given by [9]:
(A) For each n, the group of elements of G of exponent n is the cyclic group of order n. In particular the torsion subgroup $T(G)$ of G is abstractly isomorphic to the torsion subgroup of $\mathbb{S}^1$.
(B) There is a definable "circular" ordering (or orientation) R on G, satisfying, $R(x,y,z)$ implies x,y,z are pairwise distinct, $R(x,y,z) \to R(y,z,x)$, $R(x,y,z) \to R(-z,-y,-x)$ and $R(x,y,z)$ implies $R(x+w,y+w,z+w)$ for any w. Also for each $x \in G$, $R(x,y,z)$ defines a dense linear ordering without endpoints on $G \setminus \{x\}$. We write $<_x$ for this ordering on $G \setminus \{x\}$.
(C) The topology on G given by the $<_x$'s is precisely the t-topology, and every definable subset of $G \setminus \{x\}$ is a finite union of points and $<_x$-intervals. Moreover, if $a_0 \neq 0$ is such that $2a_0 = 0$, then whenever $0 <_{a_0} b <_{a_0} c$ and $d \in G$ and $0 <_{a_0} d$ and $0 <_{a_0} c+d$, then $0 <_{a_0} b+d <_{a_0} c+d$.
(D) There is an isomorphism f between the structures $(T(G),+,R)$ and $(T(\mathbb{S}^1),+,R_1)$ where R_1 is one of the two natural circular orders on $\mathbb{S}^1$.

Now let $X = G \setminus \{a_0\}$ and let $<$ denote $<_{a_0}$ on X. Define H to be the intersection of all open intervals (a,b) where $a,b \in T(G)$ and $a < 0 < b$. Then by the above H is a type-definable subgroup of G. Note also that H is torsion-free and divisible.

Claim (3): H has bounded index in G.

Proof of Claim (3): Note that G^{00} is the intersection of all $[-a,a]$ where $0 < a$ and $a \in T(G)$. Suppose $na = 0$. Then the translates of $[-a,a]$ by $a, 2a, .., (n-1)a$ cover G (by (C) for example). So boundedly many translates of G^{00} cover G.

Claim (4): $H = G^{00}$.

Proof of Claim (4): By Claim 3, it suffices to prove that H has no type-definable subgroup of bounded index. This is proved just as it was proved that $G = G^{00}$ in Case I.

We now want to prove that G/G^{00} with the logic topology is precisely the 1-dimensional compact Lie group $\mathbb{S}^1$. There are different possible approaches. Note that by Lemma 2.10 the G/G^{00} is a connected compact group. To be a connected compact Lie group it suffices that in addition, G be locally connected, namely have a neighbourhood basis of the identity consisting of connected sets. This can be proved without much difficulty. Our knowledge of the torsion yields that the only possibility is $\mathbb{S}^1$. Such an approach is possibly suitable for generalizations.

Another approach goes by first proving that the structures $(G, +, R)$ and $(\mathbb{S}^1, +, R_1)$ are elementarily equivalent. By saturation we may then assume $(G, +, R)$ to be an elementary extension of $(\mathbb{S}^1, +, R_1)$. So by "nonstandard analysis" we have the standard part map $st : G \to \mathbb{S}^1$ whose kernel can be shown to be $H = G^{00}$. Finally show the two topologies on $\mathbb{S}^1$ (logic and standard) agree.

What we will in fact do is to exhibit explicitly the homeomorphism between G/G^{00} and $\mathbb{S}^1$ (which will in practice amount to working out details of the second approach).

By (C) we already have an isomorphism f between $T(G)$ and $T(\mathbb{S}^1)$ preserving the respective circular orders, and we will "extend" f to G/G^{00}.
It will be convenient to again work with $X = G \setminus \{a_0\}$ and the ordering $<=<_{a_0}$ on X.

Claim (5): For any $a \in G$, exactly one of the following holds:
(i) $a + G^{00}$ contains a unique element of $T(G)$ which we call $t(a)$
(ii) $a + G^{00}$ contains no element of $T(G)$, $a + G^{00}$ is a convex subset of X, $B_a = \{b \in X : b \in T(G) \text{ and } b < a + G^{00}\}$ is nonempty and contains no greatest element, and $C_a = \{c \in X : c \in T(G) \text{ and } a + G^{00} < c\}$ is nonempty and contains no smallest element. Moreover $a + G_{00}$ is precisely $\{x \in X : B_a < x < C_a$. In this case let $t(a)$ denote the "cut" (B_a, C_a).

Proof of Claim (5): Suppose first that $t, t' \in a + G^{00}$ are torsion elements. Then $t - t' \in G^{00}$. But $t - t'$ is also torsion, so by definition of G^{00}, $t - t' = 0$, and $t = t'$.

Now suppose that $a + G^{00}$ contains no torsion elements. So $a \in X$. If a is either greater than all torsion elements in X or less than all torsion elements in X then it is easy (by translating by a_0) to see that $a \in a_0 + G^{00}$ so $a_0 \in a + G^{00}$, a contradiction. Thus B_a and C_a are nonempty. As in the argument we just gave (translating by a torsion element) we see that B_a has no greatest element and C_a has no least element. Let $x \in X$ be such that $B_a < x < C_a$, and assume without loss that $a < x$. Then easily $x - a \in X$. We want to show that $x - a \in G^{00}$. If not there is a torsion element t_0 such that $0 < x - a < t_0$. Fix some element $t' \in C_a$. We may choose t_0 small enough such that $t' + t_0 \in X$, and thus $t + t_0 \in X$ and $t < t + t_0$ for all $t \in B_a$. Now, as $nt_0 = 0$ for some n, it follows that $t + t_0 \in C_a$ for some $t \in B_a$. But then $x + t_0$ is in X and $> t + t_0$, so not $< C_a$, a contradiction, as $x + t_0 = a$. Claim 5 is proved.

Note that the torsion subgroup $T(\mathbb{S}^1)$ of $\mathbb{S}^1$ is dense in S_1 (with respect to the usual topology on $\mathbb{S}^1$ or equivalently with respect to the circular ordering R_1 on $\mathbb{S}^1$). We now define $h : G/G^{00} \to \mathbb{S}^1$. Let $a \in G$, then define $h(a + G^{00})$ to be $f(t(a))$ if (i) of Claim 5 holds. If (ii) of Claim 5 holds, then there is a unique $r \in \mathbb{S}^1$ such that $R_1(x, r, y)$ for all $x \in f(B_a)$ and $y \in f(C_a)$, and define $h(a+G^{00}) = r$. By Claim 5, the denseness of $T(\mathbb{S}^1)$ in $\mathbb{S}^1$ and the saturation of M we see that $h : G/G^{00}$ is a well-defined bijection. We leave it to the reader to check that it is a group isomorphism. Finally we have to check that h is also a homeomorphism, when G/G^{00} is equipped with the logic topology. Note that the circular ordering R on G induces a circular ordering R' on G/G^{00} and that h is an isomorphism between $(G/G^{00}, +, R')$ and $(\mathbb{S}^1, +, R_1)$. Thus G/G^{00} with the topology induced by R' is isomorphic/homeomorphic to $\mathbb{S}^1$. To show that this topology on G/G^{00} is the same as the logic topology, it is enough to see that any basic open neighbourhood of the identity in G/G^{00} with respect to the R'-topology, is open in the logic topology. Note that R' induces an ordering $<'$ on X/G^{00} which lifts to $<$ on X. So a basic open neighbourhood of the identity in G/G^{00} under R' is of the form $U = \{x \in G/G^{00} : a/G^{00} <' b/G^{00}\}$ which contains $0/G^{00}$. But then $Y = \{x \in X : a < x < b\}$ is definable in M, and the set of G^{00}-cosets contained in Y is open in the logic topology, and coincides with U.

This completes the proof that G/G^{00} with the logic topology is precisely $\mathbb{S}^1$.

The proof of Proposition 3.5 is complete. □

Proposition 3.6 *Conjecture 1.1 is true when G is definably simple (and noncommutative).*

Proof. Definably simple means that G has no definable proper nontrivial normal subgroup. Such groups were studied in detail in the series of papers [4], [5], and [6]. Modulo results from those papers, Proposition 3.6 will be almost immediate.

So again we are working in a saturated o-minimal structure M, and G is a definably simple group definable in M. The main result of [4] says:

Fact 3.7 *There is a real closed field R definable in M, which is 1-dimensional in the sense of M, and a definable isomorphism between G and a semialgebraic subgroup H of $GL(n, R)$ for some n.*

The proof of (2) implies (3) in Theorem 5.1 of [6] gives:

Fact 3.8 *Let $(R, +, \cdot)$ be a saturated real closed field. Let $\mathbb{R}$ be a copy of the real field which is also an elementary substructure of R. Let $H < GL(n, R)$ be a semialgebraically simple, noncommutative semialgebraic group (over R). Then H is semialgebraically isomorphic to a semialgebraic subgroup of $GL(n, R)$ which is defined over $\mathbb{R}$.*

Putting the two facts together we see that there is a (necessarily saturated) real closed field R definable in M and a semialgebraic subgroup $H < GL(n, R)$ which is defined (semialgebraically) over $\mathbb{R}$ and such that G is definably isomorphic to H. So we may assume that $H = G$.

Let $G(\mathbb{R})$ be the set of points of G with coordinates in $\mathbb{R}$. Then $G(\mathbb{R})$ is a simple real Lie group. There are two cases:

Case I. $G(\mathbb{R})$ is noncompact.

Then Theorem 6.1 of [6] says that G is abstractly simple, namely has no proper nontrivial normal subgroups. So by 2.9 (iii) G has NO proper subgroup which is type-definable in M and of bounded index. Thus $G = G^{00}$, and Conjecture 1.1 is verified for G.

Case II. $G(\mathbb{R})$ is compact.

G can be considered as the nonstandard version ${}^*G(\mathbb{R})$ of $G(\mathbb{R})$. We have the semialgebraic distance function $d(-,-)$ with values in $R_{\geq 0}$ on $GL(n, R)$ so also G. Let e be the identity element of G. We call $a \in G$ infinitesimal if $d(a, e) < r$ for all $r \in \mathbb{R}$. For each $a \in G$ there is (by compactmess of $G(\mathbb{R})$), unique $a' \in G(\mathbb{R})$ such that $d(a, a') < r$ for all $r \in \mathbb{R}$. The standard part map is the map $st : G \to G(\mathbb{R})$ which takes a to a'. This map is a surjective homomorphism and $Ker(st)$ is the set of infinitesimals $\mu(e)$ of G, which is a normal type-definable subgroup of G, of bounded index (size of the continuum). We have to show two things:
(i) the logic topology on $G/\mu(e))$ coincides with the standard topology on $G(\mathbb{R})$,

(ii) $\mu(e)$ is type-definably connected (in the sense of the ambient structure M).

(i) is well-known, but we say a few words. Both the logic topology and standard (Euclidean) topology on $G(\mathbb{R})$ are compact Hausdorff, so it suffices to show that one is stronger than the other. Let $X \subset G(\mathbb{R})$ be closed in the logic topology. So $st^{-1}(X) \subseteq G$ is type-definable in the structure M, by the set $\Sigma(x)$ of formulas say. Let $a \in G(\mathbb{R})$ be in the closure of X in the sense of the Euclidean topology. So for every positive $r \in \mathbb{R}$, there is $b \in X$ such that $d(a,b) < r$. So for every positive $r \in \mathbb{R}$, $\Sigma(x) \cup \{d(x,a) < r\}$ is consistent (in the sense of M). By the compactness theorem (of first order logic), $\Sigma(x) \cup \{d(x,a) < r : r \in \mathbb{R}_+\}$ is consistent. This yields $a' \in G$ realizing $\Sigma(x)$ such that $st(a') = a$. So $a \in X$. We have proved that X is closed in the standard topology.

Now we prove (ii) which will show that $\mu(e) = G^{00}$. Rather surprisingly we need to use some of the theory of compact Lie groups, for which we refer to [1]. $G(\mathbb{R}) < GL(n,\mathbb{R})$ is a connected compact linear Lie group, which is semialgebraic (in fact it is known that $G(\mathbb{R})$ is in fact real algebraic). By definition a maximal torus of $G(\mathbb{R})$ is a maximal closed connected commutative subgroup. Any maximal torus is (semi-) algebraic, and is in fact a product of 1-dimensional connected semialgebraic subgroups of $G(\mathbb{R})$. (For now we mean 1-dimensional in the real semi-algebraic sense.) The maximal tori of $G(\mathbb{R})$ cover $G(\mathbb{R})$, so it follows that there are a finite number $T_1, ..., T_k$ of 1-dimensional semialgebraic subgroups of $G(\mathbb{R})$ such that $G(\mathbb{R}) = T_1 \cdot T_2 \cdot \dots \cdot T_k$ (namely each element of $G(\mathbb{R})$ is of form $a_1 \cdot a_2 \cdot \dots \cdot a_k$, where $a_i \in T_i$). This transfers to G. Let us write $T_i(R)$ for the "interpretation" of T_i in G. It follows that:
(*) $\mu(e) = (\mu(e) \cap T_1(R)) \cdot \dots \cdot (\mu(e) \cap T_k(R))$.
Note that the $T_i(R)$ are 1-dimensional definably connected commutative groups in the sense of M, which are definably compact, that is fit into Case II of the proof of Proposition 3.5. It is rather easy to see that
(**) $\mu(e) \cap T_i(R)$ is precisely $T_i(R)^{00}$, which is thus type-definably connected.

Now suppose that H is a type-definable subgroup of $\mu(e)$ of bounded index. Then $H \cap T_i(R)$ has bounded index in $T_i(R)$, so by (**) must contain $\mu(e) \cap T_i(R)$. By (*), $H = \mu(e)$. We have proved (ii), and hence Proposition 3.6. $\square$

Many questions come naturally to mind concerning type-definable groups in o-minimal structures and the various forms of connectedness. For example, at the general level: Let M be an arbitrary (saturated) o-minimal structure, G a definable group in M, and H a type-definable, torsion-free, divisible, commutative subgroup of G. Is H type-definably connected (in the sense of Definition 2.8)?

We can also restrict Conjecture 1.1 to arbitrary groups definable in specific theories. Rather surprisingly, I do not know the truth of the conjectures for two of the most basic o-minimal theories, $DOAG$ the theory of divisible ordered abelian groups, and RCF the theory of real closed fields.

In the case of $DOAG$ all definable connected definable groups are commutative. Moreover, if we let V denote the underlying ordered $\mathbb{Q}$-vector space, then definably compact definably connected groups should be of the form V^n/Λ where Λ is a "lattice" and the quotient is interpreted suitably. Such things have been studied in [7] and examples given where the group G is not definably the product of 1-dimensional groups, and so one cannot simply reduce to Proposition 3.5.

In so far as RCF is concerned, one of the interesting cases is where G is of the form $A(R)^0$ where A is an abelian variety defined over the (saturated) real closed field R and $A(R)^0$ denotes the semialgebraically connected component of the group of R-rational points of A. If G is definably the product of 1-dimensional groups, then Proposition 3.5 can be used. However, this will not apply if A is a simple abelian variety of (algebraic) dimension > 1. If A is defined over $\mathbb{R}$, then we have the standard part map $st : A(R)^0 \to A(\mathbb{R})^{00}$ at our disposal, and $Ker(st)$ should coincide with G^{00} (although one still needs to prove it is type-definably connected).

More generally we have at our disposal the valuation ring V of finite elements of R, and the corresponding residue field $\mathbb{R}$. If A has "good reduction" with respect to this data, then as above we obtain a positive answer to Conjecture 1.1. (Good reduction means that A is rationally isomorphic to an abelian variety defined over V, such that when we take the image of the defining equations for A under the reduction map $\pi : V \to \mathbb{R}$, we obtain an abelian variety over $\mathbb{R}$.)

So the "difficult case" is when A is a simple abelian variety over R, of (algebraic) dimension > 1 and which does not have good reduction.

Even in the case when A is an elliptic curve (and so $G = A(R)^0$ is

1-dimensional, and definably compact), and where Proposition 3.5 gives a positive answer to Conjecture 1.1, it may be interesting to see in which part of the valued field structure $(R, \Gamma, \mathbb{R})$ the quotient G/G^{00} lives, and how this relates to the issue of good/semistable reduction.

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"Complex-like" analysis in o-minimal structures

Y. Peterzil and S. Starchenko[4]

Abstract

In these notes we survey the content of three of our recent papers ([7], [6], [8]), where we treat analogues of basic notions in complex analysis, over an arbitrary algebraically closed field of characteristic zero, in the presence of an o-minimal structure.

Setting

$\mathcal{R} = (R, <, +, \cdot, \ldots)$ is an o-minimal expansion of a real closed field, and $K = R(\sqrt{-1})$ the algebraic closure of R, identified with R^2. Subsets of K^n and maps from K^n into K will be viewed as subsets of R^{2n} and maps from R^{2n} into R^2, respectively. Thus, K and its field operations are definable in $\mathcal{R}$ and every algebraic subset of K^n is definable in $\mathcal{R}$.

1 A model theoretic result

Our original motivation for this project was the following (see [7] Theorem 3.1):

Theorem 1.1 *Let $\mathcal{K} = (K, +, \cdot, \ldots)$ be an expansion of K all of whose atomic relations are definable in $\mathcal{R}$. If $\mathcal{K}$ is a proper expansion of $(K, +, \cdot)$ then the field R is definable in $\mathcal{K}$.*

[4]The paper was written while the first author spent his Sabbatical at the University of Illinois at Urbana-Champaign. The research was partially supported by grant 1999316 US-Israel Binational Science Foundation (BSF).

By "proper expansion of $(K, +, \cdot)$", we mean that there is a definable set in $\mathcal{K}$ which is not definable in $(K, +, \cdot)$ (even with parameters).

In particular, the theorem implies that there are no proper expansions of an algebraically closed field which are stable and yet interpretable in an o-minimal structure. One can derive, for example, Chow's classical theorem on analytic subsets of $\mathbb{P}(\mathbb{C})$ using this theorem (see [7] page 340).

A weaker version of the theorem was proved earlier by D. Marker (see [4]), where the o-minimal structure $\mathcal{R}$ was assumed to be a real closed field.

Here is another curious corollary, which was recently pointed out to us by M. Tressel. Consider an algebraically closed field K of characteristic zero and a set $S \subseteq K^n$. K may contain in general infinitely many non-isomorphic maximal real closed subfields. For any such field R, K can be identified with R^2, and hence S can be viewed as a subset of R^{2n}. Let us call S *o-minimal with respect to* R if the structure $(R, +, \cdot, S)$ is o-minimal (with the natural ordering of R). For example, every algebraic variety over K is o-minimal with respect to any maximal real closed subfield. This brings up an interesting question:

Question. Characterize the structures $(K, +, \cdot, S)$ such that S is o-minimal with respect to some maximal real closed field $R \subseteq K$.

Our theorem implies:

Corollary 1.2 *For $S \subseteq K^n$, the following are equivalent:*

(1) *S is o-minimal with respect to every maximal real closed field $R \subseteq K$.*

(2) *S is o-minimal with respect to two distinct maximal real closed subfields $R_1 \neq R_2 \subseteq K$.*

(3) *S is a constructible set over K. Namely, S is definable in the field $(K, +, \cdot)$.*

In particular, unless S is constructible over K, it can be semi-algebraic with respect to at most one real closed subfield of K.

Proof. We only need to show that (2) implies (3). Indeed, if S is not definable in the field structure of K then, by Theorem 1.1, the fields R_1 and

R_2 are both definable in $(K, +, \cdot, S)$. But then, since R_2 is definable in the o-minimal structure $(R_1, +, \cdot, S)$, it must be equal to R_1, since the intersection of the two fields is infinite. $\Box$

Let us briefly review the proof of Theorem 1.1 in the particular case where the underlying real closed field R is the field of real numbers and $K = \mathbb{C}$.

Step 1. Either $\mathcal{K}$ is strongly minimal (namely, only finite and co-finite subsets of K are definable in $\mathcal{K}$), or the field $\mathbb{R}$ is definable in $\mathcal{K}$: We use here, word-for-word, an argument of D. Marker from [4].

Step 2. If $\mathcal{K}$ is strongly minimal then every definable function from $\mathbb{C}$ into $\mathbb{C}$ is a holomorphic function, up to finitely many points. (We will not touch on this step here, see [7] Theorem 3.9).

Step 3. Every function from $\mathbb{C}$ into $\mathbb{C}$ which is holomorphic outside finitely many points and definable in an o-minimal structure must be a rational function over $\mathbb{C}$.

We use at this step the following lemma which shows that no essential singularities can be defined in o-minimal structures.

Lemma 1.3 *Let $D \subseteq \mathbb{C}$ be the open unit disc. If $f : D \setminus \{0\} \to \mathbb{C}$ is holomorphic and definable in an o-minimal structure then 0 is either a removable singularity or a pole of f.*

Proof. If 0 were an essential singularity then, by classical analysis, for every open U containing 0, $f(U)$ was dense in $\mathbb{C}$. Said differently, if $G_f \subseteq (\mathbb{C} \setminus \{0\}) \times \mathbb{C}$ is the graph of f then $\overline{G_f} \cap \{0\} \times \mathbb{C} = \mathbb{C}$. But then, $\dim \overline{G_f} \setminus G_f = 2 = \dim G_f$, contradicting o-minimality. $\Box$

Now, assume that $f : \mathbb{C} \to \mathbb{C}$ is definable in an o-minimal structure and holomorphic outside finitely many points. By the above Lemma, f has only poles in $\mathbb{C} \cup \{\infty\}$. Using Liouville's Theorem it is easy to conclude that f is rational.

Using Steps 1–3, together with an observation of E. Hrushovski from [2] one can conclude Theorem 1.1.

All steps in the proof, except Step 3, generalize immediately to an arbitrary R and K. However, in order to generalize Step 3, one needs to develop analogous theory to complex analysis, for a general o-minimal expansion of a real closed field R and its algebraic closure K. The main obstacle here is of course the absence of the prime tools of classical analysis: integration and power series, since K is not locally compact anymore. Instead, we use "topological Analysis".

Our presentation here is divided roughly into 3 sections. In the first section we present the one-variable case (see [7]). Here we adapt, almost verbally, the work of Whyburn, Connell and others (see Whyburn's book [12]) on the topological foundations of complex analysis. In section 2 we present definitions and results for functions of several variables (see [6]), while in the the third section we discuss the general notion of a K-manifold. We also describe our work on 1-dimensional tori ([8]).

Related work: A sheaf theoretic approach to a similar project, in the semialgebraic setting, was carried out by Huber and Knebusch, in [3]

2 Topological analysis. Functions of one variable

2.1 Definition of K-holomorphic functions

Let $U \subseteq K$ be a definable, open set. A definable $f : U \to K$ is *K-differentiable at* $z_0 \in K$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists in } K$$

The limit is called $f'(z_0)$.

If f is K-differentiable at every $z \in U$ it is called *K-holomorphic on U*.

Examples.

• In $\overline{\mathbb{R}} = (\mathbb{R}, +, \cdot)$: Complex polynomials, complex algebraic functions (away from branch points) are $\mathbb{C}$-holomorphic . More generally, any polynomial over K is K-holomorphic.

• In $(R, +, \cdot)$, where R is real closed: Every polynomial in $K[z]$ is K-holomorphic.

• In $\mathbb{R}_{\text{an}} = (\mathbb{R}, +, \cdot, \{f | f : [-1, 1]^n \to \mathbb{R} \text{ real-analytic}\})$:

Every germ at 0 of a holomorphic function is definable $\mathbb{R}_{\mathrm{an}}$ on some neighborhood of 0 and therefore is $\mathbb{C}$-holomorphic.

• In $\mathbb{R}_{\mathrm{an,\,exp}} = (\mathbb{R}_{\mathrm{an}}, e^x)$: e^z is definable and $\mathbb{C}$-holomorphic on every horizontal strip $\{a < \mathrm{im}(z) < b\}$, $a < b \in \mathbb{R}$.

• Every branch of $\ln z$ is definable in $\mathbb{R}_{\mathrm{an,\,exp}}$.

• In an elementary extension of $\mathbb{R}_{\mathrm{an,\,exp}}$: Take $\alpha \in R$ infinitesimally close to 0, $e^{\alpha z}$ is K-holomorphic on $-1/\alpha < \mathrm{im}(z) < 1/\alpha$. The same tome e^z is *not* definable on all of $\mathbb{C}$, because the set $\{z : e^z = 1\}$ is infinite and discrete.

Claim 2.1 *Let $\mathcal{M}$ be an o-minimal expansion of the field of real numbers.*

(1) *If e^z is definable in $\mathcal{M}$ on a definable set $U \subseteq \mathbb{C}$ then $\{\mathrm{im}(z) : z \in U\}$ is bounded in $\mathbb{R}$.*

(2) *If, for some $a \in \mathbb{C}$, the function a^t is definable in $\mathcal{M}$ for all t large enough in $\mathbb{R}$, then a must be in $\mathbb{R}$.*

Proof. (1) By moving to the Pfaffian closure (see [11]), we may assume that the real exponential function e^x is definable in our structure. But now the function $e^{\mathrm{im}(z)i} = e^z/e^{\mathrm{re}(z)}$ is definable for all $z \in U$. If $\mathrm{im}(z)$ is unbounded as z varies in U, then we get the definability of e^z on the imaginary axis, which is impossible.

(2) Again, we may assume that e^x is definable in $\mathcal{M}$. Choose $b \in \mathbb{C}$ such that $e^b = a$. The function e^{bt} is definable for all t greater than some $r \in \mathbb{R}$. By (1), the set $\{bt : t > r\}$ must have bounded imaginary part, which implies that $b \in \mathbb{R}$, and hence $a \in \mathbb{R}$. □

From now on: all K-holomorphic maps are assumed definable in an o-minimal structure $\mathcal{R}$.

The following is an easy corollary of the fact that K-linear maps from $(R^2, +)$ into $(R^2, +)$ are of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Cauchy-Riemann. $f : U \to K$ is K-holomorphic on U if and only if $f = (f_1, f_2) : R^2 \to R^2$ is R-differentiable and

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}\ ;\ \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

Unfortunately, this is more or less as far as one can push the elementary approach to the theory.

2.2 The winding number

Definition 2.2 Assume that $C \subseteq K$ is a definable closed oriented curve (namely, the continuous image, under a definable map $\sigma : [0,1] \to K$ satisfying $\sigma(0) = \sigma(1)$). Assume also that $f : C \to \mathbb{S}^1$ is definable and continuous. *The winding number of* f, $W_C(f)$, is the number of counter-clockwise turns of $f(z)$ around $\mathbb{S}^1$, as z travels around C.

By o-minimality, this definition indeed makes sense and is always an integer in $\mathbb{Z}$. It can be calculated as the number of pre-images of a generic $w \in \mathbb{S}^1$, counted with direction (it turns out to be independent of w). In our paper, we defined the winding number differently, using an ad-hoc notion of a universal covering for $\mathbb{S}^1$. For a more general definition of winding number in the o-minimal context, see [1].

Properties
1. If $f : C \to \mathbb{S}^1$ is not surjective then $W_C(f) = 0$.
2. $W_C(f \cdot g) = W(f) + W(g)$.
3. (**Uniform definability of** $W_C(f)$.) If $\{f_s : s \in S\}$ is a definable family of maps from C_s into $\mathbb{S}^1$ then the map $s \mapsto W_{C_s}(f_s)$ is definable.
4. (**Homotopy invariance**.) If $f, g : C \to \mathbb{S}^1$ are *definably homotopic* then $W_C(f) = W_C(g)$.

We can now extend our definition of a winding number.

Definition 2.3 If $f : C \to K$ is definable, continuous and $w \in K \setminus f(C)$ then

$$W_C(f, w) = W_C\left(\frac{f(z) - w}{|f(z) - w|}\right).$$

2.3 Winding number of K-holomorphic maps

Lemma 2.4 *If $f : U \to K$ is K-holomorphic, $z_0 \in U$, and $f'(z_0) \neq 0$, then for every small circle C around z_0, $W_C(f, f(z_0)) = 1$.*

Proof. Assume $f'(z_0) = d$.
Compare the maps

$$\frac{f(z) - f(z_0)}{|f(z) - f(z_0)|} \text{ and } \frac{d(z - z_0)}{|d||z - z_0|}.$$

For sufficiently small z, the quotient of the two maps is close to 1, therefore its winding number is 0 (see property (1) above). So, by property (2), the two maps have the same winding number, which is easily seen to be 1. □

Notations. We will denote by D the closed unit disc in K, and by C the unit circle.

Lemma 2.5 (Main lemma) *Assume that $f : D \to K$ is continuous and K-holomorphic on $\mathring{D}$, $w \in K \setminus f(C)$. Then:*

(1) *If $w \notin f(D)$ then $W_C(f, w) = 0$.*

(2) *If $w \in f(D)$ then $W_C(f, w) > 0$.*

(3) *Every component of $K \setminus f(C)$ is either contained in $f(D)$ or is disjoint from it.*

The same remains true if f is assumed to be K-holomorphic off a definable subset of D of dimension 1.

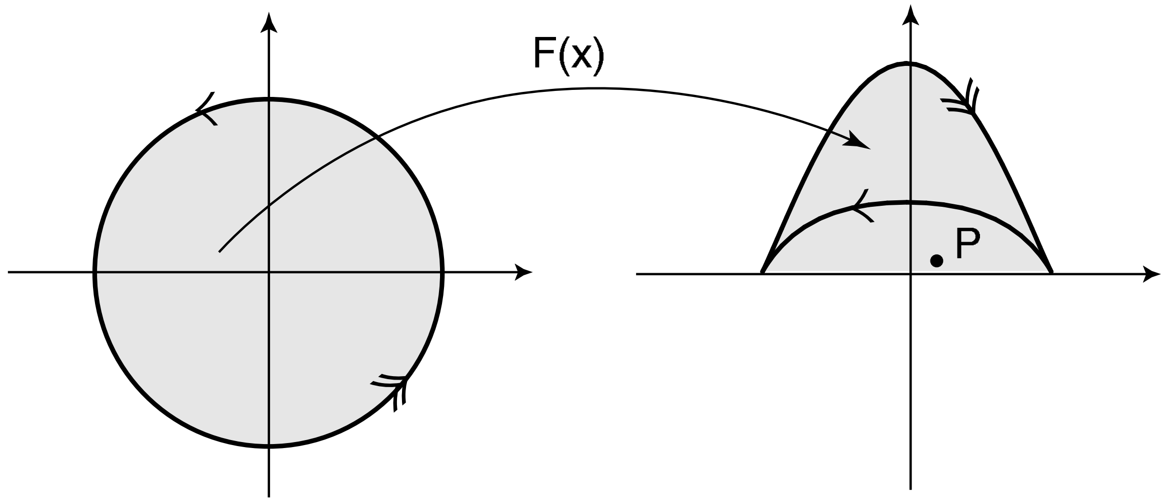
Proof. (1) Assume $w \notin f(D)$ and then shrink C to 0. Since $w \notin f(C)$ at each stage, the winding number $W_C(f, w)$ is definable and eventually equals 0.

(2) Assume $w \in W$ a definably connected component of $K \setminus f(C)$. Using o-minimality, we can show that there is $w_1 \in W$ such that $f^{-1}(w_1) = \{z_1, \ldots, z_k\}$ and $f'(z_i) \neq 0$ for each $i = 1, \ldots, k$. Each z_i contributes 1 to the winding number, by the previous lemma, so $W_C(f, w_1) = k$. By homotopy invariance, $W_C(f, w) = W_C(f, w_1) = k$.

(3) Follows from (1), (2) and homotopy invariance. □

An example of a real map where the above fails

Consider the map $F : (x, y) \mapsto (x, 2y^2 - y^3)$



P is in the range of $F(D)$ but $W_C(F, P) = 0$.

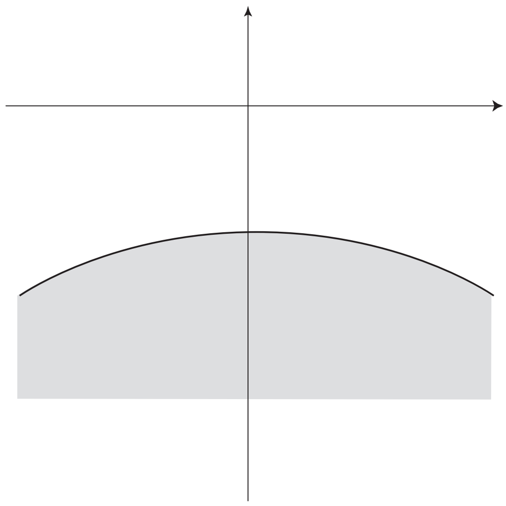
One of the main tools in the topological approach to complex analysis is the following.

Theorem 2.6 (The maximum principle) *If $f : D \to K$ is continuous and f is K-holomorphic on $\mathring{D}$ then $|f|$ attains its maximum on C. The same remains true if f is assumed to be K-holomorphic on $\mathring{D} \setminus L$, where* $\dim L = 1$.

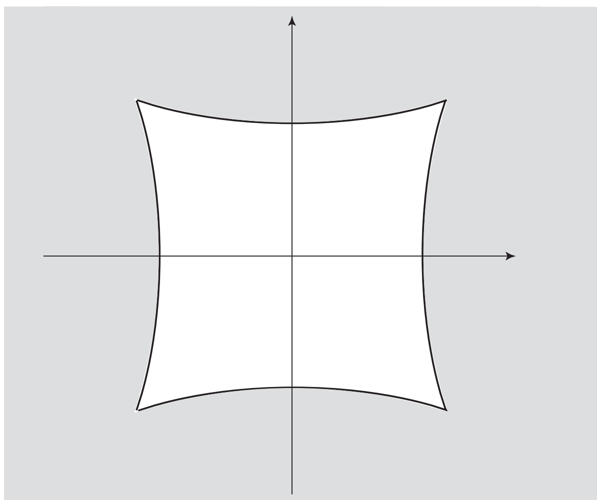
Proof. Assume $f(z) = w$ for $w \in D$. Then, either $w \in f(C)$ or $w \in W$, W a definably connected component of $K \setminus f(C)$. By the main lemma, $W \subseteq f(D)$. So, $f(z) = w \in \mathring{f}(D)$. It follows that $\max_{z \in D} |f(z)|$ is attained on $f(C)$. □

Theorem 2.7 (Identity theorem) *Assume that $f : D \to K$ is continuous on D and K-holomorphic on $\mathring{D}$ (possibly, off a definable set of dimension 1). Then, either f is constant, or for all $w \in K$, $f^{-1}(w)$ is finite.*

Proof. Without loss of generality, $w = 0$. Assume $f^{-1}(0)$ is infinite. Then, after rotations and translations we may assume that $f^{-1}(0)$ looks like the following picture (with the curve and possibly the shadowed region below it in $f^{-1}(0)$ but $0 \notin f^{-1}(0)$)



Define $H(z) = f(z)f(iz)f(-z)f(-iz)$. Now, $H^{-1}(0)$ looks like:



By Maximum Principle, H is identically zero inside the unshadowed region and so f is identically zero around 0. This can be carried out around any point therefore f is identically zero. $\square$

In fact, the same argument shows that not only can f take any value only finitely often (unless it is a constant function), but even on the boundary of D, f cannot take any value infinitely often (when extended continuously).

2.4 Other analogues of classical results

We omit the proofs of the theorems below. The first two follow easily from the Maximum principle together with the identity theorem.

Theorem 2.8 (Open mapping theorem) *A K-holomorphic map $f : U \to K$ is either constant or an open map.*

Theorem 2.9 (Liouville's theorem) *If $f : K \to K$ is K-holomorphic and $|f|$ is bounded then f is constant.*

Theorem 2.10 (Removing singularities) *If $f : D \setminus \{0\} \to K$ is K-holomorphic and bounded then 0 is a removable singularity.*

Theorem 2.11 (Strong removing of singularities) *If $f : D \to K$ continuous on D, and K-holomorphic on $\mathring{D} \setminus L$, $\dim L = 1$, then f is K-holomorphic on D.*

The classical proof of the theorem below makes heavy use of Cauchy's Theorem, so we include here a full account of its topological proof:.

Theorem 2.12 (Infinite differentiability) *If f is K-holomorphic on D then $f'(z)$ is K-holomorphic as well.*

Proof. We first show that $f'(z)$ is continuous at 0.

We consider $H(z, w) = \frac{f(z)-f(w)}{z-w}$, where we set $H(w, w) = f'(w)$.

Fix $\epsilon > 0$ and choose a circle C around 0 such that $|H(z, 0) - f'(0)| < \epsilon/2$ for all $z \in C \cup \mathring{D}_C$ (by $\mathring{D}_C$ we mean the interior of the disc carved by C).

Take a small neighborhood W of 0 which does not intersect C (thus $H(z,w)$ is continuous on $C \times W$) such that for all $w_1, w_2 \in W$ and for all $z \in C$,

$$|H(z,w_1) - H(z,w_2)| < \epsilon/2.$$

(This can be done due to the fact that C is definably compact. See Fact 2.4 in [7]).

We claim that for every $(z,w) \in \overset{\circ}{D}_C \times W$, $|H(z,w) - f'(0)| < \epsilon$. This easily implies that continuity of $f'(z)$ at 0.

Indeed, by the above, for every $z \in C$ and for every $w \in W$, $|H(z,w) - H(z,0)| < \epsilon/2$, and therefore for every such w and z, $|H(z,w) - f'(0)| < \epsilon$.

But fixing w, we may apply the Maximum principle to $H(z,w) - f'(0)$, as a function of z, and conclude that for all $z \in \overset{\circ}{D}_C$, $|H(z,w) - f'(0)| < \epsilon$, thus ending the proof of the continuity of f'.

To prove that $f'(z)$ is K-holomorphic at 0, we may reduce to the case where $f'(0) = 0$ and therefore need to show that $f'(z)/z$ has a limit as z tends to 0. Consider the function $h(z) = f(z)/z$, with $h(0) = 0$. By the "strong removing of singularity" theorem, h is K-holomorphic at 0, hence $h(z)/z = f(z)/z^2$ has a limit as z tends to 0. Moreover, by what we just showed, $h'(z)$ is continuous at 0, hence it has a limit as z tends to 0. By the rules of differentiation,

$$h'(z) = \frac{f'(z)}{z} - \frac{f(z)}{z^2},$$

thus $f'(z)/z$ has a limit as z tends to 0. □

2.5 Poles and zeroes

Definition 2.13 For K-holomorphic f in a punctured neighborhood of 0 and $n \in \mathbb{Z}$ we say that *the order of f at 0 is n* if $\lim_{z\to 0} f(z)/z^n$ exists in K and different than 0.

Theorem 2.14 (No "essential" singularities)
Assume that f is K-holomorphic in a punctured disc around 0.

(1) *If f is nonconstant then there exists $n \in \mathbb{Z}$ such that f has order n at 0. (In particular, there is no definable function like z^α for infinite $|\alpha|$).*

(2) *The order of f at z_0 equals $W_C(f, 0)$ for all small circles C around z_0.*

(3) *If $0 \notin f(C)$ then $W_C(f, 0) = \#$ zeroes $- \#$ poles in $\mathring{D}_C$.*

2.6 No new entire functions

Using the analogue Louiville's theorem, together with (1) above, the theorem below is proved just like its classical version which was proved in the introduction.

Theorem 2.15 *If $f : K \to K$ is K-holomorphic outside a finite set then f is a rational function.*

2.7 Formal power series

Assume that f is K-holomorphic on a disc $D \setminus \{0\}$ and $n = ord_0(f)$.

Existence: For every $i \geq n$ there is $a_i(f) \in K$, such that $\forall k \geq n$, $f(z) - \Sigma_{i=n}^{k} a_i(f) z^i$ has order $k + 1$ at 0.

Uniqueness: If g is K-holomorphic on $D \setminus \{0\}$ and for all $i \in \mathbb{Z}$, $a_i(f) = a_i(g)$ then $f = g$ on D.

WARNING: Although the map into the ring of formal power series is injective, power series do NOT necessarily converge in K.

Question Assume that $\Sigma_{n=0}^{\infty} a_n x^n$ is the series associated to a K-holomorphic function at 0, and that all $a_n \in \mathbb{Q}$ (or $\mathbb{R}$). Does the series necessarily converge in $\mathbb{R}$?

2.8 Definability results

Assume that $F = \{f_s : s \in S\}$ is a definable family of K-holomorphic functions on $U_s \setminus \{z_s\}$.

Theorem 2.16 (Uniform definability of order) *The map $s \mapsto ord_{z_s} f_s(z)$ is definable. In particular, the order is bounded as s varies in S.).*

Proof. This follows from the uniform definability of the winding number. □

Definition 2.17 For f K-holomorphic on a punctured disc around z_0, we let $res_{z_0} = a_{-1}(f)$, at z_0.

The following is a corollary of the fact that the order of f_s at z_s is bounded.

Theorem 2.18 (Uniform definability of residue) *The map $s \mapsto res_{z_s} f_s$ is definable.*

Theorem 2.19 (Classical integration) *If $F(z,w) : D \times W \to K$ is definable in an o-minimal structure and for every $w \in W$, $F(z,w)$ is meromorphic in z then the function*

$$G(w) = \frac{1}{2\pi i} \int_C F(z,w)dz$$

is definable.

Remark. Although the order of functions is bounded when they are uniformly defined, it does not mean that the degree of polynomials is bounded, when they are uniformly defined on some open set. The following example is due to A. Piekosz (see [9]): Let $\{a_n : n \in \mathbb{N}\}$ be an arbitrary sequence of complex numbers, all in the disc $D_{1/2} = \{|z| < 1/2\}$. Let

$$f(z,w) = \Sigma_{n=1}^{\infty} z^n (w - a_1) \cdots (w - a_n).$$

The function f is holomorphic on the closure of the set $D_{1/2}^2$, thus it is definable in $\mathbb{R}_{\text{an}}$. Note that whenever $w = a_n$, the function $f(z,w)$ is polynomial of degree n in z, therefore we have a definable family of functions on $D_{1/2}$ which includes polynomials of unbounded degrees.

Question. Is there a definable family of functions $\{f_s : s \in S\}$ from U to K, in an o-minimal structure, such that *for every* $s \in S$, the function $f_s(z)$ is a polynomial in z, and such that the degrees of the polynomials are unbounded in S?

(Note that if such a family existed then the structure could not be ω-saturated).

2.9 The missing ingredient: Analytic continuation

Given a K-holomorphic $f : U \to K$ definable in an o-minimal structure $\mathcal{M}$, when can we extend it *definably in* $\mathcal{M}$ beyond the boundary of U?

We clearly cannot do it always, since there might be points on the boundary where the function has "poles". However, this can happen at only finitely many points;

We prove in [7] Theorem 2.57 (D is still assumed to be the closed unit disc):

Theorem 2.20 (Boundary behavior) *If $f : \mathring{D} \to K$ is K-holomorphic then $f(z)$ has a limit in $K \cup \{\infty\}$ at every boundary point of D. Moreover, ∞ can occur in at most finitely many points.*

One tool is still available for definable analytic continuation.

Theorem 2.21 (The Schwartz reflection principle) *If $f(z)$ is definable on an open subset of the upper half plane, whose boundary contains part of the x-axis, and if $f(z)$ takes real values on this part of the boundary then the function $F(z) = \overline{f}(\overline{z})$ is a K-holomorphic continuation of $f(z)$ below the x-axis.*

The Schwartz Principle, together with some positive answers to the following question, could help in producing definable analytic continuation, at least locally.

Question (A. Wilkie). Which "Riemann mappings" are definable in an o-minimal structure? E.g., Let $U \subsetneq \mathbb{C}$ be a simply connected, open, semialgebraic set. By Riemann's theorem, U is bi-holomorphic with the open disc. Is this (almost unique) bi-holomorphism definable in some o-minimal structure?

3 Functions of several variables

We now move to describe definitions and basic properties of K-holomorphic functions in several variables. We develop the theory through reduction to the 1-variable case, using 1-fibers of definable sets in K^n (see [6]).

Definition 3.1 Let $U \subseteq K^n$ be a definable open set. A definable $f : U \subseteq K^n \to K$ is *K-holomorphic on U* if it is continuous on U, and K-holomorphic in each variable separately.

By Osgood's Lemma, the definition is equivalent, over $\mathbb{C}$, to classical definitions, using convergent power series.

One of the main features which distinguishes real multivariable calculus from the complex one is the following important tool. It is an immediate corollary of the fact that the zeroes of a K-holomorphic function can be counted using the winding number, together with the homotopy invariance of the winding number.

Fact 3.2 (Counting zeroes) *For C a definable simple closed curve, assume that $\{f_{\overline{w}} : \overline{w} \in S\}$ is a definable, continuous family of K-holomorphic functions on $C \cup \mathring{D}_C$ with no zeroes on C, and that S is definably connected. Then the number of zeroes of $f_{\overline{w}}$ in $\mathring{D}_C$ is the same for all $\overline{w} \in S$.*

Corollary 3.3 ("Hurwitz theorem") *For $U \subseteq K$ definable and open, assume $f(\overline{w}, z) : R^k \times U \to K$ is continuous, and for every $\overline{w} \in R^k$, $f(\overline{w}, -)$ is K-holomorphic in z. Assume also that $F(\overline{w}_0, -)$ has order p at z_0. Then, for all sufficiently small $U_1 \subseteq U$ containing z_0, there is $W_1 \subseteq W$ containing $\overline{w}_0$, such that $\forall \overline{w} \in W_1$, $f(\overline{w}, -)$ has precisely p zeroes, counted with multiplicity, in U_1.*

3.1 Basic properties

If $U \subseteq K^n$ is open, definably connected, and $f : U \to K$ is K-holomorphic, $U \subseteq K^n$ then:

(i) Every partial derivative of f is K-holomorphic on U.

(ii) The zero set of f has dimension $\leq 2n - 2$, or else $f \equiv 0$.

(iii) If all partial derivatives of f vanish on U then $f \equiv 0$.

(iv) (**Maximum Principle 1**). If, for some $z_0 \in U$, $|f(z_0)|$ is maximal in U then $f \equiv 0$.

(v) (**Maximum Principle 2**). If $D_i, i = 1, \ldots, n$ are closed discs in K whose boundaries are $C_1, \ldots, C_n$, and if $f : D_1 \times \cdots \times D_n \to K$ is K-holomorphic then $|f|$ attains its maximum on $C_1 \times \cdots \times C_n$.

We can now prove a generalization of the 1-variable theorem for entire functions.

Theorem 3.4 *If $f : K^n \to K$ is K-holomorphic then f is a polynomial function.*

Proof. Induction on n. Write $f = f(\overline{w}, z)$. By the 1-variable case, for every $\overline{w} \in K^{n-1}$, $f(\overline{w}, -)$ is a polynomial of degree $n(\overline{w})$.

Since being K-holomorphic is a first order property, this remains true in elementary extensions, so $n(\overline{w})$ is bounded on W. I.e., there is an n such that for all $\overline{w} \in K^{n-1}$, $f(\overline{w}, z) = \Sigma_{k=0}^{n} a_k(\overline{w}) z^k$.

Now, $a_k(\overline{w}) = \partial^k / \partial z^k f(\overline{w}, 0)$. Thus, each $a_k(\overline{w})$ is K-holomorphic on K^{n-1}. By induction, each a_k is a polynomial and therefore so is f. $\square$

3.2 Removing singularities

The following theorems are analogues of classical theorems on the removing of singularities.

Theorem 3.5 *Let $U \subseteq K^n$ be definable and open, $L \subseteq U$ a definable set, $f : U \setminus L \to K$ a K-holomorphic function.*
Case A. $\dim L \leq \dim U - 1$. *If f is continuous on all of U then it is K-holomorphic on all of U.*
Case B. $\dim L \leq \dim U - 2$. *If f is locally bounded on L then f is K-holomorphic on all of U.*
Case C. $\dim L \leq \dim U - 3$. *In this case f is necessarily K-holomorphic on all of U.*

3.3 The ring of germs of K-holomorphic functions

Definition 3.6 Let $\mathcal{O}_n$ be the ring of germs of (definable) K-holomorphic functions at $0 \in K^n$.

Properties.

(i) $\mathcal{O}_n$ is a subring of $K[[\overline{z}]]$.

(ii) $\mathcal{O}_n$ is a local ring.

(iii) Weierstrass Preparation and Division Theorems hold in $\mathcal{O}_n$.

(iv) $\mathcal{O}_n$ is Noetherian.

(When the underlying field is $\mathbb{C}$, these results are basically contained in the paper of Van Den Dries: "On the elementary theory of restricted analytic functions")

One corollary of the Weierstrass Preparation Theorem is:

Corollary 3.7 *If f is K-holomorphic on an open and definably connected $U \subseteq K^n$ then either f vanishes on U or the dimension of the zero set of f is exactly $2n-2$.*

Remark. The definition of the polynomial h in WPT used only the zero set of $f(\overline{w}, z)$ and the multiplicity of $f(\overline{w}, -)$ at each point.

It turns out that the multiplicity at each point can be recovered from the zero set of f alone, and therefore we have the following:

Given the zero-set Z of a K-holomorphic function f, one can recover uniformly, at each point $\overline{z} \in Z$, a K-holomorphic function $h_{\overline{z}}$, whose zero set around $\overline{z}$ equals Z. $h_{\overline{z}}$ is definable in the structure $(R, <, +, \cdot, Z)$. In particular, in every elementary extension of $(R, <, +, \cdot, Z)$, Z is still locally the zero set of a K-holomorphic function.

3.4 Boundary behavior

The boundary version of the identity theorem for K-holomorphic functions of one variable (see comment after Identity Theorem 1) easily extends to the multi-variable theorem (see 2.13 (2) in [6]).

Theorem 3.8 *Let $V \subseteq K^n$ be a definably connected open set, $f : U \to K$ a K-holomorphic function. Let Z be the set of all points z_0 in the topological closure of V such that the limit of f(z), as z tends to z_0 in V, exists and equals to 0. If $\dim Z \geq 2n-1$ then f is the constant zero function on V.*

3.5 K-meromorphic functions

Definition 3.9 A definable partial function $f : U \subseteq K^n \to K$ is *K-meromorphic on U* if for every $\overline{z} \in U$, there are K-holomorphic functions g, h on a neighborhood of $\overline{z}$ such that $f = g/h$ on this neighborhood (and their domain of definition is the same).

A question arises whether a K-meromorphic function remains so in every elementary extensions. The answer is positive due to the following result, whose proof we do not include here.

Theorem 3.10 *For $U \subseteq K^n$ open. If $f : U \setminus L \to K$ is K-holomorphic and* $\dim L \leq 2n - 2$ *then f is K-meromorphic on U.*

Corollary 3.11 *If f is K-meromorphic on U then it remains so in elementary extensions.*

Proof. At a neighborhood of every $\overline{z} \in U$, $f = g_{\overline{z}}/h_{\overline{z}}$. f is K-holomorphic in this neighborhood outside the zero set of $h_{\overline{z}}$ which is of dimension $2n-2$. Thus, f is K-holomorphic on U outside a definable set of dimension $2n-2$. This is a first order property which remains true in elementary extensions. Applying the last theorem, we obtain that f remains K-meromorphic in elementary extensions as well. $\square$

4 K-groups and nonstandard tori

We consider in this section analogues of complex manifolds and complex analytic groups in the context of definable sets and K-holomorphic maps. We mainly focus on an analogue to the classical construction of elliptic curves as one-dimensional tori, namely, as the quotient of $(\mathbb{C}, +)$ by a 2-lattice. Classically, any such quotient is isomorphic, as a complex analytic group, to a nonsingular cubic projective curve (with its group structure), and vice-versa. We describe here a parameterized version of this construction, in the o-minimal context, which allows us to view the family of all one-dimensional complex tori as definable in $(\mathbb{R}, <, +, \cdot)$. Our main theorem (see [8]) says that, in the structure $\mathbb{R}_{\mathrm{an,exp}}$, every smooth nonsingular projective cubic over K, is K-isomorphic to a one-dimensional torus and the family of these isomorphisms is given uniformly. However, there is no uniform way to identify every one-dimensional complex torus with an algebraic curve, thus giving rise in elementary extensions to "nonstandard tori". This last phenomenon shows that in contrast to the classical case there is no definable version of "Riemann Existence Theorem" in arbitrary o-minimal structures (see [5] for a similar question in the model theoretic context of compact complex manifolds).

Definition 4.1 A *definable K-manifold M of dimension n* is a Hausdorff topological space M covered by finitely many definable charts $(U_i, \varphi_i,)$ such that each φ_i is a homeomorphism from U_i onto open definable $\varphi_i(U_i) \subseteq K^n$; every transition map $\varphi_i \circ {\varphi_j}^{-1}$ is definable and K-holomorphic on $\varphi_j(U_j \cap U_i)$.

A definable manifold M is *definably connected* if it is not a union of two proper disjoint definable open subsets. (Equivalently, every definable locally constant function is constant.)

A definable manifold M is *definably compact*, if for every definable $f : (0,1) \longrightarrow M$ the limit $\lim_{x \to 0^+} f(x)$ exists in M. (Equivalently, M is definably homeomorphic to a closed and bounded subset of R^k.)

Given two definable K-manifolds M and N, and an open definable subset $U \subseteq M$ a *definable K-holomorphic map* $f : U \to N$ is a definable map, which, when read through the charts, is K-holomorphic. We call such f a *K-biholomorphism* if it is a bijection and its inverse is K-holomorphic as well.

As in the classical case we have that the only entire K-holomorphic functions on a definably connected definably compact K-manifold functions are constant functions.

Claim 4.2 *Let M be a definably compact definable K-manifold. Every definable K-holomorphic $f : M \longrightarrow K$ is constant.*

Proof. Since M is definably compact, $|f|$ attends its maximum value at some $m_0 \in M$. By multi-variable Maximum Principle, the set $U = \{m \in M\, f(m) = f(m_0)\}$ is open. It is also closed. Since M is definably connected, $U = M$. $\square$

If M and N are definable K-manifolds then the product $M \times N$ also has a natural structure of a definable K-manifold. *A definable K-group G*

is a definable K-manifold together with a definable K-holomorphic group operation on it.

Lemma 4.3 (Rigidity lemma) *Let X, Y, Z be definably connected definable K-manifolds, $F : X \times Y \to Z$ a definable K-holomorphic function. Assume that X is definably compact, and that for some $y_0 \in Y$ and $z_0 \in Z$, $F(X \times \{y_0\}) = \{z_0\}$. Then for all $y \in Y$, $F(-, y)$ is a constant function in the first variable.*

Proof. Let $W \subseteq Z$ be a small definable open neighborhood of z_0 definably biholomorphic to an open subset of K^n. Since X is definably compact, there is an open neighborhood $V \subseteq Y$ of y_0 such that $F(X, y) \subseteq W$ for every $y \in V$. Let $y \in V$. Since X is definably compact, every holomorphic function on X is constant, so $F(-, y)$ is constant in the first variable on all of X. But then $\partial F / \partial x$ vanishes on $X \times V$ and therefore it vanishes on $X \times Y$, proving the desired result. □

As in the classical case we obtain the following corollary.

Corollary 4.4 *Let G be a definable K-group which is definably compact. Then,*

(1) *G is an abelian group.*

(2) *If H is another definable K-group and $f : G \to H$ a K-holomorphic function then there is $h_0 \in H$ such that $x \mapsto f(x)h_0^{-1}$ is a group homomorphism.*

Proof. (1) Consider the map $F : (x, y) \mapsto xyx^{-1}y^{-1}$ from $G \times G$ into H. Since $F(x, e) = e$, $F(-, y)$ is constant for every $y \in G$.
(2) Let $h_0 = f(0)$ and $F : x \mapsto f(x)h_0^{-1}$, so $F(0) = e$. We need to show that $F(x + y) = F(x)F(y)$, i.e. the map $h : (x, y) \mapsto F(x + y)^{-1}F(x)F(y)$ is a constant map. Obviously, $h(0, y) = h(x, 0) = e$. □

4.1 One-dimensional K-tori

Before we define one-dimensional K-tori let us consider the simpler case of a one-dimensional R-torus and illustrate the difference between the classical construction and the point of view taken here.

4.1.1 One-dimensional R-torus S_1.

Let S_1 be the the abelian group on $[0,1)$ with addition

$$x+y=\begin{cases} x+y & \text{if } x+y<1,\\ \\ x+y-1 & \text{if } x+y\geq 1.\end{cases}$$

Obviously it is an $\mathcal{R}$-definable group. As a definable group manifold, it requires only two definable charts, (U_1,φ_1), (U_2,ϕ_2), where $U_1=(0,1)$ with ϕ_1 being the identity map, and $U_2=[0,\frac{1}{3})\cup(\frac{2}{3},1)$ with

$$\phi_2(x)=\begin{cases} x+1 & \text{if } x\in[0,\frac{1}{3}),\\ \\ x & \text{if } x\in(\frac{2}{3},1).\end{cases}$$

In the classical case, $R=\mathbb{R}$, S_1 is usually described as the quotient group $\mathbb{R}/\mathbb{Z}$ with the quotient topology. Since $\mathbb{Z}$ is not definable in an o-minimal structure, this clearly cannot be done in our context. Moreover, if R is a non-archimedean real closed field then the interval $[0,1)$ is not fundamental anymore: it does not contain a representative for $\mathbb{Z}$-class of any infinitely large element.

Here is another classical construction that produces S_1 and which can be used in arbitrary real closed fields.

Gluing along a diffeomorphism. S_1 can be also seen as the interval $[0,1]$ with 0 and 1 glued along the diffeomorphism $\alpha: x\mapsto x+1$ as follows.

Let $\sim$ be the equivalence relation on $[0,1]$ generated by the set $\{(x,y)\in[0,1]^2\; y=\alpha(x)\}$. $[0,1)$ is a fundamental domain for $[0,1]/\sim$, and we can identify them. Since α commutes with addition on $[0,1]$, i.e $\alpha(x+y)=x+\alpha(y)=\alpha(x)+y$, the addition operation induces a group structure on $[0,1)$. We can endow the group with the quotient topology of $[0,1]/\sim$, obtaining a topological group identical to S_1 above.

Thus, this construction also allows as to get a definable manifold structure on S_1 in any real closed field.

Remark Although in the classical case the topological group S_1 described above is isomorphic to the circle group $\mathbb{S}^1$, this isomorphism is not semi-algebraic, and therefore we do not identify these two groups from a model theoretic point of view.

In general, for G a definable group, let $F \subseteq G$ be the closure of an $\mathcal{R}$-definable definably connected open set, whose boundary consists of two definably connected components Γ_1, Γ_2. Let α be an $\mathcal{R}$-definable smooth (with respect to R) permutation of G, with $\alpha(\Gamma_1) = \Gamma_2$ such that $\alpha(\mathring{F}) \cap \mathring{F} = \varnothing$, and α is a bijection between definable disjoint open neighborhoods of Γ_1 and Γ_2. Then we can glue Γ_1 and Γ_2 along α and obtain an $\mathcal{R}$-definable smooth manifold M with universe $\Sigma = \mathring{F} \cup \Gamma_1$. If F is definably compact then M is definably compact as well.

4.1.2 One-dimensional K-tori.

The classical construction of one-dimensional tori is usually carried out as follows. Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$, and $\Lambda_{\omega_1,\omega_2} = \{m_1\omega_1 + m_2\omega_2 \; m_1, m_2 \in \mathbb{Z}\}$ the subgroup of $(\mathbb{C}, +)$ generated by ω_1, ω_2.

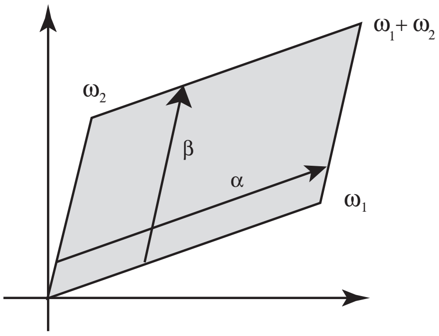
Let E_{ω_1,ω_2} be the quotient group $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$ and $\pi : \mathbb{C} \longrightarrow E_{\omega_1,\omega_2}$ the natural projection. E_{ω_1,ω_2}, with the quotient topology, is a compact group. It also inherits from $\mathbb{C}$ the structure of a complex analytic group.

As it was pointed out above, in the case of an arbitrary K one needs to take a different approach in order to define the family one-dimensional K-tori.

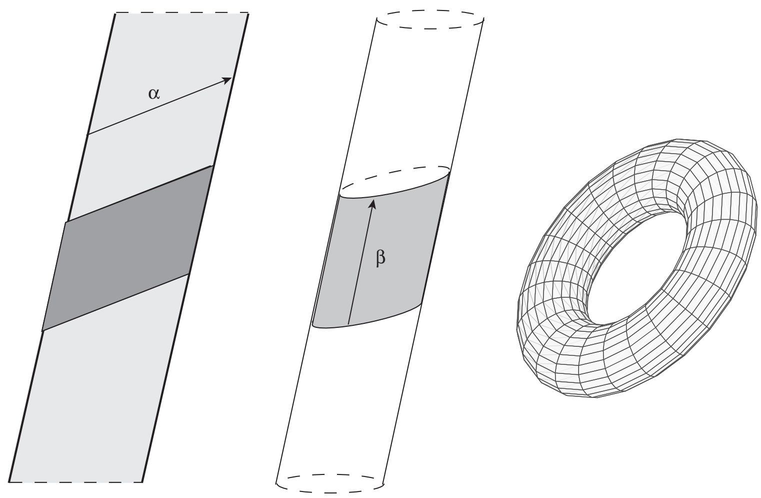
Let $\omega_1, \omega_2 \in K$ be linearly independent over R. Consider the definable closed region

$$F = \{t_1\omega_1 + t_2\omega_2 \in K \;\; t_1, t_2 \in R \quad \text{with} \quad 0 \le t_1, t_2 \le 1\}$$

Let $\alpha, \beta : K \longrightarrow K$ be $\alpha : z \mapsto z + \omega_1$, $\beta : z \mapsto z + \omega_2$. To glue F along both α, β we will first glue along α and then along β.



The region F



Gluing along α and then along β

Since both α, β are K-biholomorphisms, we obtain an $\mathcal{R}$-definable K-manifold whose underlying set can be taken as

$$E_{\omega_1,\omega_2} = \{t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in R, 0 \leq t_1, t_2 < 1\}.$$

This manifold turns out to be definably compact.

Both α, β commute with addition on K and thus it induces a definable K-group structure on this manifold. We will denote this K-manifold by E_{ω_1,ω_2} and call it *a one-dimensional K-torus*. Notice that this construction can be dome uniformly in ω_1, ω_2 and thus we obtain a definable family of all K-tori.

4.1.3 Definable isomorphisms between K-tori

Every definable group in an o-minimal structure admits a definable R-manifold structure which is moreover unique (see [10]). Namely, every definable group isomorphism is smooth with respect to R. However the K-holomorphic structure usually is not unique:

As is easily seen, every definable K-torus E_{ω_1,ω_2} is R-definably isomorphic, as a topological group, to $S_1 \times S_1$, so all K-tori are definably isomorphic to each other as topological groups. Our goal is to describe when two K-tori are definably K-biholomorphic. The characterization we get is identical to the classical one.

Remark. Every torus E_{ω_1,ω_2} is definably isomorphic to a torus $E_{1,\tau}$, with τ in the upper half plane $\mathfrak{H}(K) = \{z \in K : \mathrm{im}(z) > 0\}$. Thus it is sufficient to consider only tori $E_{1,\tau}$, with $\tau \in \mathfrak{H}(K)$. We will denote such a torus by E_τ.

As in the classical case, $\mathrm{SL}(2, R)$ acts on $\mathfrak{H}(K)$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

Theorem 4.5 *Let $\tau, \tau' \in \mathfrak{H}(K)$ and $F : E_\tau \longrightarrow E_{\tau'}$ a definable K-holomorphic group isomorphism. Then $\tau = A\tau'$ for some $A \in \mathrm{SL}(2, \mathbb{Z})$.*

Idea of the proof. E_τ and $E_{\tau'}$ are $\mathcal{R}$-definably isomorphic to $S_1 \times S_1$.

Since $\mathcal{R}$ is o-minimal, every definable group homomorphism from S_1 into S_1 has form $x \mapsto nx$ for some $n \in \mathbb{Z}$.

It is not hard to describe all $\mathcal{R}$-definable $\mathcal{R}$-homomorphisms from E_τ into $E_{\tau'}$ and then using the Cauchy-Riemann condition deduce the theorem. $\square$

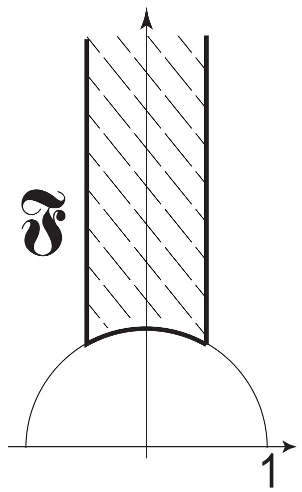
4.2 Nonstandard one-dimensional K-tori

Consider the following subset of $\mathbb{C}$:

$$\mathfrak{F} = \{z \in H \quad -\frac{1}{2} \leq \mathrm{re}(z) \leq \frac{1}{2} \text{ and } |z| \quad 1\}$$

In the classical case, every one-dimensional complex torus is biholomorphic to a torus of the form E_τ, for $\tau \in \mathfrak{F}$, and, in turn, every such torus is biholomorphic to a smooth projective cubic. The converse is also true, namely, every smooth projective cubic is biholomorphic to a one-dimensional complex torus.

This remains only partially true for K-tori over arbitrary real closed fields.



Theorem 4.6 *Let $\mathcal{R}$ be a model of $\mathbb{R}_{\mathrm{an,exp}}$. For $\tau \in \mathfrak{H}(K)$, E_τ is definably K-biholomorphic to a smooth projective algebraic curve if and only if $A\tau \in \mathfrak{F}(K)$, for some $A \in \mathrm{SL}(2, \mathbb{Z})$. Moreover, every smooth nonsingular cubic is K-biholomorphic with E_τ, for some $\tau \in \mathfrak{F}(K)$.*

Note that when $K = \mathbb{C}$ every $\tau \in \mathfrak{H}(\mathbb{C})$ satisfies the condition of the theorem and thus we just obtain the classical theorem. The main ingredient in the proof of the above theorem is finding in $\mathbb{R}_{\mathrm{an,exp}}$ a definable parametrization of the Weierstrass $\wp$-functions, as τ varies in $\mathfrak{F}$.

One corollary of the above theorem is the existence of so-called nonstandard tori:

Corollary 4.7 *In every proper elementary extension of* $\mathbb{R}_{\mathrm{an,exp}}$ *there is a definable one-dimensional* K*-torus that is not definably* K*-biholomorphic to any smooth projective algebraic curve.*

In a subsequent work, still in preparation, we examine the model theoretic properties of these nonstandard tori, when equipped with all K-analytic structure. We prove that these are locally modular, strongly minimal groups.

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The elementary theory of elliptic functions I: the formalism and a special case

Angus Macintyre

Abstract

In this paper we consider the elementary theory of all Weierstrass functions, with emphasis on definability and decidability. The work can be seen as a refinement of the work of Bianconi [2], using ideas of Wilkie and me [9] to get effective model-completeness. The novelty is the subsequent use of a Conjecture of Grothendieck-André [3] to get decidability in many cases.

1 Introduction

In the last twenty years, mathematical logic has begun to interact with analytic function theory. The trend has been to adapt to analytic settings notions and methods from the extensive model theory of fields, where model theory and algebraic geometry interact.

Much of the action has involved the old notion of model-completeness and the newer notion of o-minimality (directly applicable only in *real* situations). The more intricate notions of stability, at least in first-order model theory, seem rarely to apply in natural situations beyond the first-order (for notable exceptions, see [10]). A promising development, begun by Zilber [15], suggests that stability for non-elementary classes may be very relevant.

There are of course theories of analytic functions over complete valued fields K. Here I concentrate on the real and complex cases. The other cases are of great interest, and very substantial results are known on the model-theoretic side ([4] and [8]), but we do not discuss them here.

I note first that for $K = \mathbb{C}$ essentially one example is known of an entire f such that f is not polynomial and $Th((K, +, \cdot, f))$ does not interpret arithmetic. This is the case when f is a Liouville function, to which attention was drawn by the conjectures of Zilber [15], and which was proved $\omega-stable$, and decidable, by the work of Wilkie and Koiran [7].

For real analytic $f : \mathbb{R} \to \mathbb{R}$ many nonalgebraic examples are known where $Th(\mathbb{R}, +, \cdot, f)$ is o-minimal and so does not interpret arithmetic. However, the real exponential is the only known "interesting" model-complete example. It is decidable assuming Schanuel's Conjecture [9], and, again, is the only known decidable example.

It will be convenient in what follows to call a theory (or structure) *tame* if it does not interpret arithmetic. O-minimality is the most familiar condition implying tameness.

The theory of the complex exponential is not tame, as arithmetic gets interpreted via the periods, but Zilber has drawn attention to the possibility that one can still make a penetrating analysis of definitions in this setting [15].

Of course, most of the most fundamental functions are only meromorphic. Let us consider the Weierstrass $\wp$ function for a lattice Λ. If we consider simply the theory of $(\mathbb{C}, +, \cdot, \wp)$ we have arithmetic interpreted (essentially via the periods), but Zilber's vision suggests that this will not block a satisfying analysis of definitions [15].

In this paper we consider the elementary theory of all Weierstrass functions, with emphasis on definability and decidability. The work can be seen as a refinement of the work of Bianconi [2], using ideas of Wilkie and me [9] to get effective model-completeness. The novelty is the subsequent use of a conjecture of Grothendieck-André [3] to get decidability in many cases.

The paper is intended as the first of at least two. Here I establish the foundations, and prove decidability in one special case. In the sequel(s) I (and perhaps collaborators) will concentrate on global issues as the lattice Λ varies, linking to the work [12] of Peterzil and Starchenko.

2 Generalities

2.1 The general setting

We fix a real-closed field K, and let $L = K(i)$, the algebraic closure of K. (Usually $K = \mathbb{R}$, but for global problems we need the general case). K is o-minimal, and we generally enrich it to an o-minimal $K^\sharp$. The latter will be chosen so that in $K^\sharp$ (for $K = \mathbb{R}$) one can interpret some interesting structure from complex analysis. In this paper, $K^\sharp$ will be a model of some subtheory (maybe in a smaller language) of $Th(\mathbb{R}_{\mathrm{an,\,exp}})$.

2.2 Lattices and their Weierstrass functions

A lattice Λ in L is an additive subgroup of L generated by elements ω_1, ω_2 which are linearly independent over K. Just as in the conventional case of $K = \mathbb{R}$, ω_1, ω_2 is determined up to multiplication by an element of $\mathrm{GL}(2, \mathbb{Z})$. L^* acts on the set of Λ by scalar multiplication. The orbits are the orbits are the equivalence classes for similarity. As in the standard case, any Λ is similar to one generated by 1 and τ, where $\tau \notin K$, and "the imaginary part " of τ (relative to K) is positive. Note that τ is unique given Λ .

We write $\mathcal{H}$ for the upper half-plane of L (this is obviously a semi-algebraic subset of K^2). For $\tau \in \mathcal{H}$ we let Λ_τ be the lattice generated by 1 and τ. Let $\mathcal{E}_\tau$ be the quotient group of the additive group of L by Λ_τ, and $E_\tau = \{t_1 + t_2 : t_1 , t_2 \in K, 0 \le t_i < 1\}$ the standard set of representatives for $\mathcal{E}_\tau$. Here we follow mainly the notation of Peterzil and Starchenko [12]. $\mathcal{H}$ is semi-algebraic, as is $E^{\mathcal{H}} = \{ (\tau, z) \in \mathcal{H} \times L : z \in E_\tau\}$.

On $L = \mathbb{C}$, $E^{\mathcal{H}}$ is naturally a semi-algebraic complex manifold, and this remains true in general L ,with the interpretation given in [12]. Now $\mathcal{E}_\tau$ is also a 1-dimensional complex torus, and so compact, for $L = \mathbb{C}$. The issue of a uniform meaning for this, for general L and τ, will be discussed below.

The E_τ are the fibres of the natural L-holomorphic projection $E^{\mathcal{H}} \to \mathcal{H}$. Now, in 3.6 of [12], it is shown how to give, uniformly, L-holomorphic structure on the fibres E_τ, so that $+$ is holomorphic (in fact semi-algebraic).

The next step is the interesting one, and complications arise, as shown in [12]. Classically, the $\mathcal{E}_\tau$ have holomorphic projective embeddings $g_\tau : \mathcal{E}_\tau \to \mathbb{P}^2(\mathbb{C})$ given essentially by $g_\tau = (\wp_\tau(z), \acute{\wp}_\tau(z), 1)$ if $z \notin \Lambda_\tau$ and $= (0, 1, 0)$ if $z \in \Lambda_\tau$ where $\wp_\tau$ is the Weierstrass $\wp$ function for Λ_τ, usually defined via an infinite sum over Λ_τ [1]. Classically, each τ has a corresponding $\wp_\tau$, satisfying a familiar differential equation, and thereby identifying $\mathcal{E}_\tau$ holomorphically with a nonsingular projective curve (of genus 1).

The issue of uniformity here is delicate [12]. In fact, classically, the function $\wp(\tau, z)$ on $\mathcal{H} \times \mathbb{C}$ defined by $\wp(\tau, z) = \Lambda_\tau(z)$ is meromorphic as a function of two variables [12]. This function is of course undecidable, but a model theory along the lines of Zilber's Conjecture for exp is not to be excluded [15].

Classically, $\mathcal{E}_{\tau_1}$ and $\mathcal{E}_{\tau_2}$ are biholomorphic if and only if τ_1 and τ_2 are in the same orbit for the action of SL(2, $\mathbb{Z}$) on the upper half-plane [12]. Moreover, the semi-algebraic

$$\mathcal{F} = \{\tau \in \mathcal{H} : -\frac{1}{2} \leq \mathrm{re}(\tau) < \frac{1}{2}, |\tau| \geq 1\}$$

contains exactly one representative from each orbit under the action of SL(2, $\mathbb{Z}$) [12], and thus it is natural from a logical point of view (there are better viewpoints) to consider $\wp(\tau, z)$ merely on $\mathcal{F} \times \mathbb{C}$ (or its generalization to L if possible), and this is what we do henceforward.

$\mathcal{E}^{\mathcal{F}}$ is defined as $\pi^{-1}(\mathcal{F})$, where π is the natural projection from $\mathcal{H} \times \mathbb{C}$ to $\mathcal{H}$. From now on, the binary $\wp$ is to be construed as having domain $\mathcal{E}^{\mathcal{F}}$.

An important result from [12] is that this $\wp$ is definable in $\mathbb{R}_{\mathrm{an,\,exp}}$, and so is tame. Moreover, $\wp$ then makes sense in any model of $Th(\mathbb{R}_{\mathrm{an,\,exp}})$. There is a striking converse [12], namely that exp is definable in any o-minimal expansion of $(\mathbb{R}, +, \cdot)$ including the above $\wp$.

The last result of [12] of which we must take account is that for general K, and $\tau \in \mathcal{H}$, $\mathcal{E}_\tau$ is definably K-biholomorphic with a nonsingular projective curve over L if and only if τ is in the SL(2, $\mathbb{Z}$)-orbit of an element of $\mathcal{F}$. Since for general K not every element of $\mathcal{H}$ is in such an orbit, one gets, in [12], exotic nonalgebraic tori in any proper extension of $\mathbb{R}$ (and, presumably, these tori have no nonconstant K-meromorphic functions).

2.3 The formalism

With these preliminaries done, I can define the formalism for the elementary theory of Weierstrass $\wp$-functions.

The language is that for real-closed fields K, with distinguished constants α and β (coding $\tau = \alpha + i\beta$ in the algebraic closure L of K, with L identified with K^2), together with two binary function symbols (for real and imaginary part of $\wp$) re$\wp$ and im$\wp$. E_τ is defined as in the preceding discussion, and

$$\wp = \mathrm{re}\wp + i\mathrm{im}\wp$$

is a function from $E_\tau - \{(1,0), (\alpha, \beta)\}$ to L.

We are interested in the interpretations where $\wp$ is the Weierstrass function for the lattice E_τ, and the elementary theory of Weierstrass $\wp$-functions is defined as T_W, the set of sentences true for all such interpretations. T_τ, for fixed τ in the $\mathcal{F}$ of a model of $Th(\mathbb{R}_{\mathrm{an,\,exp}})$, is defined as the complete extension where $\alpha + i\beta$ is interpreted as τ. For the vigilant reader, i is unambiguous, interpreted as $(0,1)$ in K^2.

We address such questions as:

1. What are the complete extensions of T_W?
2. Is T_W model-complete?
3. Is T_W o-minimal?
4. Is T_W decidable?
5. The analogues of 2 to 4 for individual T_τ.
6. What is the nature of the set of τ for which a fixed sentence Φ holds, with $\alpha + i\beta$ interpreted as τ?

Most of the above will be considered in subsequent papers. For now, I set out the foundational components of the analysis, and consider only Question 5.

3 Interpreting $\wp_\tau$ via a fragment

3.1 Restriction to a compact

Our main source of o-minimal theories with no (differential-algebraic) constraint on the primitives is $Th(\mathbb{R}_{\mathrm{an}})$. For the present purpose, the following is the basic observation.

Suppose U is an open set in $\mathbb{C}$, and A is a closed rectangle contained in U. Let f be an analytic function on U, with real and imaginary parts u and v respectively. Then the real-closed field $\mathbb{R}$, enriched by the restrictions of u and v to A, is o-minimal. This is proved by observing that u and v are real-analytic on U, and using [4].

3.2 Gabrielov version

A very useful refinement of this observation was obtained by Gabrielov [5]. Namely, the real field enriched by all partial derivatives of u and v (and so, by Cauchy-Riemann, by the real and imaginary parts of all derivatives of f) is o-minimal and model-complete.

3.3 The Weierstrass case. Fixing a rectangle

Fix a τ as above. We define the rectangle A_τ inside E_τ by specifying its vertices, namely $\frac{1+\tau}{8}$, $\frac{3+\tau}{8}$, $\frac{1+3\tau}{8}$ and $\frac{3+3\tau}{8}$. Then $\wp_\tau$ is analytic on an open set containing A_τ , and so the preceding results apply to this situation. The reason for the choice of the vertices will be explained later. It has to do with issues of uniformity in the model-completeness, as τ varies.

3.4 The differential equation and the addition formula

One crucial point about the $\wp_\tau$ is that they satisfy a differential equation and an addition formula. The latter implies that $\wp_\tau$ on E_τ can be interpreted in the theory of the restrictions to A_τ of the real and imaginary parts of $\wp_\tau$ and so is o-minimal. Antecedents of this remark are in Bianconi's thesis [2], as is the systematic use of the differential equation to get a model-completeness result (not exactly the one I prove below).

The following facts about $\wp_\tau$ and $\wp'_\tau$ are basic [1]. $\wp_\tau$ has a pole of order 2 at each point of Λ_τ and no other poles. For all constants c, $\wp_\tau - c$ has either exactly two zeros modulo Λ_τ and each is of multiplicity one, or has a unique double root modulo Λ_τ. The latter case can happen only at a zero of $\wp'_\tau$. Now, $\wp'_\tau$ has exactly three zeros modulo Λ_τ, at $\frac{1}{2}$, $\frac{\tau}{2}$ and $\frac{1+\tau}{2}$, each of multiplicity one. So the only possibility, modulo Λ_τ, for a double zero of $\wp_\tau - c$ is when c is one of $\wp_\tau(\frac{1}{2})$, $\wp_\tau(\frac{\tau}{2})$, or $\wp_\tau(\frac{1+\tau}{2})$, numbers usually written (up to permutation) as e_1, e_2, e_3. Standard calculations then yield the differential equation

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

with the right-hand side usually written as

$$4\wp^3(z) - g_2\wp(z) - g_3.$$

The discriminant of $4\wp^3(z) - g_2\wp(z) - g_3$ is $\Delta = g_2^3 - 27g_3^2$, and the j-invariant of Λ_τ is by definition

$$\frac{1728\, g_2^3}{\Delta}.$$

Note that the e_i, the g_i, Δ, and j are all rational in $\wp(\frac{1}{2})$, $\wp(\frac{\tau}{2})$, $\wp(\frac{1+\tau}{2})$, uniformly. Note, too, that the above shows that $\wp_\tau - (c)$ has a double zero exactly when $c \in \{e_1, e_2, e_3\}$. Recall that it is standard that $\Delta \neq 0$, that is, the e_i are distinct.

Now we can explain the Addition Formula. This says

$$\begin{aligned}\wp(z_1 + z_2) &= -\wp(z_1) - \wp(z_2) \\ &+ \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2\end{aligned}$$

in the sense that one side is defined if and only if the other is, and then both sides are equal. It is quite important for later logical work that this point is fully understood.

One should note also the related formula

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2.$$

Another important point is that $\wp$ is an even function [1]. The following is worth noting:

$$\begin{vmatrix} \wp(z_1) & \wp'(z_1) & 1 \\ \wp(z_2) & \wp'(z_2) & 1 \\ \wp(z_1 + z_2) & -\wp'(z_1 + z_2) & 1 \end{vmatrix} = 0$$

3.5 Exploiting the addition formula

Our strategy now is to show how to interpret, uniformly in τ, $\wp_\tau$ on all of E_τ from $\wp_\tau$ on A_τ. The basic idea, from Bianconi [2], is to work on additive sums of A_τ such as $A_\tau + A_\tau$, and $-A_\tau$, using the Addition Formula and the variants above.

It is readily verified that the vertices of $A_\tau + A_\tau$ are $\frac{1+\tau}{4}$, $\frac{3+\tau}{4}$, $\frac{1+3\tau}{4}$ and $\frac{3+3\tau}{4}$.

Similarly, the vertices of $A_\tau + A_\tau + A_\tau$ are $\frac{3(1+\tau)}{8}$, $\frac{3(3+\tau)}{8}$, $\frac{3(1+3\tau)}{8}$ and $\frac{3(3+3\tau)}{8}$.

Finally, the vertices of $-A_\tau - A_\tau - A_\tau$ are $\frac{-3(1+\tau)}{8}$, $\frac{-3(3+\tau)}{8}$, $\frac{-3(1+3\tau)}{8}$ and $\frac{-3(3+3\tau)}{8}$.

Observe that the Addition Formula allows us to interpret (over the theory of real-closed fields) $\wp_\tau$ on $A_\tau + A_\tau$ quite simply in terms of $\wp_\tau$ and $\wp'_\tau$ on A_τ. In fact the interpretation of the graph of $\wp_\tau$ can be given in either existential or universal form (a remark important for a future claim on model-completeness), because of the equational nature of the Addition Formula. Moreover, the determinantal identity gives us a similar interpretation of $\wp'_\tau$ (in this case there is no chance that $\wp(z_1) - \wp(z_2)$, the coefficient of $-\wp'(z_1 + z_2)$, vanishes).

Similar, but slightly more complicated, arguments apply to $A_\tau + A_\tau + A_\tau$ and the interpretation of $\wp$ and $\wp'$ thereon in terms of $\wp$ and $\wp'$ on A_τ. Again, the interpretation is both existential and universal.

Finally, the same applies to the case of $-A_\tau - A_\tau - A_\tau$. In the latter case, we go on to consider

$$-A_\tau - A_\tau - A_\tau + E_\tau.$$

Because of the periodicity, we have an interpretation, again both existential and universal, of $\wp_\tau$ and $\wp'_\tau$ thereon, in terms of the same functions on A_τ.

Now, a simple calculation shows that E_τ is covered by the union of A_τ, $A_\tau + A_\tau$, $A_\tau + A_\tau + A_\tau$, and $-A_\tau - A_\tau - A_\tau + E_\tau$.

Each is a semi-algebraic set, and the preceding discussion shows that we have given a uniform interpretation, both existential and universal, of the theory of $\wp_\tau$ on E_τ in the theory of $\wp$ and $\wp'$ on A_τ. Recall that the latter is an o-minimal expansion of the theory of real-closed fields.

3.6 First theorems

If we put together the observations so far we get a theorem essentially proved in Bianconi's thesis [2]. The proof is, however, slightly different, and has, for future developments in a constructive direction, some advantages over Bianconi's. In effect, we have proved:

Theorem 3.1 *($K = \mathbb{R}$) For each $\tau \in \mathcal{H}$ the theory T_τ is o-minimal.*

A much deeper theorem is

Theorem 3.2 *The (incomplete) theory T_W is o-minimal.*

Proof. Peterzil and Starchenko [12] show that the binary Weierstrass function $\wp(\tau, z)$ on E_τ, for $\tau \in \mathcal{F}$, is interpretable in $\mathbb{R}_{\mathrm{an,exp}}$. Since the latter is o-minimal, it follows that the theory of the class of $\wp_\tau$ on E_τ, for

$\tau \in \mathcal{F}$, is o-minimal. But this is the theory T_W, by standard facts about $\mathrm{SL}(2,\mathbb{Z})$ action on $\mathcal{H}$. □

Thus T_W is at least tame. I am currently collaborating with Speissegger and Starchenko in an effort to understand whether the theory of the binary Weierstrass function on $\mathcal{F}$ is model-complete or decidable. This appears to lead into delicate issues concerning the Manin map in the theory of abelian varieties.

3.7 Model completeness for the local theories T_τ

Because of the uniform nature of our interpretation, and the emphasis on the fact that it is both existential and universal, the model completeness problem for the T_τ reduces to that for the theories of $\wp_\tau$ on A_τ.

That the latter theories are model-complete follows from a (nonconstructive) result of Gabrielov, on the model-completeness of sets of real analytic primitives, closed under partial derivatives, on compacts [5].

The main point in our setting is the differential equation for $\wp_\tau$, expressing $\wp'_\tau$ algebraically in terms of $\wp_\tau$. Since we are working in extensions of the theory of real-closed fields, we view this as giving the real and imaginary parts of $\wp'_\tau$ semi-algebraically in terms of the real and imaginary parts of $\wp_\tau$. From this observation, the model-completeness of the individual $\wp_\tau$ on A_τ follows easily.

We have set things up so that $\wp_\tau$ is $1-1$ on A_τ. For it is well-known that

$$\wp(z_1) = \wp(z_2) \iff (z_1 + z_2) \in \Lambda$$

and clearly A_τ is disjoint from $-A_\tau + \Lambda_\tau$. Note, too, that we have arranged that $\wp'_\tau$ has no zeros on A_τ.

There are complex issues of uniformity that will not be confronted here (for example, a lower bound for $|\wp'_\tau|$ on A_τ). The hope is to be able to prove uniform and/or constructive model-completeness for $\wp_\tau$ on A_τ. Why is there hope even for individual τ? The idea is to use either very strong results of Gabrielov and Vorobjov [6] or constructive versions of the Wilkie-Macintyre result from [9]. Both are for Pfaffian primitives on compact polydiscs, and we are not quite in that situation here. The differential equation for $\wp_\tau^{-1}$ does arise from a Pfaffian chain of complex functions, but the Hovanskii Finiteness results and their constructive elaborations by [6] are for real functions. I have an outline of a proof, definitely not uniform as it stands, that may show that the theory of $\wp_\tau^{-1}$ on a suitable polydisc may be interpreted in a real Pfaffian structure, and thereby yield model-completeness.

In the remainder of this paper I will give details on a special case where one can apply [6] and [9] to get constructive model-completeness.

4 The case $\tau = i$

This is a variant of the first case discusseed by Bianconi in his thesis [2]. The elliptic curve has complex multiplication by i. It is, however, not defined over $\mathbb{Q}$. But is is isomorphic to the (compactification of)

$$y^2 = x(x^2 - 1),$$

which is defined over $\mathbb{Q}$ but has period lattice generated by Ω and Ωi, where Ω is the transcendental number

$$2\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}.$$

By using the usual Weierstrass sum for $\wp$ one sees that $\wp$ takes real values at real arguments. The coeffficients of the differential equation for $\wp$ are real. Moreover, if x is real then $\wp'(x)$ is real, and on any connected interval avoiding the 2-torsion points the differential equation has the constant choice of positive square root or the constant choice of negative square root.

Again, consideration of the Weierstrass series shows that $\wp(iz) = -\wp(z)$, and so this function too is real-valued on the real points of its domain of definition. Now we can apply the Addition Formula to get, for real x and y

$$\wp(x + iy) = -\wp(x) - \wp(iy)$$

$$+\frac{1}{4}(\frac{\wp'(x) - \wp'(iy)}{\wp(x) - \wp(iy)})^2$$

which clearly gives us an interpretation (existential and universal) of $\wp_\tau$ on A_τ in terms of $\wp_\tau$ on the interval $[\frac{1}{8}, \frac{3}{8}]$. So we are reduced to the problem of constructive model-completeness and decidability for $\wp_\tau$ on the interval $[\frac{1}{8}, \frac{3}{8}]$.

To get into a position to apply the results of [6] or [9] one more trick is needed. By our choice of A_τ, $\wp$ is monotone on $[\frac{1}{8}, \frac{3}{8}]$. I leave it to the reader to check that $\wp$ is in fact decreasing. Let $[a, b]$ be the image of $[\frac{1}{8}, \frac{3}{8}]$ under $\wp$.

Note that a and b are transcendental in the present case. For future use we discuss some detail around this. One can show that the unique zero of $\wp$ in the period parallelogram is a double zero at $\frac{1+i}{2}$ (a nice exercise). It follows that the constant term in the cubic from the differential equation is 0, i.e one of the e_j is 0. Now Siegel ([11]) showed that our curve cannot be defined over a number field, and so at least one of the other e_k is transcendental. In fact, our earlier discussion about $\wp(iz)$ shows that the other two e_k sum to 0, so each is transcendental. These are the values of $\wp$ at two 2-torsion points. Now our end points $\frac{1}{8}$ and $\frac{3}{8}$ are 8-torsion points, and their sum is a 2-torsion point. It follows easily from the Addition Formula that each of a and b is transcendental, but $\mathbb{Q}(a, b)$ has transcendence degree 1.

In fact, we can be rather more explicit about a. It is obviously algebraic over the transcendental $\wp(\frac{1}{2})$. Now under the isomorphism, induced by multiplication by Ω, the lattice spanned by 1 and i goes to that spanned by Ω and Ωi, and the corresponding Weierstrass functions are related in a familiar way. Moreover the Weierstrass function for the new lattice takes the values $1, -1$ at its 2-torsion points $\frac{\Omega}{2}$ and $\frac{i\Omega}{2}$, and thus one shows easily that a is algebraic over the transcendental Ω.

We consider $\wp^{-1}$ on $[a, b]$. This function satisfies the differential equation

$$f' = (2\sqrt{(f - e_1)(f - e_2)(f - e_3)})^{-1}$$

with, as before, a definite choice of positive or negative square root. Notice that then

$$f'' = -\frac{1}{4}(f')^3,$$

so that f' is Pfaffian on $[a, b]$, and so f is. In particular one has an explicit Pfaffian chain for $\wp^{-1}$ on an open interval including $[a, b]$. Now, it is again obvious that $\wp_\tau$ on the interval $[\frac{1}{8}, \frac{3}{8}]$ is (both existentially and universally) interpreted in the theory of $\wp^{-1}$, $(\wp^{-1})'$ and $(\wp^{-1})''$ on $[a, b]$. Finally, we have reduced our problems to the study of a Pfaffian system of the kind studied in [6].

It is important to spell out the coefficients of the cubic $(x-e_1)(x-e_2)(x-e_3)$ in the above. By the earlier remark about the the zeros of $\wp$ one sees that the cubic is

$$4x(x^2-(\wp(\frac{1}{2})^2)),$$

and, crucially for us, that its coefficients are algebraic over Ω.

Thus we see that our Pfaffian data, namely the differential equation, and the endpoints a and b, is all algebraic over Ω, or, equivalently, over $\wp(\frac{1}{2})^2)$. So, when dealing with this system (in which we will interpret $\wp_i$), we will work with a constant (one of the above) over which all the data is algebraic. Alternatively, if we wish to stick exactly to the formalism of [6] or [9], we can rescale our function to be defined on $[-1,1]$. I see no point in doing so.

4.1 Applying Gabrielov-Vorobjov

The Gabrielov-Vorobjov papers provide a constructivization of various parts of the elementary theory of subanalytic sets. The main results are for Pfaffian primitives on compact polydiscs. Model-completeness is not the main emphasis (which is rather on more specific topics of geometric interest), but constructive results close to model-completeness can be read off from their work.

In this subsection familiarity with [6] is assumed. The asssumption there, not very restrictive, is that one is working over the interval $[-1,1]$. In the example above, we were working over the interval $[a, b]$. We don't bother to scale to get into the case $[-1,1]$ as it is obvious that the arguments go over.

Naturally, in matters of model-completeness, the chosen language is crucial. So let us spell out the essentials.

We have as base the usual language for semi-algebraic geometry (with addition, multiplication, zero, one, and order). Here quantification is unbounded (but also redundant, by Tarski). Beyond that, one has the function symbols for the Pfaffian primitives of various arities. Each such primitive is analytic on an open set containing an appropriate power of $[-1,1]$. The corresponding function symbol has the interpretation given by Denef and van den Dries [4], i.e the standard definition on the power of $[-1,1]$, and 0 elsewhere. This rather unattractive convention induces a classification of the occurrences of variables in a formula. Namely, an occurrence of v inside the scope of one of the Pfaffian function symbols is called *bounded*. The meaning is of course that if one quantifies this occurrence of the variable,essentially the quantification is over $[-1, 1]$, because for values of v outside this interval the value of the Pfaffian term is automatically 0. There are numerous ways to elaborate this remark, and I feel no need to make pages out of this now. The following should be sufficient, and this much should be obvious to anyone familiar with [6] and [9].

Because of what we know after the fact about these Pfaffian theories (mainly the model-completeness results of Gabrielov and Wilkie, and the constructive versions of Gabrielov and Vorobjov) one really only needs to consider quantifier-free and existential formulas. First one notes that the only compositions of terms one really needs involve only polynomials and basic Pfaffian functions. For this to be true literally one must work in the category of existential formulas (all such nuisances would be removed by a more scheme-theoretic approach to definability), but since this is the category most relevant to our work nothing is lost. Thus the only terms one needs to consider are of the form

$$F(v_1, \ldots, v_n, P_1, \ldots, P_k)$$

where F is a polynomial over Q and the P_i are terms got by applying Pfaffian primitives to some of the variables v. (The reader familiar with [9] will recognize such things, which I now call *simple terms*. This now constrains the atomic formulas ones has, etc. Now one imposes a sorting of variables into bounded and unbounded, with the above motivation. Namely we require that variables in the scope of a P as above are bounded. Only these need to be quantified, because of Tarski's theorem. Thus we are led to the following notion of existential formula as one of shape

$$\exists \overline{y} \Psi(\overline{y}, \overline{x})$$

where the $\overline{y}$ are bounded variables, and the quantifier-free formula Ψ is built up booleanly from atomic formulas of types $u = v$ and $u > v$ where u and v are simple terms.

The Wilkie-Gabrielov model-completeness for restricted Pfaffians says that the negation of an existential formula is equivalent to an existential formula. Unfortunately, the constructive version is a bit weaker, as we now explain.

We begin by fixing the meaning of "existential", for the rest of this paper, to be that just given. We can then formulate an unconditional result, of model-completeness type, which can be read off from the analysis of [6]. Namely, let us define *a very nearly quantifier-free formula* to be a Boolean combination of atomic formulas and existential sentences. Then we define *a very nearly existential formula* as one in prenex form with only existential quantifiers in the prenex and very nearly quantifier-free formulas following the prenex. Then the following is implicitly proved in [6] :

Theorem 4.1 *Suppose the Pfaffian chain is defined using constants $\overline{c}$, and that we have as primitives also the $\overline{c}$ and the first partials of the functions in the chain. Then, constructively, every formula is equivalent to a very nearly existential formula.*

So, constructively, our theory above is unconditionally "very nearly model-complete".

4.2 Going to étale form

There is one final refinement needed before we can approach decidability. It involves a phenomenon first detected by Wilkie [13] in his proof of model-completeness for restricted Pfaffian situations. We will follow the analogous development from [9]. Essentially one is giving a characterization of definable functions in the restricted Pfaffian theories.

One of the main results of Gabrielov-Vorobjov involves (relatively) effective weak stratification of restricted semi-Pfaffian sets. Essentially it says the following:

There is an algorithm which, given a finite conjunction of equations and inequations and order inequalities in $\mathbb{R}^n$, defined by simple terms over $\mathbb{Q}$ (i.e the data of a semi-Pfaffian set), provides a finite list of such conjunctions extending the original one, each of which defines an effectively nonsingular manifold in a weak stratification of the original set, so that the original set is the union of these strata. There is no claim that the individual strata are nonempty. But they can be chosen nonempty if one has an algorithm for deciding the truth of existential sentences. Moreover, relative to the same algorithm one can decide the dimension (in the o-minimal sense) of any semi-Pfaffian set.

The essential point is the meaning of "effectively nonsingular". The content is that if the stratum has codimension k in the original set, then among the closed conditions explicitly defining it are k equations such that on the set in question their Jacobian is nonzero. Indeed the algorithm produces these equations.

We need the relative version of the above, in which the free variables are split into two groups $\overline{x}$ and $\overline{y}$, and the stratification is "over $\overline{x}$", in the sense that we stratify the set in $\overline{y}$-space thought of as defined over $\overline{x}$. In effect one is just doing the previous constructive procedure uniformly over the Pfaffian theory with constants $\overline{x}$. The outcome is the following basic result :

There is an algorithm which, given a conjunction

$$\Psi(\overline{y},\, \overline{x})$$

of equations, inequations and inequalities between simple terms in the n variables $\overline{y}$ and the m variables $\overline{x}$, produces finitely many such formulas Ψ_j whose disjunction is equivalent to Ψ (in the basic Pfaffian structure) and such that each disjunct Ψ_j is effectively nonsingular in $\overline{y}$ in the sense that it contains, for some specified $k \leq n$, k equations, together with an inequation expressing that the Jacobian, with respect to some subset of the $\overline{y}$ of size k, is nonzero. Moreover, for any $\overline{\alpha}$, the set defined by $\Psi_j(\overline{y},\, \overline{\alpha})$ has, if it is nonempty, codimension k in the set defined by $\Psi(\overline{y}, \overline{\alpha})$.

It follows that constructively existential formulas are equivalent to disjunctions of ones in *étale* form

$$\exists \overline{y} \Psi(\overline{y},\, \overline{x})$$

where $\overline{x}$ and $\overline{y}$ have lengths m and n respectively, Ψ is a conjunction of equations, inequations and order-inequalities in simple terms, and for some $k \leq n$ Ψ contains k equations and the nonvanishing of their Jacobian over some subset of $\overline{y}$ of size k, and moreover the codimension condition is satisfied.

This is of course related to an important feature first noted by Wilkie in [13], and crucial for decidability in [9]. One formulation is Theorem 2.1 of [9], saying essentially that if V is the zero set in K^n of a simple term, and if S is a nonempty definable subset of V closed in K^n and open in V then S contains a nonsingular zero of a system of n simple terms in n unknowns.

4.3 An obstruction

The results of Gabrielov and Vorobjov give remarkably good bounds for their various algorithms, but leave untouched a basic obstruction to proving decidability. Everything is relative to an oracle for the existential theory, and certainly gives decidability relative to that over a huge range of problems. But, it seeems not to give one any special hold on the decidability of the existential theory. Anyone familiar with [6] and [9] will see that one cannot simply put them together to get decidability in the above special case. The advantage of the Wilkie-Macintyre method [9] is that it is done relative to an explicit recursive set of axioms, and this does allow one, as we shall see below, to go all the way to decidability in special cases. (It is of course general recursive decidability). One may hope to get decidabilty with the kinds of bounds occurring in [6], but currently I see no way to organize the proof to combine it with the endgame of [9].

4.4 Axioms

Given that we are dealing here with an exponential of a compact Lie group, it is not so surprising that one can mimic the axiomatization given in [9] for the real exponential. Wilkie in his big paper [13] spells out the ineffective version of model-completeness in restricted Pfaffian situations. Now I show how to do this effectively against a recursive set of axioms in the special case treated above.

Here complete familiarity with [6] and [9] will be assumed. In fact, very few changes need to be made to [9]. Various conventions are possible, just as with restricted exponentiation. In [9] we chose to work with a total real analytic function instead of the truncation, but little depends on it. Here I choose to take $\wp^{-1}$ on $[a, b]$ and put equal to 0 off the interval.

The schemata $A1 - A4$ are clearly true in the present situation for the appropriate language. $A5$ requires an obvious modification, to express the differential equation for $\wp^{-1}$ on $[a, b]$

In [9] one then introduces the basic class of functions $M_{n,r}$, and we need an analogue here. The $M_{n,r}$ are total, and closed under taking partial derivatives. We should rather begin with formal objects that take account of the crucial distinction between bounded and unbounded variables. Thus, as well as bounded variables x_i we should have unbounded variables w_j, and our basic formal entities should be polynomials in the x_i , $(2\sqrt{(x_i - e_1)(x_i - e_2)(x_i - e_3)})^{-1}$, w_j and the $\wp(x_i)$. Thus, over a subfield k, of whatever structure we are in, we should consider $M_{n,n',r}(k)$ the ring of formal polynomials over k in $x_1, \ldots, x_n$, $(2\sqrt{(x_1 - e_1)(x_1 - e_2)(x_1 - e_3)})^{-1}$, ..., $(2\sqrt{(x_r - e_1)(x_r - e_2)(x_r - e_3)})^{-1}$, $y_1, \ldots, y_{n'}$, $\wp(x_1), \ldots, \wp(x_r)$.

We use the same notation for the corresponding ring of functions on $[a,b]^n \times (K^{n'})$ or on $K^{n+n'}$ where we use the extension by zero interpretation.

That the classes are closed under partial differentiation is obvious. There is no difficulty in obtaining the analogue of $A6$ from [9], the Hovanski estimates. And now we come to the most delicate part, the analogue of $A7$, which I think of as the effective Lojasiewicz Inequalities. I will use a mixture of ideas from [6] and [9]. What one has to prove is the following for $K = \mathbb{R}$;

Theorem 4.2 (Axiom A7) *There is a recursive function τ from $\mathbb{N}^4$ to the set of finite subsets of the set of rational numbers such that for all n, n', r, $m \in \mathbb{N}$ and $f_1, \ldots f_{n+n'} \in M_{n,n',r}(K)$ having total degree at most m, and for all $A \in K$ and continuous definable $\phi_1, \ldots, \phi_{n+n'} : \{\alpha \in K : \alpha > A\} \to K$ such that $\phi_1, \ldots, \phi_n$ take values in $[a,b]$ and $\phi_{n+n'}$ is the identity, and satisfying*

$$f_i(\phi_1(\alpha), \phi_2(\alpha), \ldots, \phi_{n+n'}(\alpha)) = 0$$

(for $i = 1, \ldots, n + n' - 1$ and $\alpha > a$) and

$$\det \frac{\delta(f_1, \ldots, f_{n+n'-1})}{\delta\,(x_1, \ldots, x_n, y_{n+1}, \ldots, y_{n+n'-1})}(\phi_1(\alpha), \ldots, \phi_{n+n'}(\alpha)) \neq 0$$

(for $\alpha \in K$ and $\alpha > A$), then there exists a $q \in \tau(n, n', r, m)$ such that setting

$$h(y) = f_{n+n'}(\phi_1(y), \ldots, \phi_{n+n'-1}(y), y)$$

then either h is identically zero for $y > A$ or $y^q h(y)$ tends to a finite nonzero limit as y tends to infinity.

Just as in [9], once one has this we get a recursive set of axioms true for $\wp$ and which is model-complete. The first idea of the proof is to change the range of the α to an open interval on one side of the origin. This is a trivial move in our formalism where we distinguish between bounded and unbounded variables.

Basically we consider each variable y_j such that ϕ_j does not tend to a finite limit as α tends to plus infinity. It then follows that ϕ_j tends to plus or minus infinity as α tends to infinity (by o-minimality). These are the bad variables. For all other y_l and the x variables, the corresponding ϕ tends to a finite limit as α tends to infinity. Now we change variables. Each bad variable y_j is replaced by its reciprocal y_j^{-1} and every other variable v is replaced by $v - \gamma$, where γ is the limit of the ϕ corresponding to v. In addition, one multiplies each f_i by the product of all v^m where v is a bad variable. In this way one is led to the variant of $A7$ in which α is tending to 0^+ , each ϕ_j , f tends to a finite limit as α tends to 0^+, and $f_{n+n'}$ is one of the projection functions x_j or y_k. One should note that after the change of variables one is dealing with f which may no longer be in $M_{n,n',r}(K)$, and not merely because the m has changed as it does in Section 3 of [9]. The f get transformed to functions of the form which differs in shape from the original f only in the fact that the variables x and y (and in the corresponding terms for $\wp^{-1}$ and its derivative) are replaced by various "translates" $\pm x + \gamma$ and $y + \gamma$. For the proof I now sketch it suffices to note that these new f are Pfaffian, of complexity bounded by our data n, n', r, m.

Note that our reduction is not quite as strong as that done in Section 3 of [9]. We compensate by using a very general result of Gabrielov and Vorobjov on complex multiplicities [6]. Note that the new functions above (got by transforming the original f) are complex analytic in an open disc around the origin. By an observation of van den Dries (used heavily in [9]) each (transformed) ϕ_j is analytic in a rational power near the origin. Precisely, for each j there is a θ_j, real analytic around the origin, and an integer λ_j, so that

$$\phi_j(\alpha) = \theta(\alpha^{\lambda_j^{-1}})$$

for positive α near 0, where the positive root is taken. In fact, we may assume that all the λ_j are the same integer d, and that d is chosen as small as possible. By working in a sufficiently small neighbourhood of the origin, we may assume that

$$\phi_j(\alpha^d) = \theta(\alpha)$$

for positive α near the origin. Thus we have

$$f_i(\theta_1(\alpha), \theta_2(\alpha), \dots, \theta_{n+n'-1}(\alpha), \alpha^d) = 0$$

for $i < n + n'$ and for positive α near the origin, and the Jacobian (with respect to $x_1, \dots, y_{n+n'-1}$) is nonzero at $\theta_1(\alpha), \theta_2(\alpha), \dots, \theta_{n+n'-1}(\alpha), \alpha^d$.

Now, by analytic continuation, we may replace the real variable α by the complex variable z and get the equations $f_i = 0$ holding in a complex open disc around the origin. Moreover, by exactly the argument used in [9] at this point, we may assume that the above Jacobian, as a function of z is zero at most at zero (by shrinking our disc if need be).

The point now is to convert this data into a family of complex Pfaffian systems, and apply Gabrielov-Vorobjov. Consider, for variable w in the disc, the Pfaffian system

$$f_i(z_1, \dots, z_{n+n'-1}, w) = 0$$

($i = 1, \dots, n + n' - 1$) with the condition that the Jacobian with respect to

$$z_1, \dots, z_{n+n'-1}$$

is nonzero. By the preceding, for any nonzero w in the disc the element $\theta_1(w), \theta_2(w), \dots, \theta_{n+n'-1}(w)$ is a solution.

Now we reach the crucial point. If η is a dth root of 1 then we get another solution by replacing w by ηw. The argument in [9] (page 455) now shows (using the minimality of d) that these solutions are distinct for w arbitrarily close to the origin.

Now we use the multiplicity result of [6]. The complexity of the Pfaffian system is bounded by a computable function of n, n', r, m, and so by [6] we bound d computably in terms of this data.

Now we conclude the proof of $A7$ with only a minor modification of that in [9]. We are not trying to be as neat as in [9], and in particular are using nothing like the functional equation of exp. Our substitute is to work with a more general class of Pfaffian functions (and in particular this is why we use the $\pm x$). As in [9] we can work with ϕ_k^{-1}, where k is the subscript corresponding the choice of f_n, and then reparametrize as in [9] to get the corresponding d for this. From all this, one gets constructively a finite set containing the growth rates, and the most difficult part of the proof is done.

4.5 Applying Bertolin's work

Just as Schanuel's Conjecture is the last item in the proof of decidability of the real exponential, so in the elliptic case (or at least the special one considered here) one uses in the end game a conjecture from transcendence theory. In fact, there is a very general conjecture of André on 1-motives, generalizing a conjecture of Grothendieck on the nature of the comparison map between the algebraic De Rham cohomology and the singular cohomology of projective nonsingular maps over $\mathbb{Q}$, which implies the transcendence estimates we need (and which also implies Schanuel's Conjecture). The transcendence estimates are proved in Bertolin's paper [3], assuming the André Conjecture.

We now show that the special theory we consider is decidable, if André's Conjecture is true. The argument is very general, and based on the concluding argument in [9].

The problem is entirely to decide if an existential sentence is true. We first show that the set of true such sentences is recursively enumerable. By our recursive model-completeness result it is enough to show that true existential sentences are proved from a recursive set of true sentences.

The argument is a variant of the Newton Approximation argument from [9]. It goes as follows.

We are working on $I = [a, b]$ for the a and b introduced earlier. We have already stressed that our Pfaffian data is algebraic over Ω, and we should once and for all now fix constants, one for Ω and one for an element algebraic over it, so that all our data is in the recursive real field l generated by Ω and the other element. We can use the Taylor expansion of $\wp^{-1}$ about the (algebraic) midpoint of this interval. This has coefficients algebraic over a, and the sequence of coefficients is given by a recursive sequence of algebraic functions of a. Moreover, there is an algorithm which, given a positive integer n provides an initial segment of the Taylor series uniformly approximating $\wp^{-1}$ on I to within $\frac{1}{n}$. In fact, and this is crucial, there is an algorithm, which when given any multivariable polynomial over $\mathbb{Q}$ in arguments x_j and $\wp^{-1}(x_j)$ provides the analogous uniform Taylor expansion. Moreover, all this is derivable from the recursive set of axioms produced above (if anyone doubts this, that person may simply add a recursive set of axioms from which the above facts are derivable).

Prior to getting to the main arguments, note that the set of basic $\wp^{-1}$ polynomial systems, with only bounded variables, unsolvable in the structure, is also recursively enumerable. For such a system will have a supnorm bounded away from zero by a rational, and this will be revealed by semi-algebraic data.

What about the variant where some of the variables are bounded, some not? The latter only occur in semi-algebraic form, and can be eliminated, but possibly at the cost of introducing inequations or order-inequalities amongst the bounded variables. In fact, this creates very nontrivial problems, which will require André's Conjecture for their solution.

Still, these remarks are useful, since they clearly show that we have only to decide systems of equations and inequalities with bounded variables. The coefficients of such systems can be assumed to come from the field l (note that $\wp^{-1}$ takes rational values at a and b. With this convention we are reduced to looking for systems solvable in the interior of cartesian powers of $[a, b]$ (including the special case where there are no variables, clearly a problem linked to transcendence theory).

The advantage of having to solve in an open set is that we may appeal to a very important observation of Wilkie [13], related to the stratification procedures in [6]. The result is that if we have a system over a subfield k, in n variables, consisting of equations and inequations, solvable in the interior of $[a, b]^n$ then there is a system of n equations in n unknowns, with a nonsingular zero in the above interior, solving also the original system.

Now, because of the constructive model-completeness, we will get decidability if we can show that the set of solvable systems (with the above restrictions) is recursively enumerable. For this, the game we will play is very similar to that of [9], with the slight complication that here we look for bounded solutions, and in [9] one looked for unbounded, to get decidability for the global exponential.

A convenient way to use the Newton Approximation method of [9] is to map the real line bijectively to (a, b) by a continuous semi-algebraic map G, and to replace $\wp^{-1}$ by its composition with G. One then gets a Newton Approximation result for the global functions, and comes back via G^{-1}. One then gets an analogue of Theorem 4.1 of [9]. It is important that we can choose G so that G' is bounded above in absolute value. Note that G' has no zeros, but there is no way to bound it below by a positive real.

The result is:

Theorem 4.3 *There is an algorithm which, given positive integers n and*

N, and an n-tuple F of functions $f(x_1, \ldots, x_n, \wp^{-1}(x_1), \ldots, \wp^{-1}(x_n))$ where each f is polynomial over k, produces a positive integer θ such that if $\overline{\alpha} \in I^n$ is in $(a,b)^n$, and $\|G^{-1}(\alpha)\| < N$, $\|F(\overline{\alpha})\| < \theta^{-1}$ and

$$|\mathrm{Jac}(F)(\overline{\alpha})| > N^{-1}(G'(G^{-1}\alpha_1), \ldots, G'(G^{-1}\alpha_n))^{-1},$$

then there exists $\overline{\gamma} \in I^n$ such that $F(\overline{\gamma}) = \overline{0}$ and $\|\overline{\alpha} - \overline{\gamma}\| < \sup |G'| \, N^{-1}$.

In the above, $\| \cdot \|$ denotes the supnorm.

Now, this Theorem is applicable to bounded Hovanski systems in the following sense. If we have such a system $F = 0$ and a nonsingular zero in the interior of (a,b), then under G^{-1} this is transformed into a nonsingular zero of $F(G) = 0$. Near to the latter there will be a rational point and an N so that the usual hypotheses of Newton Approximation are satisfied, and these will translate into the hypotheses given above.

Now fix a standard G with the above properties once and for all, and consider the recursive set of axioms corresponding to the above version of Newton Approximation (obviously true in our intended model).

Just as in [9] the point now is that if one has a Hovanski system over k solvable in the interior of $[a,b]$ then our axioms prove the existence of such a solution. Moreover, they even locate each real solution uniquely.

For a general system, Wilkie's observation shows that if we have a system, in n variables consisting of equations $F = 0$ and inequations $R \neq 0$ then if it has a solution at all in $(a,b)^n$ then some Hovanski system over k provides a solution to the system. Note that at this moment we do not assume anything about the constructive nature of this.

The argument till now is parallel to that for the real exponential. Now we have to evoke the André Conjecture for our elliptic curve. The point is to consider a solution $(x_1, \ldots, x_n, \wp^{-1}(x_1), \ldots, \wp^{-1}(x_n))$ of the Hovanski system. We consider this as a solution in affine $2n$ space of the corresponding polynomial system over k (exactly as in [9]). Now comes the crucial issue of the dimension over k of this solution, and this is where Bertolin's results become applicable.

Here is the enumeration procedure (which, alas, loses most of what we have gained from the delicate work of Gabrielov-Vorobjov). It involves provability from the recursive set of axioms isolated above.

Fix a system $\mathcal{S}$ of equations and inequalities over k of the usual type in n variables. We are going to show that our axioms prove that if the system is solvable in $\mathbb{R}$ then our axioms prove that it is solvable.

The optimal situation is when $\mathcal{S}$ has no solution in $\mathbb{R}$ where the $\wp^{-1}(x_i)$ together with 1 are linearly dependent over $\mathbb{Q}(i)$. We first show that it is enough to consider such systems.

For consider a nontrivial formal linear dependence over $\mathbb{Q}(i)$ between the $\wp^{-1}(x_i)$ and 1 over $\mathbb{Q}(i)$. If we think of the $\wp^{-1}(x_i)$ as real variables (reasonable in our settin) this induces a nontrivial linear relation over $\mathbb{Z}$ between the $\wp^{-1}(x_i)$ and 1. Now, using the Addition Theorem, we can formally eliminate one pair of variables x_j and y_j (just as is done in [9] at a similar stage in the proof) and obtain a new system, in one fewer variable, which is also solvable, and such that our axioms prove that if it is solvable so is $\mathcal{S}$. By continuing this way, we eventually reduce to the case we want. We now assume $\mathcal{S}$ has this property.

Now let $\mathcal{H}$ be a Hovanski system over k in n variables with a solution in common with $\mathcal{S}$. We can assume that the equations of $\mathcal{H}$ are among the equations of $\mathcal{S}$. By our assumption on $\mathcal{S}$ no such common solution has its $\wp^{-1}$ part linearly dependent with 1 over $\mathbb{Q}(i)$. Fix such a common solution $(x_1, \ldots, x_n)$ for the rest of our analysis.

Each Hovanski system $\mathcal{H}$ has an associated polynomial system $\mathcal{H}^*$ in affine $2n$ space, and the solution $(x_1, \ldots, x_n)$ of $\mathcal{H}$ yields the solution $(x_1, \ldots, x_n, \wp^{-1}(x_1), \ldots, \wp^{-1}(x_n)$ of $\mathcal{H}^*$. Moreover, an easy calculation shows that this solution is nonsingular. From this in turn it follows that the dimension of the solution over k is at most n. We will be able to show, using Bertolin's work, that, generically, this dimension is exactly n. In such a case it follows that our $2n$ tuple is a generic point of some irreducible component of $\mathcal{H}^*$ (over k) of dimension n.

At this point we use what Bertolin [3] calls "Conjecture Elliptico-Torique", and which is a consequence of the André Conjecture. An interesting feature of this conjecture (and one we expect to exploit more in future) is the presence in it of quasi-periods ,the j-invariant, and integrals of the first and second kind on the curve. Another interesting feature is that to get what we want for our original Weierstrass function we do best to pass to an analogous problem for the isomorphic lattice got by multiplying by Ω. I appreciated this point fully only after discussions with Daniel Bertrand, whom I heartily thank.

So, let us start with points $z_1, \ldots, z_n$ in $\mathbb{C}$ (the tangent space of whatever curve we are dealing with) and corresponding points $P_1, \ldots, P_n$ on the curve (got from the standard exponential map involving $\wp$ and $\wp'$). Following Bertolin we write $p_1, \ldots, p_n$ for the integrals of the first kind associated with $z_1, \ldots, z_n$. These are defined up to periods, and in the above notation can be construed as $z_1, \ldots, z_n$. Again following Bertolin, we write $d_1, \ldots, d_n$ for the integrals of the second kind associated with $P_1, \ldots, P_n$. These again are defined only up to quasi-periods, and for our purposes should be thought of as $\zeta(z_1), \ldots, \zeta(z_n)$, where ζ is the Weierstrass zeta-function. In the present paper one need know almost nothing about this function, but we intend to say more about it in a future publication. To complete the presentation of characters to state the Conjecture, we write ω_1, ω_2 for generators of the periods, η_1, η_2 for the corrresponding quasi-periods, j for the j-invariant, and k for $\mathbb{Q}$ if there is no complex multiplication, and the field of complex multiplication otherwise. Of course, in our case, j is 12^3 and k is $\mathbb{Q}(i)$.

Now the Conjecture says :

Suppose that the k-dimension of the space generated by the $p_1, \ldots, p_n$ and the periods is D plus the dimension of the space generated by the periods. Then the transcendence degree over $\mathbb{Q}$ generated by the periods, the quasi-periods, the j-invariant, the P_i, p_i and the d_i, is at least

$$2D + 2.$$

(The Conjecture is more general, and can involve ordinary logarithms, again something to which we plan to return in a subsequent paper).

Now we assume we are in a CM case, and that the j-invariant is algebraic (true in our present example). Moreover, we assume that our curve is defined over an algebraic number field, not the case for our original curve, but true for its isomorphic copy got by multiplying the lattice by Ω. Then the transcendence degree of the field generated by the periods and quasi-periods is 2, by a result of Chudnovsky.

So let us now assume that in the above $D = n$. Then by trivial counting we get that the transcendence degree ,over any period, of the field generated by the $P_1, \ldots, P_n$ and the $p_1, \ldots, p_n$ is at least n.

Now we can go back to our original curve and get what we need. Start with $z_1, \ldots, z_n$ in $\mathbb{C}$. Now multiply the z_i by Ω , and let $P_1, \ldots, P_n$ be the corresponding points on the isomorphic curve got by multiplying the lattice by Ω. Let p_i have the usual meaning (on the new curve). Suppose that the $\Omega\, z_1, \ldots, \Omega\, z_n$ and Ω generate a $\mathbb{Q}(i)$ space of dimension $n + 1$.(Note that this is the same assumption as the one that 1 and the z_i generate a space of dimension $n + 1$ over $\mathbb{Q}(i)$). Then we can deduce by the previous discussion that the field generated over $\mathbb{Q}(\Omega)$ by $\Omega\, z_1, \ldots, \Omega\, z_n$ and $P_1, \ldots, P_n$ has transcendence degree at least n over $\mathbb{Q}(\Omega)$. To finish, we need only identify $P_1, \ldots, P_n$ in terms of the data on the original curve. Now, by the transformation rules for relating Weierstrass functions for isomorphic lattices, one gets that (essentially)

$$P_i \;=\; (\Omega^2\, \wp(z_i),\; \Omega^2\, \wp'(z_i),\; 1)$$

so we conclude that the transcendence degree over $\mathbb{Q}(\Omega)$ (and so over l) of the field generated by $z_1, \ldots, z_n$ and the $\wp(z_1), \ldots, \wp(z_n)$ is at least n, provided that that 1 and the z_i generate a space of dimension $n + 1$ over $\mathbb{Q}(i)$).

This is exactly what we need to get decidability by a variant of the method of [9].

To complete the proof we have to do some Newton Approximation, in the style of [9]. Firstly, fix a rational point and an N so that the Newton data isolates our fixed solution above, within a standard neighbourhood on which the Jacobian is bounded below in absolute value by a positive rational. Now, our point is on exactly one of the irreducible components mentioned above, and for each of the others one of its defining polynomials will not vanish in a subneighbourhood of $(x_1, \ldots, x_n, \wp^{-1}(x_1), \ldots, \wp^{-1}(x_n))$. There is a fixed compact subneighbourhood that will work simultaneously for all such polynomials, and a uniform rational lower bound for the absolute value of the polynomials (and the Jacobian of the Hovanski system) on the neighbourhood. That this lower bound holds on the standard neighbourhood in question will be provable from our axioms, by uniform approximation of our functions by polynomials (and then Tarski's Theorem). Thus our axioms will leave open only one possibility, V say, for an irreducible component of dimension n on which our point lies. In fact, the same argument applies to any solution (even in a nonstandard model) lying in the standard compact neighbourhood. But it leaves open the possibility that some such points have lower dimension.

Consider the polynomials occurring in the equations of $\mathcal{S}$. Since

$$(x_1, \ldots, x_n, \wp^{-1}(x_1), \ldots, \wp^{-1}(x_n))$$

is a zero of each, and is generic for V, each of them vanishes on V in any field.

The inequalities of $\mathcal{S}$ hold at our point, and so hold in a neighbourhood of our point. By increasing N if need be, we can suppose, by the usual approximation, that our axioms prove that the functions do not vanish on our compact standard neighbourhood.

So, we will be done if we show that our axioms prove that there is a point of V in our standard compact neighbourhood. This requires one last trick, already used in [9]. Namely, there are polynomials (over k)

$$P_0, P_1, \ldots, P_n$$

so that P_0 is not identically zero on V and on the Zariski open subset of $2n$ space defined by the nonvanishing of P_0 V is defined by the vanishing of $P_1, \ldots, P_n$. Now our point will be a solution of the Hovanski system given by $P_1, \ldots, P_n$, and a provable instance of Newton Approximation will reveal that inside our compact neighbourhood there is a Hovanski solution of the new system at which P_0 does not vanish (again use approximation). This will give a point of V, and we are done.

We have proved:

Theorem 4.4 *Assuming André 's Conjecture, the Weierstrass function for the lattice generated by* 1 *and* i *is decidable.*

5 Concluding remarks

Naturally, we expect a large-scale generalization of the preceding. Firstly, one can surely, with little extra effort, deal with other specific examples. Secondly, I think it likely that one can add some restricted exponential and keep decidability (modulo the Conjecture). Maybe one can even add the full real exponential in the above case. Thirdly, I expect to do something with both integrals of the second kind and with the Jacobi elliptic functions. Fourthly, I expect to be able to deal with uniformities as the lattice varies, both for definability and decidability.

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O-minimal expansions of the real field II

P. Speissegger[5]

Abstract

In my tutorial at the Euro-conference in Model Theory and its Applications in Ravello, Italy in May 2002, I surveyed (among others) the construction of the Pfaffian closure of an o-minimal expansion of the real field. This note mentions an application of the o-minimality of the Pfaffian closure to the theory of o-minimal structures. I also describe the state of affairs (as I know it) concerning the model completeness conjecture for the Paffian closure and try to formulate an open question testing the limit of the Pfaffian closure's applicability.

These notes are a continuation of [17], and I assume the reader is familiar with the latter; in particular, I continue to use notations and terminology introduced there. Here I discuss an application of the o-minimality of the Pfaffian closure to o-minimal expansions of the additive group of reals, and I try to describe two related open problems. The first of these problems is the "old" question whether the Pfaffian closure of an o-minimal expansion of the real field is model complete. One aspect of this question is to recognize those leaves of a given 1-form that have the Rolle property, which might also play a major role in understanding the foliation associated to a definable 1-form. The latter is the second topic I discuss, because it provides a way to test the limits of what is definable in the Pfaffian closure of an o-minimal structure.

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1 Adding multiplication

The application discussed here involves o-minimal expansions of the additive ordered group of real numbers.

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Theorem 1.1 (Peterzil et al. [14]) *Let $\mathcal{R}$ be an o-minimal expansion of $(\mathbb{R}, <, +, 0)$. Then $(\mathcal{R}, \cdot)$ is o-minimal.*

The Pfaffian closure enters the picture in the following form:

Lemma 1.2 *Let $\mathbb{R}'$ be an o-minimal expansion of the real field, and let $(\mathbb{R}, <, *)$ be an ordered group definable in $\mathbb{R}'$ and $h : (\mathbb{R}, <, +) \longrightarrow (\mathbb{R}, <, *)$ an isomorphism. Then $(\mathbb{R}', h)$ is o-minimal.*

Proof. We assume for simplicity that $*$ and h are C^1 (the general case uses Pillay [16] to reduce to the C^1 case). Then $h'(0) \neq 0$ and for all $x \in \mathbb{R}$,

$$\begin{aligned} h'(x) &= \lim_{\epsilon\to 0} \tfrac{h(x+\epsilon)-h(x)}{\epsilon} \\ &= \lim_{\epsilon\to 0} \tfrac{h(x)*h(\epsilon)-h(x)*1_*}{h(\epsilon)-1_*} \cdot \tfrac{h(\epsilon)-h(0)}{\epsilon} \\ &= \tfrac{\partial(u*v)}{\partial v}(h(x), 1_*) \cdot h'(0). \end{aligned}$$

Hence h is definable in $\mathcal{P}(\mathbb{R}')$ and $(\mathbb{R}', h)$ is o-minimal. □

Proof. [Sketch of proof of Theorem 1.1] We assume for simplicity that $\mathcal{R}$ has a pole, that is, there is a definable homeomorphism between a bounded and an unbounded interval. (The case where $\mathcal{R}$ has no pole is similar but more involved and uses Edmundo [3].) Since for every $\lambda \in \mathbb{R}$, the linear function $x \mapsto \lambda x : \mathbb{R} \longrightarrow \mathbb{R}$ is definable in $\mathcal{R}$, every bounded interval is definably homeomorphic to $\mathbb{R}$. It follows from Laskowski and Steinhorn [9, Theorem 3.8] that there are definable maps $\oplus, \otimes : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $(\mathbb{R}, <, \oplus, \otimes)$ is a real closed field. Since $(\mathbb{R}, <, \oplus, \otimes)$ is Archimedean, there is an isomorphism

$$\phi : (\mathbb{R}, <, +, \cdot) \longrightarrow (\mathbb{R}, <, \oplus, \otimes);$$

let $\mathbb{R}'$ denote the pullback of $\mathcal{R}$ via ϕ, and let $*$ denote the pullback of $\oplus$ via ϕ. Then $\mathbb{R}'$ is an o-minimal expansion of the real field, so by Lemma 1.2, there is an isomorphism

$$h : (\mathbb{R}, <, +) \longrightarrow (\mathbb{R}, <, *)$$

such that $(\mathbb{R}', h)$ is o-minimal. Hence $f = \phi \circ h : (\mathbb{R}, <, +) \longrightarrow (\mathbb{R}, <, \oplus)$ is an isomorphism such that $(\mathcal{R}, f)$ is o-minimal. Let $\odot$ be the pullback of $\otimes$ via f. Then there is a nonzero $s \in \mathbb{R}$ such that $x \odot y = \frac{xy}{s}$ for all $x, y \in \mathbb{R}$, so $\cdot$ is definable in $(\mathcal{R}, f)$. □

2 Model completeness

Let $\mathcal{R}$ be an o-minimal expansion of the real field. The main open question about its Pfaffian closure $\mathcal{P}(\mathcal{R})$, first posed in essence by Gabrielov [5], is the following:

Question 2.1 *Is $\mathcal{P}(\mathcal{R})$ model complete?*

From the proof of o-minimality in [17], we know that every set definable in $\mathcal{P}(\mathcal{R})$ is a Λ^∞-set (in the terminology of [17]). Thus, one way of trying to answer Question 1 is to show that every Λ^∞-set is, up to projection, a finite union of Rolle leaves over $\mathcal{R}$. In other words, given $A \subseteq \mathbb{R}^{m+n}$ and 1-forms $\omega_1, \dots, \omega_q$ definable in $\mathcal{R}$, one would (essentially) need to find finitely many 1-forms $\eta_1, \dots, \eta_k$ definable in $\mathcal{R}$ such that for every choice of Rolle leaves L_i of $\omega_i = 0$, every Hausdorff limit of the fibers $(A \cap L_1 \cap \cdots \cap L_q)_a$, $a \in \mathbb{R}^m$, is (the projection of) a finite union of Rolle leaves of the η_j's.

To be more specific, we need the following terminology:

Definition 2.2 Let G_n^p denote the Grassmannian of all p-dimensional vector subspaces of $\mathbb{R}^n$. We view G_n^p as an embedded submanifold of $\mathbb{R}^{n^2}$ by identifying each $V \in G_n^p$ with the unique orthonormal $n \times n$-matrix whose kernel is V (see Bochnak, Coste and Roy [1] for details); in particular, G_n^p is definable in $\mathcal{R}$.

Let $M \subseteq \mathbb{R}^n$ be a C^1 manifold, and let $d : M \longrightarrow G_n^p$. The map d is called a *p-distribution* on M, and d is *tangent to* M if $d(x) \subseteq T_xM$ for all $x \in M$. A manifold $N \subseteq M$ is an *integral manifold of* d if $T_xN = d(x)$ for all $x \in N$. A *leaf of* d is a maximal, connected, integral manifold of d.

Example 2.3 Let α be a nonsingular q-form on an open $U \subseteq \mathbb{R}^n$, with $q \le n$. (See Spivak [19] for details on such α; the notions of $\ker \alpha$, integral manifolds, etc. generalize from those for 1-forms, and α is *nonsingular* if $\ker \alpha(z)$ has dimension $n - q$ for all $z \in U$.) Then the map $x \mapsto \ker \alpha(x) : U \longrightarrow G_n^{n-q}$ is an $(n-q)$-distribution d_α on U, and any integral manifold (leaf) of $\alpha = 0$ is an integral manifold (leaf) of d_α.

As a particular instance of this example, let $\Omega = \{\omega_1, \dots, \omega_q\}$ be a finite, *transverse* collection of 1-forms on U, that is, $\bigcap_{i=1}^q \ker \omega_i(x)$ has dimension $(n-q)$ for all $x \in U$. Then the form $\alpha = \omega_1 \wedge \cdots \wedge \omega_q$ is a nonsingular q-form on U, and if L_i is a leaf of $\omega_i = 0$, for $i = 1, \dots, q$, then every connected component of $L_1 \cap \cdots \cap L_q$ is a leaf of d_α.

Any procedure to prove model completeness of $\mathcal{P}(\mathcal{R})$ as outlined above has to address the following two problems:

(P1) Given $A \subseteq \mathbb{R}^{m+n}$ and 1-forms $\omega_1, \dots, \omega_q$ on an open $U \subseteq \mathbb{R}^{m+n}$, all definable in $\mathcal{R}$, we need to find $(n-1)$-distributions $d_1, \dots, d_k$, definable in $\mathcal{R}$ and only depending on A and $\omega_1, \dots, \omega_q$ (not on any particular Hausdorff limit), such that for every choice of Rolle leaves L_i of $\omega_i = 0$, every Hausdorff limit of the fibers $(A \cap L_1 \cap \cdots \cap L_q)_a$, $a \in \mathbb{R}^m$, is the projection of a Boolean combination of leaves of the d_j's.

(P2) Once the d_j as in (P1) are found, we still need to know that the leaves of the d_j's in question are Rolle leaves.

In collaboration with Lion [12], we carried out a weak version of such a procedure; it avoids the above two problems, but also does not give the desired model completeness. Nevertheless, this weak version gives a solution to a weak form of (P1), as described below.

Indeed, we show that (P1) holds as long as we do not require the d_j to be $(n-1)$-distributions, but allow arbitrary p-distributions (for various p). This is sufficient for the following weak model completeness result: Let $\mathcal{L}$ be the language consisting of all C^1 cells $C \subseteq \mathbb{R}^n$ that are definable in $\mathcal{P}(\mathcal{R})$ and for which there exist

(i) a C^1 submanifold $M \subseteq \mathbb{R}^n$ definable in $\mathcal{R}$, and

(ii) a p-distribution d on M definable in $\mathcal{R}$,

such that C is an integral manifold of d.

Theorem 2.4 (Corollary 5.2 in [12]) *The Pfaffian closure $\mathcal{P}(\mathcal{R})$ is model complete in the language $\mathcal{L}$. More precisely, given a set $S \subseteq \mathbb{R}^n$ definable in $\mathcal{P}(\mathcal{R})$, there are $C_1, \dots, C_k \in \mathcal{L}$ such that $C_j \subseteq \mathbb{R}^{n_j}$ with $n_j \geq n$ and $\dim(C_j) = \dim(\Pi_n(C_j))$ for $j = 1, \dots, k$, and such that $S = \Pi_n(C_1) \cup \cdots \cup \Pi_n(C_k)$.*

As a corollary, we obtain

Corollary 2.5 ([12]) *If $\mathcal{R}$ admits analytic (resp. C^∞) cell decomposition, then so does $\mathcal{P}(\mathcal{R})$.*

Proof. [Sketch of proof] Assume that $\mathcal{R}$ admits analytic cell decomposition (the proof is similar for C^∞ cell decomposition), and let $S \subseteq \mathbb{R}^n$ be definable in $\mathcal{P}(\mathcal{R})$. By routine cell decomposition arguments, it suffices to show that S is a finite union of analytic manifolds definable in $\mathcal{P}(\mathcal{R})$. This is done by induction on $p = \dim(S)$; the case $p = 0$ is trivial, so we assume that $p > 0$ and the claim holds for lower values of p. By Theorem 2.4, we may assume that $S = \Pi_n(C)$, where $C \subseteq \mathbb{R}^N$ belongs to $\mathcal{L}$ and satisfies $\dim(C) = \dim(\Pi_n(C)) \leq p$. Since C is a cell, it follows that $C = \mathrm{graph}(g)$, where $g : \Pi_n(C) \longrightarrow \mathbb{R}^{N-n}$ is a definable C^1 function and $\Pi_n(C)$ is a C^1 cell; in particular, $\Pi_{n|C} : C \longrightarrow \Pi_n(C)$ is a C^1 diffeomorphism.

By the inductive hypothesis, we may assume that $\dim(C) = p$. Let $M \subseteq \mathbb{R}^N$ be a C^1 manifold and $d : M \longrightarrow G^p_N$ a distribution, both definable in $\mathcal{R}$, such that C is an integral manifold of d. Using analytic cell decomposition in $\mathcal{R}$ and the inductive hypothesis, we may reduce to the case where both M and d are analytic. It follows from the theory of differential equations that C is an analytic manifold. Hence $\Pi_{n|C}$ is an analytic diffeomorphism and $\Pi_n(C)$ an analytic manifold. □

Here is another application of Theorem 2.4: Lion et al. [11] show that if $f : (a, \infty) \longrightarrow \mathbb{R}$ is of class C^k and satisfies an ordinary differential equation

$$f^{(k)}(t) = G\left(t, f(t), f'(t), \ldots, f^{(k-1)}(t)\right), \text{for all } t > a, \tag{1}$$

where G is definable in $\mathcal{R}$, and if f is *non-oscillatory over* $\mathcal{R}$, then f is eventually bounded by an iterate of the exponential function. Instead of defining "non-oscillatory over $\mathcal{R}$", it suffices to say here that every one-variable function definable in some o-minimal expansion of $\mathcal{R}$ is non-oscillatory over $\mathcal{R}$. Thus, we obtain

Corollary 2.6 ([11]) *If $\mathcal{R}$ is polynomially bounded, then $\mathcal{P}(\mathcal{R})$ is exponentially bounded.*

Proof. [Sketch of proof] Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be definable in $\mathcal{P}(\mathcal{R})$. By Theorem 2.4 there are $n \in \mathbb{N}$, $a \in \mathbb{R}$, $F = (F_1, \ldots, F_n) : (a, \infty) \longrightarrow \mathbb{R}^n$ definable in $\mathcal{P}(\mathcal{R})$, $M \subseteq \mathbb{R}^{n+1}$ a C^1 manifold definable in $\mathcal{R}$ and $d : M \longrightarrow G^1_{n+1}$ definable in $\mathcal{R}$, such that $\mathrm{graph}(F)$ is an integrable manifold of d and $F_1 = f_{|(a,\infty)}$. By o-minimality and a routine argument using Lie derivatives, and after increasing a if necessary, it follows that $f_{|(a,\infty)}$ is of class C^k for some k and satisfies 1 for some G definable in $\mathcal{R}$. Since f is non-oscillatory over $\mathcal{R}$, the corollary follows. $\square$

Corollaries 2.5 and 2.6 show that the model completeness of $\mathcal{P}(\mathcal{R})$ in the language $\mathcal{L}$ can be useful from a geometric point of view. However, from other points of view—such as that of associating a measure of complexity to the sets definable in $\mathcal{P}(\mathcal{R})$, or that of estimating the number of connected components of definable sets in some explicit way, see [4] and Zell [20]—the language $\mathcal{L}$ has some serious drawbacks:

- Instead of distributions of codimension 1 (coming from 1-forms), we need to allow distributions of arbitrary codimension.
- We are unaware of a natural notion of "Rolle leaf" for distributions of codimension greater than 1.
- The integral manifolds making up the language $\mathcal{L}$ are not in general leaves of the corresponding distribution.

What these drawbacks amount to is that we do not know of a description of the predicates in $\mathcal{L}$ that would be "intrinsic to $\mathcal{R}$", say, in the way that a leaf L of a 1-form ω definable in $\mathcal{R}$ is uniquely determined by knowing ω and any point of L.

3 Foliations

Let $\mathcal{R}$ be an o-minimal expansion of the real field. Let $\omega = a_1 dx_1 + \cdots + a_n dx_n$ be a definable 1-form of class C^1 on an open set $U \subseteq \mathbb{R}^n$, and let $S(\omega) = \{x \in U : \ker \omega(x) = \mathbb{R}^n\} = \{x \in \mathbb{R}^n : a_1(x) = \cdots = a_n(x) = 0\}$ be the *set of singularities* of ω. From the definition of $\mathcal{P}(\mathcal{R})$, we know that every Rolle leaf in U of $\omega = 0$ is definable in $\mathcal{P}(\mathcal{R})$.

Question 3.1 *Are there natural conditions on ω and/or U that guarantee that every leaf of $\omega = 0$ is a Rolle leaf?*

Besides being of interest in its own right, this question is closely tied to Problem (P2).

Definition 3.2 Assume that ω is nonsingular and $\omega \wedge d\omega = 0$ on U (such an ω is called *integrable on* U). Then by Froebenius' Theorem every $x \in U \setminus S(\omega)$ belongs to a unique leaf of $\omega = 0$. In this situation, the collection of all leaves of $\omega = 0$ is called the *foliation associated to* $\omega = 0$.

Question 3.3 *Assume that ω is nonsingular and integrable. Under what additional conditions on ω and/or U is the foliation associated to ω given as a family of sets definable in some o-minimal expansion of $\mathcal{R}$?*

Question 3.3 is expected to be much harder to answer than Question 3.1; indeed, a "good" answer to Question 3.3 could have important implications for Hilbert's 16th problem.

The following is a topological criterion for the Rolle property, which we will use in our brief discussion of Questions 3.1 and 3.3:

Fact 3.4 (Khovanskii [8]) *Assume that $U \setminus S(\omega)$ is simply connected, and let $L \subseteq U \setminus S(\omega)$ be an embedded leaf of ω that is closed in $U \setminus S(\omega)$. Then L is a Rolle leaf of ω in U.*

Proof. [Sketch of proof] By Theorem 4.6 and Lemma 4.4 of Chapter 4 in [7], the set $U \setminus S(\omega)$ has exactly two connected components U_1 and U_2, such that $\mathrm{bd}(U_i) \cap (U \setminus S(\omega)) = L$ for $i = 1, 2$. The argument of Example 1.3 in [18] now shows that L is a Rolle leaf in U. $\square$

We now return to the discussion of Questions 3.1 and 3.3:

Example 3.5 Assume that U is connected and simply connected and that ω is nonsingular and *closed*, that is, $d\omega = 0$ on U. Fix an $a \in U$; then the integral $g(x) = \int_a^x \omega, \quad x \in U$, computed along any C^1 path connecting a to x, defines a C^1 function $g : U \longrightarrow \mathbb{R}$ such that $dg = \omega$, that is, the graph $\mathrm{graph}(g)$ of g is a closed, embedded leaf of $dx_{n+1} - \omega = 0$ in $U \times \mathbb{R}$. It follows from Fact 3.4 that $\mathrm{graph}(g)$ is a Rolle leaf of $dx_{n+1} - \omega = 0$, so g is definable in $\mathcal{P}(\mathcal{R})$. Therefore, and because ω is nonsingular, for every $c \in \mathbb{R}$ the set $g^{-1}(c)$ is a closed, embedded submanifold of U. Since every leaf of $\omega = 0$ is a connected component of $g^{-1}(c)$ for some $c \in \mathbb{R}$, it follows from Fact 3.4 again that every leaf of $\omega = 0$ is a Rolle leaf.

In fact, this argument shows that the foliation associated to $\omega = 0$ is definable in $\mathcal{P}(\mathcal{R})$ as well: it is the family of all connected components of the definable family of sets $\{g^{-1}(c) : \ c \in \mathbb{R}\}$.

Unfortunately, the case where ω is closed is not very interesting in connection with Hilbert's 16th problem: there can be no limit cycle among the level sets of the function g above, that is, there is no compact level set L of g such that all nearby level sets of g are not compact.

If ω is not closed, things get more complicated:

Example 3.6 Moussu and Roche [13] prove, based on Haefliger [6], that if ω is nonsingular and analytic and $U \subseteq \mathbb{R}^n$ is open and simply connected, then every leaf of $\omega = 0$ in U is a Rolle leaf. As a consequence, if $\mathcal{R}$ admits analytic cell decomposition and $\mathcal{C}$ is a decomposition of $\mathbb{R}^n$ into analytic cells such that $\omega_{|C}$ is analytic for all $C \in \mathcal{C}$, then for every $C \in \mathcal{C}$, every leaf of $\omega_{|C} = 0$ is a Rolle leaf in C.

The first statement of Example 3.6 is false when "analytic" is replaced by "C^∞": Lion [10] gives an example of a nonsingular 1-form ω on $\mathbb{R}^3$ of class C^∞, such that ω is definable in the expansion of the real field by the exponential function, and such that there is a leaf of $\omega = 0$ that is not a Rolle leaf.

Lion's example, however, still seems to satisfy the second assertion of Example 3.6. This suggests the following:

Conjecture 3.7 *There is a finite decomposition $\mathcal{C}$ of $\mathbb{R}^n$ into cells definable in $\mathcal{R}$ and compatible with $S(\omega)$, such that for every open $C \in \mathcal{C}$ with $C \cap S(\omega) = \emptyset$, every leaf of $\omega_{|C} = 0$ is a Rolle leaf in C.*

Proof. [Proof for the case $n = 2$] Let $\mathcal{C}$ be a cell decomposition of $\mathbb{R}^2$ definable in $\mathcal{R}$ and compatible with the sets $A_i = \{x \in \mathbb{R}^2 : \ a_i(x) = 0\}$ for $i = 1, 2$; we claim that this $\mathcal{C}$ works.

Let $C \in \mathcal{C}$ be open. Note that for $i = 1, 2$, the map $a_{i|C}$ has constant sign (because C is connected); in particular, $\mathcal{C}$ is compatible with $S(\omega)$. If $a_{2|C} = 0$, then the leaves of $\omega_{|C} = 0$ are vertical segments and therefore Rolle leaves. So we assume that $a_2(x) \neq 0$ for all $x \in C$.

Let $L \subseteq C$ be a leaf of $\omega_{|C} = 0$, and let $a \in \Pi_1(C)$. If $L \cap C_a$ contains two distinct points, then by Rolle's Theorem there is an $z \in L$ such that $T_z L = \ker dx(z)$, that is, $a_2(z) = 0$, a contradiction. Therefore, L is the graph of a C^1 function $f : (c, d) \longrightarrow \mathbb{R}$; in particular, L is a connected, closed and embedded submanifold of C. By Fact 3.4, the leaf L is a Rolle leaf of $\omega_{|C} = 0$. $\square$

The crucial point of the proof above is the use of Rolle's Theorem, which is not available if $n \geq 3$.

Lion's counterexample makes essential use of the function $e^{-1/x}$. This raises the following

Question 3.8 *If ω is nonsingular, C^1 and definable in a polynomially bounded o-minimal structure, and if $U \subseteq \mathbb{R}^n$ is open and simply connected, is then every leaf of $\omega = 0$ in U a Rolle leaf?*

While Conjecture 3.7 seems insufficient to answer Question 3.3 in a satisfactory way, it does provide, in combination with Khovanskii Theory, the tools to show that in each open $C \in \mathcal{C}$ such that $C \cap S(\omega) = \emptyset$, the foliation associated to $\omega_{|C}$ is *piecewise trivial.* For more details and a proof in the analytic case, see Chazal [2].

4 A rectification

Let $(R, <, +, -, \cdot, 0, 1)$ be a real closed field. A function $f : R^2 \longrightarrow R$ is called *harmonic* if f is of class C^2 and $(\partial^2 f/\partial x^2)(x, y) + (\partial^2 f/\partial y^2)(x, y) = 0$ for all $(x, y) \in R^2$.

In my tutorial in Lisbon, I claimed that the following observation was a corollary of the o-minimality of the Pfaffian closure: *any harmonic function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ definable in an o-minimal expansion of the real field is rational.* While this is indeed a corollary of the o-minimality of the Pfaffian closure, a more general statement can be obtained by a more direct argument:

Proposition 4.1 (Miller) *Let $\mathcal{R}$ be an o-minimal expansion of a real closed field $(R, <, +, -, \cdot, 0, 1)$, and let $f : R^2 \longrightarrow R$ be definable and harmonic. Then f is a polynomial function.*

Proof. Let K be associated to R as in Peterzil-Starchenko [15]. Since f is harmonic, the function $g = \partial f/\partial x - i\partial f/\partial y : K \to K$ is definable and holomorphic (in the sense of [15]). By Theorem 2.47 of [15], g has to be a polynomial over K. Now let G be a formal antiderivative of g, that is, G

is a polynomial over K such that $G' = g$, and let h be the real part of G. Clearly, h is polynomial (and hence definable in $\mathcal{R}$). By the Cauchy-Riemann equations, $G' = \partial h/\partial x - i\partial h/\partial y$. Therefore $\nabla h = \nabla f$, and since both f and h are definable C^1 functions, it follows that $f = h + c$ for some $c \in R$; in particular, f is polynomial. $\square$

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On the gradient conjecture for definable functions

Adam Parusiński[6]

Abstract

We present the main ideas of the proof of gradient conjecture of R. Thom in the analytic case and we discuss which of them can be carried over to the o-minimal case.

1 Analytic gradient conjecture

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be real analytic, $U \subset \mathbb{R}^n$ open. Consider the trajectories of gradient ∇f of f

$$\frac{dx}{dt}(t) = \nabla f(x(t)),\ t \in [0, \beta).$$

Theorem 1.1 (Łojasiewicz [9], [10]) *If a trajectory $x(t)$ has a limit point $x_0 \in U$ (i.e. $x(t_\nu) \to x_0$ for some sequence $t_\nu \to \beta$), then the length of $x(t)$ is finite.*

As a corollary we see that such a limit point is unique.

It is easy to see that if $\nabla f(x_0) = 0$ then $\beta = \infty$. Note that if $\nabla f(x_0) = 0$ then the trajectory is, in general, not analytic at x_0. The original proof

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of theorem 1.1 is based on the Łojasiewicz inequality for gradient. For a different proof see [2], [3].

We shall suppose that $x_0 = 0$ and $f(0) = 0$. Then, being increasing, f has to be negative along the trajectory.

Łojasiewicz inequality for gradient [8]: If $f(0) = 0$ then there is $\rho < 1$ and $c > 0$ such that in a neighbourhood of 0

$$|\nabla f| \geq c|f|^{\rho}. \tag{2}$$

We recall how the Łojasiewicz inequality gives theorem 1.1. Suppose that the trajectory $x(t)$ is contained in a neighbourhood of 0 where 2 holds and reparametrise it by its arc-length s

$$\dot{x} = \frac{dx}{ds} = \frac{\nabla f}{|\nabla f|}. \tag{3}$$

Then

$$\frac{df}{ds} = \langle \nabla f, \dot{x} \rangle = |\nabla f| \geq c|f|^{\rho}.$$

On $f < 0$:

$$\frac{d(-|f|^{1-\rho})}{ds} \geq c(1-\rho) > 0.$$

By integrating the both sides along the trajectory we get:

Łojasiewicz's bound: $\text{length}\{x(t); 0 \leq t < \beta\} \leq const|f(x(0))|^{1-\rho}$.

As a corollary we see that the flow of ∇f defines locally a continuous retraction $f^{-1}(c-\varepsilon, c] \to f^{-1}(c)$.

Gradient Conjecture of René Thom [13], [14]:
The trajectory $x(t)$ of a gradient vector field has a tangent at its limit point.

In other words the conjecture states that the following limit of secants exists

$$\lim_{t\to\infty} \frac{x(t) - x_0}{|x(t) - x_0|}.$$

Thom in [13] proves the conjecture in some cases including the homogeneous one. Thom's main idea is based on the blowing-up the origin and the

Łojasiewicz argument. Let us illustrate Thom's argument briefly. Expand in spherical coordinates (r, θ) at $x_0 = 0 \in \mathbb{R}^n$, with $\theta \in S^{n-1}$,

$$f = r^m F_0(\theta) + r^{m+1} F_1(\theta) + \ldots, \quad F_0 \not\equiv const \tag{4}$$

(the case $F_0 \equiv const \neq 0$ is easy). Let $\widetilde{x}(t)$ denote the projection of $x(t)$ onto S^{n-1}. Thom observed that in the homogeneous case $f = r^m F_0(\theta)$, $\widetilde{x}(t)$ coincides with a trajectory of $\nabla_\theta F_0$ on S^{n-1}. Hence in this case:
- the length of $\widetilde{x}(t)$ in S^{n-1} is finite.
- Gradient Conjecture holds for $x(t)$.

Split ∇f into the sum of its radial component $\partial_r f = \frac{\partial f}{\partial r}\frac{\partial}{\partial r}$ and the spherical one $\nabla' f = \nabla f - \frac{\partial f}{\partial r}\frac{\partial}{\partial r} = r^{-1}\nabla_\theta f$. Then

$$\begin{aligned} \partial_r f &= m r^{m-1} F_0(\theta) + \cdots \\ \nabla' f &= r^{-1}\nabla_\theta f = r^{m-1}\nabla_\theta F_0 + \cdots \end{aligned}$$

Note that Thom's argument still works in the region where $\nabla_\theta F_0$ is sufficiently big, more precisely where $\nabla_\theta F_0 \gg r$. Indeed then the higher order terms do not mess up the Łojasiewicz argument. This shows the following result attributed in [11] to Thom and Martinet.

Thom-Martinet Theorem. *The limit set of $\widetilde{x}(t)$ is contained in $Sing F_0 = \{\theta; \nabla_\theta F_0(\theta) = 0\}$.*

Corollary: *$F_0(x(t)$ has a limit $a_0 \leq 0$.*

The paper [11] is an excelent source of information on the gradient conjecture. One may find there also the following result attributed in [11] to Thom and Kuiper.

Thom-Kuiper Theorem. *If $a_0 < 0$ then Gradient Conjecture holds.*

We present below a proof of Thom-Kuiper Theorem. We show later how this proof generalizes and leads to a proof of the conjecture given in [6]. Firstly we introduce the notion of a control function whose idea comes back, probably, also to Thom.

A *control function* is a function g defined on the trajectory $x(t)$ such that:

- g is bounded ;
- g is increasing fast, for instance such that

$$\frac{dg}{d\widetilde{s}} \geq const|g|^{\rho}, \quad \rho < 1,$$

where $\widetilde{s}$ denotes the arc-length parameter on $\widetilde{x}(t)$. If such a function exists then the length $\widetilde{x}(t)$ is finite and the gradient conjecture holds for $x(t)$.

Proof. [Idea of Proof of Thom -Kuiper Theorem]
Try $g = F_0 - a_0$ as a control function. If, for instance, $|\nabla_\theta F_0| \geq r^\delta$, $\delta < 1$, then $r^m|\nabla_\theta F_0|$ is a dominant term in $\nabla_\theta f$ and

$$\frac{d(F_0 - a_0)}{d\widetilde{s}} = |\nabla_\theta F_0| + O(r) \geq const\,|F_0 - a_0|^{\rho}, \quad \rho < 1.$$

In the region where $|\nabla_\theta F_0| \leq r^\delta$, $\delta < 1$, and $|F_0| \geq |a_0|/2 > 0$, the radial part of the gradient dominates the spherical one: $|\partial_r f| \gg |\nabla' f|$. In such region the trajectory goes very fast to the origin and one may take $g = -r^\alpha$, $\alpha > 0$ small, as a control function

$$\frac{d(-r^\alpha)}{d\widetilde{s}} = -\alpha r^\alpha \frac{a_0 + O(r)}{|\nabla_\theta F_0 + O(r)|} \geq const \cdot (r^\alpha)^\rho,$$

$\rho = \frac{\alpha-\delta}{\alpha}$, if $|\nabla_\theta F_0| \leq r^\delta$, $\delta > 0$.

In general we have to find a common control function. It is not entirely obvious, but also not particularly difficult, to show that $g = (F_0 - a_0) - r^\alpha < 1$, for $\alpha > 0$ small, is such a function. $\square$

1.1 Main steps of the proof of the gradient conjecture

The gradient conjecture has been proved recently in [6]. We present below the main points of this proof. For the details the reader is refered to [6]. We have seen above that the only case left is when $a_0 = 0$. In this case many terms of the expansion 4 may contribute to the size of the radial and spherical components of ∇f. One may try to follow Thom's suggestion and to continue to blow-up. But it is difficult to control the behaviour of the gradient vector field after several blowings-up since the metric changes drastically. Instead we note that along a trajectory for which $a_0 = 0$ the asymptotic size of f is much smaller then r^m and as is shown in [6], has to be of the size r^l, where l is a rational number.

All the construction presented below are for the germs at the origin.

Step I. *Characteristic exponents.*

Consider the set

$$W^\varepsilon = \{x; f(x) \neq 0, \varepsilon|\nabla' f| \leq |\partial_r f|\},$$

defined for $\varepsilon > 0$ small and fixed.

For any connected component of W of W^ε there is $l \in \mathbb{Q}^+$ such that

$$W \subset U_l = \{x;\, c < \frac{|f(x)|}{r^l} < C\}, \quad C, c > 0.$$

The set L of such exponents is finite and $m = mult_0 f = min\, L$.

Step II. *Hierarchy of attractors.*

On each trajectory $F = \frac{f}{r^k}$, k being any positive exponent, increases in the complement of W. As a corollary one concludes that a trajectory can pass from U_l to $U_{l'}$ only if $l < l'$. Any trajectory that tends to 0 has to end-up in one of U_l.

The next step would be to expand f in U_l as in 4 but formally it is not possible (U_l is not a cone for instance). Therefore while looking for a control function, F_0 is replaced simply by $F = \frac{f}{r^l}$ and the notion of a critical value is replaced by the one of an asymptotic critical value.

Step III. *Asymptotic critical values.*

Let F be a meromorphic function, $F = \frac{f}{r^l}$ for instance. We say that $a \in \mathbb{R}$ is an *asymptotic critical value* of F at the origin if there exists a sequence $x \to 0$ such that

(a) $|\nabla_\theta F(x)| = |x||\nabla' F(x)| \to 0$,

(b) $F(x) \to a$.

By a curve selection lemma one may show that in this definition one may repalce (a) by the condition $|x||\nabla F(x)| \to 0$, see e.g. section 5 of [6].

For such asymptotic critical values both Sard Theorem and Ehresmann Theorem hold, i.e. the set of asymptotic critical values of F is finite and the family of fibres of f is locally topologically trivial over the set of non-critical values. This can be proven directly or one can use the following argument of [6] proposition 5.1. Let $X = \{(x,t); F(x) - t = 0\}$ be the graph of F. Consider X and $T = \{0\} \times \mathbb{R}$ as a pair of strata in $\mathbb{R}^n \times \mathbb{R}$. Then the (w)-condition of Kuo-Verdier at $(0,a) \in T$ reads

$$1 = |\partial/\partial t(F(x) - t)| \leq C|x||\partial/\partial x(F(x) - t)| = Cr|\nabla F|. \tag{5}$$

In particular, $a \in \mathbb{R}$ is an asymptotic critical value if and only if the condition (w) fails at $(0,a)$. The set of such values is discrete by the genericity of (w) condition. (It is finite since $a \neq 0$ is an asymptotic crititcal value of F iff a^{-1} is an asymptotic critical value of $\frac{1}{F}$.)

We have the following version of Thom-Martinet Theorem.

Proposition 1.2 *For a trajectory $x(t)$ included entirely in U_l*

$$\lim_{t\to\infty} F(x(t)) = a_0,$$

where $a_0 < 0$ is an asymptotic critical value of F.

Step IV. *Conclusion.*

We repeat the argument of our proof of Thom-Kuiper theroem. Suppose the trajectory $x(t)$ is contained in U_l, $F = \frac{f}{r^l}$, and

$$\lim_{t\to\infty} F(x(t)) = a_0 < 0.$$

Then

$$g = (F_0 - a_0) - r^{\alpha},$$

$\alpha > 0$ small, is a control function. This shows that

- The length of $\widetilde{x}(t)$ is finite.
- Gradient Conjecture holds for $x(t)$.

2 Trajectories of the gradient of definable functions

Suppose that $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ open definable, is a C^1 definable function in an o-minimal structure that we fix. Consider a trajectory $x(s)$ of ∇f parameterized by the arc-length s and that has a limit point $x_0 \in \overline{U}$. It is known by [4] that then its length is finite and hence $x(s) \to x_0$ as $s \to s_0$,

Conjecture: *The trajectory $x(s)$ has a tangent at its limit point.*

In [5] the following partial results are obtained.

Theorem 2.1

(a) The length of trajectory has the same asymptotic as the distance to the limit point

$$\frac{|x(s) - x_0|}{|s - s_0|} \to 1 \text{ as } s \to s_0.$$

(b) The gradient conjecture holds for $n = 2$.

(c) The gradient conjecture holds for polynomially bounded o-minimal structures.

We shall comment below on the major similarities and differences between the analytic and the o-minimal cases.

2.1 The length of the trajectory is finite

We recall the main result of [4] that in our argument replaces the Łojasiewicz inequality for gradient.

Theorem 2.2 *Suppose moreover U bounded. Then there exists $c > 0, \rho > 0$, and a continuous definable $\Psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that*

$$|\nabla(\Psi \circ f)(x)| \geq c, \tag{6}$$

for $x \in U$ and $f(x) \in (-\rho, \rho)$.

Note that in particular in the polynomially bounded o-minimal structures we obtain the Łojasiewicz inequality for gradient.

The construction of Ψ is elementary. We recall it briefly. Suppose for simplicity that $f \geq 0$. Choose a definable curve $\gamma(t) : (\mathbb{R}_{\geq 0}, 0) \to \overline{U}$, such that $\gamma(t) \in U$ for $t > 0$, $f(\gamma(t)) = t$, and that minimizes the norm of the gradient on the levels of f

$$|\nabla f(\gamma(t))| \leq 2 \inf\{|\nabla f(x)|; f(x) = t\}, \tag{7}$$

in $\overline{U}$.

Change the parameter of γ by $\gamma(s) = \gamma(s(t))$ so that $|\frac{d\gamma}{ds}(0)| = 1$ and $\gamma(s)$ is definable of class C^1 (for instance we may use the distance to $\gamma(0)$ as the parameter). Then we define Ψ as the inverse function of $s \to f(\gamma(s))$ that is

$$\Psi(f(\gamma(s))) = s. \tag{8}$$

Hence for arbitrary $x \in U$, $t = f(x)$ close to 0, and $s = s(t)$,

$$|\nabla(\Psi \circ f)(x)| \geq \frac{1}{2}|\nabla(\Psi \circ f)(\gamma(t))| \geq 1/4\langle\nabla(\Psi \circ f)(\gamma(s)), \gamma'(s)\rangle = 1/4, \tag{9}$$

as required.

Since the gradients ∇f and $\nabla(\Psi \circ f)$ are parallel their trajectories coincide. Consequently, by integrating both sides of 6 we obtain a bound for the length of trajectory.

(Note that this bound actually shows that the length of trajectory is bounded by the length of the definable curve γ of the above proof. In fact it is not necessary to use 6 to show it. It can be done directly as in [2], [3].)

2.2 The main arguments

By theorem 2.2 we may assume that $|\nabla f| \geq 1$ that we shall do and we consider a trajectory $x(s)$ approaching the origin. We shall also assume that $f(x(s)) \to 0$ as $s \to s_0$. Then

$$|f(x(s))| \geq \text{length}\{x(s'); s \leq s' < s_0\} \geq |x(s)|. \tag{10}$$

Step I. *Characteristic functions.*

The characteristic exponents are replaced naturally by the characteristic functions but the method of proof of [6] does not work in an arbitrary o-minimal structure. The argument of [5] is different and is based on the existence of quasi-convex decomposition of definable sets.

In order to simplify the exposition and assume that the set W^ε is contained in $\{|f(x)| \geq |x|\}$, that we can do by 10. Consider

$$W^\varepsilon = \{x; f(x) \neq 0, |\partial_r f| \geq \varepsilon |\nabla' f|\} \cap \{|f(x)| \geq |x|\}, \tag{11}$$

for $\varepsilon > 0$ small and fixed. For any connected component of W of W^ε define $\varphi(r) = \inf\{|f(x)|; x \in W \cap S(r)\}$. Then there exists $C, c > 0$ such that such that

$$W \subset U_\varphi = \{x; c\varphi(r) < |f(x)| < C\varphi(r)\}. \tag{12}$$

We explain how to show 12. Let $\gamma(r)$ be a definable curve in W parametrized by the distance to the origin. In the spherical coordinates $\gamma(r) = r\theta(r)$, $|\theta(r)| \equiv 1$. Then $r|\theta'(r)| \to 0$ as $r \to 0$ and by definition of W^ε

$$\frac{df(\gamma(r))}{dr} = \partial_r f + \langle \nabla' f, r\theta'(r)\rangle \simeq \partial_r f, \tag{13}$$

since the first term in the sum dominates the second one. Hence on $\gamma(r)$

$$|f(\gamma(r))| \geq r\left|\frac{df(\gamma(r))}{dr}\right| \geq r(\varepsilon'/\varepsilon)|\partial_r f| \geq \varepsilon' r|\nabla' f|, \tag{14}$$

for any $\varepsilon' < \varepsilon$. (Here, for the first inequality, we use $f(\gamma(r)) \geq r$.) By existence of a quasi-convex decomposition of W, cf. [5], there exists a constant $M > 0$ such that for every $x, x' \in W$, that satisfy $|x| = |x'| = r$, there is a continuous definable curve $\xi(t)$ joining x and x' in $W \cap S(r)$ and of length $\leq Mr$. Then

$$|\frac{d}{dt} f(\xi(t))| = |\langle \nabla' f, \xi'(t)\rangle| \leq |\nabla' f||\xi'(t)| \tag{15}$$

and hence by 14

$$|\frac{d}{dt} \ln |f(\xi(t))|| \leq \frac{\leq |\nabla' f||\xi'(t)|}{|f(x(t)|} \leq \frac{(\varepsilon')^{-1}}{r}|\xi'(t)|. \tag{16}$$

By integration along $\xi(t)$,

$$|\ln |f(x)| - \ln |f(x')|| \leq M' = M(\varepsilon')^{-1},$$

which gives

$$|\frac{f(x)}{f(x')}| \leq e^{M'}. \tag{17}$$

Now 12 follows easily.

Step II. *Hierarchy of attractors.*

This part is similar to the analytic case. The function $F = \frac{f}{\varphi(r)}$ is increasing on the trajectory in the complement of W^ε. This implies that the trajectory $x(s)$ has to end up in one of U_φ.

The next step is to consider the behaviour of $F = \frac{f}{\varphi}$ on the trajectory.

Step III. *Asymptotic critical values.*

The definition and the properties of asymptotic critical values are the same and our previous argument works since the Kuo-Verdier (w) condition is generic also in the o-minimal set-up, see e.g. [7] or [1].

Then the analog of proposition 1.2 holds and for a $x(t)$ entirely included in U_φ

$$\lim_{t\to\infty} F(x(t)) = a_0,$$

where $a_0 < 0$ is an asymptotic critical value of $F = \frac{f}{\varphi}$.

On may also simplify the picture by applying section 2.1 to $f|_{U_\varphi}$. Then we may actually suppose that $\varphi \equiv r$ and that the trajectory is entirely included in

$$U = \{x;\, -Cr < f(x) < -cr\}.$$

By restricting to a smaller neighbourhood of the origin we may choose C, c as close to $-a_0$ as we wish.

As a corollary we obtain (a) of theorem 2.1. The argument is simple, see also [6] corollary 6.5. By checking on each definable curve as in 13 it is easy to see that on U

$$r|\nabla f| \geq |f| - o(r). \tag{18}$$

Then

$$\frac{df}{ds} = |\nabla f| \geq |f|r^{-1} - o(1) \geq c - o(1). \tag{19}$$

Integrate both sides and bound $|f|$ by Cr. We get

$$\sigma(s) \leq (C/c)r + o(r). \tag{20}$$

Since while taking $r \to 0$ we may choose $C/c \to 1$, (a) of theorem 2.1 follows.

Step IV.
The next question is: in what area is $F - a_0$ a control function (in the above notation, in particular $F = \frac{f}{r}$)? First of all the defintion of a control function given in section 1 is not well adapted to the o-minimal structures. Our new defintion is the following. A *control function* is a function g defined on the trajectory $x(t)$ such that:
- g is bounded ;
- g is increasing fast, that is there exists a continuous definable $\Psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that

$$\frac{d(\Psi \circ g)}{d\widetilde{s}} \geq const > 0. \tag{21}$$

As shown in [5], $F - a_0$ is a good control function on the set where the spherical part of the gradient $\nabla' f$ is not too small. Let us be more precise. First note that

$$\frac{dF}{d\widetilde{s}} = \frac{1}{|\nabla' f|}\Big(|\nabla' f|^2 + |\partial_r f|^2 (1 - \frac{f}{r\partial_r f})\Big) = r|\nabla' F| + r\partial_r F \frac{\partial_r f}{|\nabla' f|}. \tag{22}$$

The sign of expression $(1 - \frac{f}{r\partial_r f})$ can be arbitrary so to bound the size of it is crucial for the argument both in [5] and [6]. It is easy to see that $(1 - \frac{f}{r\partial_r f})$ goes to zero on W^ε (and also that $F - a_0$ is a control function away of W^ε). Let us choose a continuous definable function $\omega(r)$, $\omega(0) = 0$, such that

$$|1 - \frac{f}{r\partial_r f}| \leq \frac{1}{2}\omega^2(r) \quad \text{on } W^\varepsilon. \tag{23}$$

Suppose that the o-minimal structure is polynomially bounded. Now the argument in [5] goes as follows. On the set

$$U_1 = \{x \in U;\, \omega|\partial_r f| \leq |\nabla' f|\}$$

$F - a_0$ is a good control function on U_1 and $-r$ is a good control function on its complement $U \setminus U_1$. Both cases can be "glued" to define a single control function that is of the form $g = \Psi(F - a_0) - \widetilde{\alpha}(r)$.

2.3 The general case

For the last step we need not only that the function $\omega(r)$ defined in 23 goes to 0 as $r \to 0$ but also that $\frac{\omega(r)}{r}$ is locally integrable at 0. This is of course the case in any polynomially bounded o-minimal structure but not in general. For instance $\alpha(r) = (-\ln r)^{-1}$ defined for $r > 0$ satisfies $\alpha(r) \to 0$ as $r \to 0$ but $\frac{\alpha}{r}$ is not locally integrable at 0. Indeed $\alpha(r)$ satisfies

$$r\alpha'(r) = \alpha^2(r). \tag{24}$$

Hence $\frac{\alpha}{r} = \frac{\alpha'}{\alpha}$ is not integrable.

It is interesting to note that the expression $(1 - \frac{f}{r\partial_r f})/r$ can be always bounded by a locally integrable function of r. In other words we may always assume that the function ω^2/r integrable at 0. But, in general, it does not imply that ω/r is integrable at 0. Indeed, in the above example, $\alpha^2/r = \alpha'(r)$ is of course integrable at 0.

2.4 Gradient conjecture on the plane

We show that the gradient conjecture holds in the o-minimal case for $n = 2$. Write in polar coordinates (r, θ)

$$f(r, \theta) = f(r\cos\theta, r\sin\theta).$$

Denote $\partial_\theta f = \partial f/\partial\theta$. By a standard argument, see e.g. [6] Proposition 2.1, it suffices to show that the trajectory $x(t)$ does not spiral that is that θ is bounded on the trajectory.

This is fairly obvious in the analytic case since we may write

$$f = f_0(r) + r^m F_0(\theta) + \cdots, \tag{25}$$

where $F_0 \not\equiv const$. For some $\varepsilon > 0$ both sectors

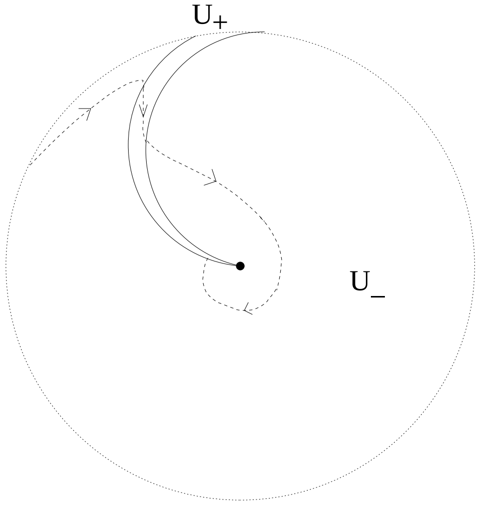
$$A_+(\varepsilon) = \{x = (r, \theta); F_0'(\theta) > \varepsilon\}, \quad A_-(\varepsilon) = \{x = (r, \theta); F_0'(\theta) < -\varepsilon\}$$

are not empty. It follows that the trajectory crosses $A_+(\varepsilon)$ anti-clockwise and $A_-(\varepsilon)$ clockwise, and therefore the trajectory cannot spiral.

In the general, even subanalytic, case the expansion 25 does not hold. Consider instead

$$U_+ = \{\partial_\theta f \geq 0\}, \quad U_- = \{\partial_\theta f \leq 0\}.$$

Both of them are non-empty and it is clear that the trajectory cannot spiral if each $U_\pm$ contains a non-empty sector of the form $\{\theta_1 < \theta < \theta_2\}$. But even for f subanalytic it may happen that one of $U_\pm$ does not contain a sector, see the picture below.



In what follows we shall assume that $U = \{x; -Cr < f(x) < -cr\}$ contains a punctured disc centered at the origin, U_- contains a non-empty sector but U_+ does not. If we show that, however, U_+ contains a definable curve which $x(t)$ crosses anti-clockwise then we are done.

Denote

$$\psi(r) = \min_{x \in S(r) \cap U_-} f(x) = \min_{x \in S(r) \cap U_+} f(x)$$
$$\varphi(r) = \max_{x \in S(r) \cap U_-} f(x) = \max_{x \in S(r) \cap U_+} f(x).$$

Case 1. $\frac{\varphi(r) - \psi(r)}{r^2}$ is integrable.

Then $|\partial_\theta f| = r|\nabla' f|$ is too small for $x(t)$ to cross U_-. More precisely since

$$|f(r,\theta_1) - f(r,\theta_2)| = -\int_{\theta_1}^{\theta_2} \partial_\theta f \, d\theta,$$

there is a non-empty sector $U'_- \subset U_-$ on which

$$|\partial_\theta f| \leq const \cdot (\varphi(r) - \psi(r)).$$

Since

$$\Big|\frac{d\theta}{ds}\Big| = \frac{|\nabla' f|}{r|\nabla f|} \leq \frac{|\partial_\theta f|}{r^2} \leq const \cdot \frac{\varphi(r) - \psi(r)}{r^2}$$

and the right-hand side is integrable. This means that the trajectory cannot cross U'_- if it remains in a sufficiently small neighborhood of the origin.

Case 2. $\frac{\varphi(r)-\psi(r)}{r^2}$ is not integrable.

Then one may show that for

$$\lambda(r) = \sup_{x \in S(r) \cap U_+} \frac{|\nabla' f|}{|\partial_r f|}, \tag{26}$$

$\frac{\lambda(r)}{r}$ is not integrable.

On the other hand let $\gamma(r) = r\theta(r)$, $|\theta(r)| \equiv 1$, be a definable curve in U_+. An easy geometric argument shows that $x(t)$ crosses the image of γ anti-clockwise if and only if on $\gamma(r)$

$$\partial_\theta f > r^2 \theta'(r) \partial_r f(r). \tag{27}$$

Let $\gamma(r)$ be definable curve on which $\lambda(r) = \frac{|\nabla' f|}{|\partial_r f|}$. Then $\frac{\lambda(r)}{r} \gg \theta'(r)$ and hence we have

$$\partial_\theta f = \lambda_\gamma(r) r |\partial_r f| \gg r^2 \theta'(r) \partial_r f.$$

Thus Γ is crossed anti-clockwise and the trajectory cannot spiral.

Moreover, in two dimensional case, by [12], if f is defined in an o-minimal structure $\widetilde{\mathbb{R}}$ then the trajectory $x(t)$ is definable in the pfaffian closure of $\widetilde{\mathbb{R}}$.

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Algebraic measure, foliations and o-minimal structures

Jean-Marie Lion[7]

Abstract

Let $\mathcal{F}_\lambda$ be a family of codimension p foliations defined on a family M_λ of manifolds and let X_λ be a family of compact subsets of M_λ. Suppose that $\mathcal{F}_\lambda$, M_λ and X_λ are definable in an o-minimal structure and that all leaves of $\mathcal{F}_\lambda$ are closed. Given a definable family Ω_λ of differential p-forms satisfaying $i_Z\Omega_\lambda = 0$ for any vector field Z tangent to $\mathcal{F}_\lambda$, we prove that there exists a constant $A > 0$ such that the integral of $|\Omega_\lambda|$ on any transversal of $\mathcal{F}_\lambda$ intersecting each leaf in at most one point is bounded by A. We apply this result to prove that p-volumes of transverse sections of $\mathcal{F}_\lambda$ are uniformly bounded.

Remarques Le travail dont il est question dans cet exposé va paraître dans *Publicacions Matemàtiques* [2]. Nos résultats sont vrais dans un cadre o-minimal très général (voir [4] ou [6] pour une introduction à la o-minimalité). Dans un soucis subjectif de simplicité je les présenterai ici dans un cadre semi-algébrique. Dans un soucis plus objectif de simplicité je les présenterai *en codimension un*. Pour plus de détails je renvoie à l'article à paraître [2].

Plan

Dans la partie 1 nous présentons les résultats de Łojasiewicz, D'Acunto et Kurdyka qui sont à la source de notre réflexion. Nous donnons quelques arguments de leur démonstration. Dans la partie 2 est exposé notre résultat et le lien avec celui de D'Acunto et Kurdyka. Nous en profitons pour expliquer ce qu'est un feuilletage défini par une forme différentielle. Une esquisse de démonstration est tracée dans la partie 3. La partie 4 est une micro-introduction aux formes de contact. La dernière partie est consacrée à la

[7]Collaboration avec Frédéric Chazal (Université de Bourgogne - Dijon, France).

description du feuilletage de Reeb, premier exemple dynamiquement riche de feuilletage. Une bibliographie permet de trouver des textes de référence pour les sujets exposés.

1 Les résultats de Łojasiewicz, D'Acunto et Kurdyka

Notre travail a pour origine des résultats de D'Acunto et Kurdyka qui ont eux pour source un théorème que Łojasiewicz a prouvé afin de répondre à une question de René Thom. Je rappellerai ces résultats maintenant.
Soit $B = \{x \in \mathbb{R}^n, ||x|| < 1\}$ la boule unité de $\mathbb{R}^n$, $d \in \mathbb{N}$ un entier et $P \in \mathbb{R}[X_1, ..., X_n]$ un polynôme de degré au plus d. On note $\nabla P = \sum_{i=1}^{n} \frac{\partial P}{\partial x_i} \frac{\partial}{\partial x_i}$ le champ gradient associé au potentiel P. Enfin on désigne par γ un morceau de trajectoire du champ ∇P contenu dans B.
Łojasiewicz montre

Théorème 1.1 (Łojasiewicz [17]) *Il existe $K_P > 0$ qui majore la longueur de tout morceau de trajectoire γ de ∇P contenu dans la boule B.*

Il obtient ce résultat comme corollaire d'une des célèbres inégalités qui portent son nom : *si $\{\nabla P = 0\} \cap \overline{B} \subset \{P = 0\}$ alors il existe $c > 0$ et $\theta \in]0, 1[$ tels que pour tout $x \in B$ on a $c|P|^\theta \leq ||\nabla P(x)||$.*

Cette inégalité permet de contrôler la longueur des morceaux de trajectoire à l'aide d'une fonction de contrôle bornée définie sur la boule unité.

Récemment D'acunto et Kurdyka ont généralisé ce résultat de la façon suivante, *effective et uniforme* par rapport au degré.

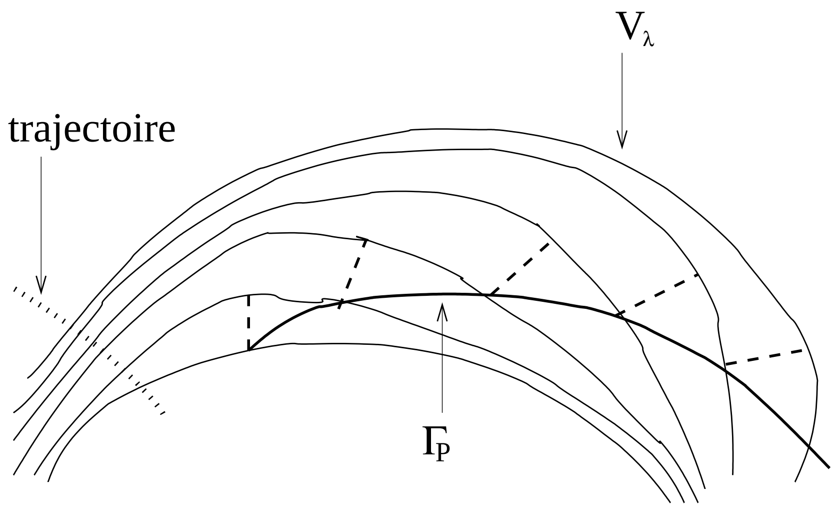
Théorème 1.2 (D'acunto et Kurdyka [5]) *Il existe une courbe semi-algébrique $\Gamma_P \subset B$ qui dépend semi-algébriquement de P et dont la longueur majore la longueur de tout morceau de trajectoire γ de ∇P contenu dans la boule B.*

Or il résulte du théorème de Bezout combiné à la formule de Cauchy-Crofton [19] qu'il existe une constante $K_{d,n}$ qui majore la longueur des courbes Γ_P

lorsque P décrit l'ensemble des polynômes réels de degré au plus d. Ainsi D'Acunto et Kurdyka obtiennent

Corollaire 1.3 (D'Acunto et Kurdyka [5]) *La constante $K_{d,n}$ précédente majore la longueur de tout morceau de trajectoire γ de ∇P contenu dans la boule B quelque soit le polynôme P de degré au plus d.*

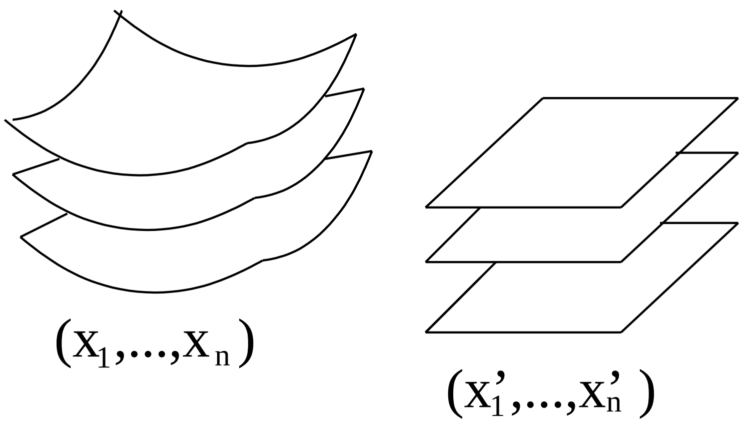
Décrivons brièvement la preuve de D'Acunto et Kurdyka. L'idée principale est de rechercher là où les niveaux de P voisins de P sont les plus éloignés. Soit $V_\lambda = \{P = \lambda\} \cap B$ un niveau de P et soit $V_{\lambda+d\lambda}$ un niveau voisin. Ils remarquent qu'en suivant une idée de "fiber cutting lemma" à la Gabrielov [10], on peut choisir par un procédé semi-algébrique un point M_λ sur V_λ tel que si M est un autre point de V_λ alors la longueur du morceau de trajectoire de gradient qui va de V_λ à $V_{\lambda+d\lambda}$ en passant par M_λ est supérieure ou égale à la longueur de celui qui passe par M. Le point M_λ est un point ou $||\nabla P||_{|V_\lambda}$ est minimal. La réunion des points M_λ est la courbe semi-algébrique Γ_P recherchée. Dans les "bons cas" (en particulier sans effet de bord) on a $\Gamma_P = \{dP \wedge d||\nabla P||^2 = 0\} \cap B$.



La courbe Γ_P majore la longueur d'un morceau de trajectoire γ contenu dans B : en effet le morceau infinitésimal de la trajectoire γ qui va de V_λ à $V_{\lambda+d\lambda}$ en passant par M est plus court que le morceau de trajectoire de gradient qui va de V_λ à $V_{\lambda+d\lambda}$ en passant par M_λ et ce dernier est plus court que le morceau de la courbe Γ_P qui va de V_λ à $V_{\lambda+d\lambda}$ en passant par M_λ (merci Pythagore).

2 Version feuilletée

Dans notre travail nous obtenons une version feuilletée des résultats qui précèdent. On considère par exemple une 1-forme différentielle $\omega = P_1 dx_1 + ... + P_n dx_n$ à coefficients des polynômes P_j de degrés au plus d. On note Ω l'ouvert semi-algébrique $\Omega = B \setminus \{\omega = 0\}$ c'est à dire $\Omega = B \setminus \{P_1 = ... = P_n = 0\}$. En chaque point x de Ω la forme ω définit un hyperplan H_x qui est le noyau de la forme linéaire $\omega(x)$. On sait grâce à Frobénius que pour qu'il existe des sous-variétés de codimension un de Ω tangentes au champ d'hyperplans associé à ω il faut et il suffit que $\omega \wedge d\omega \equiv 0$. Dans la partie 4 on rappellera ce qui se passe localement lorsque $\omega \wedge d\omega \not\equiv 0$. Ici on suppose que $\omega \wedge d\omega \equiv 0$. Alors il existe une partition (infinie continue)$\mathcal{F}_\omega$ de Ω en hypersurfaces immergées injectivement dans Ω et tangentes à $\ker \omega(x)$ en tout point $x \in \Omega$. C'est facile à voir quand on a observé que la condition de Frobénius garantit qu'au voisinage de tout point de Ω il existe un système de coordonnées locales $(x'_1, ..., x'_n)$ (pas unique et qui dépend du point) et il existe une fonction f non nulle dans ce voisinage tels que $\omega = f dx'_n$.



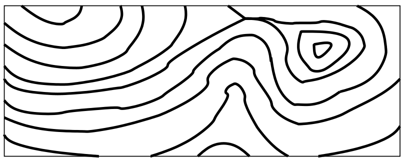
La partition $\mathcal{F}_\omega$ s'appelle *feuilletage* . Les hypersurfaces injectivement immergées s'appellent les *feuilles* du feuilletage. Ce ne sont pas nécessairement des sous-variétés fermées de Ω (voir l'exemple de Reeb [8] dans la partie 5). Cependant pour notre théorème nous le supposerons. Pour en savoir plus sur les feuilletages on peut regarder par exemple [3] ou [12]. Considérons encore une seconde 1-forme différentielle $\theta = g_1 dx_1 + ... + g_n dx_n$ à coefficients des fonctions semi-algébriques. On suppose de plus que si $x \in \Omega$ alors $\ker \omega(x) \subset \ker \theta(x)$. Dans ce cadre notre résultat est le suivant :

Théorème 2.1 (F. Chazal et J.-M. Lion [2]) *Il existe une courbe semi-algébrique $\Gamma_{\omega,\theta} \subset B$ qui dépend semi-algébriquement de ω et de θ telle que si C est une courbe C^1 par morceaux contenue dans Ω alors soit $\int_C |\theta| \leq \int_{\Gamma_{\omega,\theta}} |\theta|$ soit il existe une feuille coupée deux fois par C.*

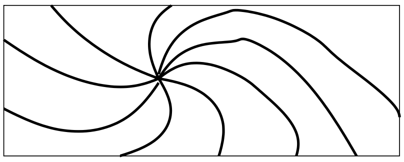
On retrouve le théorème de D'Acunto et Kurdyka en posant $\omega = dP$ et $\theta = \dfrac{dP}{||\nabla P||}$ et en remarquant qu'un morceau de trajectoire de gradient γ coupe au plus une fois chaque niveau de P et a pour longueur l'intégrale $\int_\gamma |\theta|$.

Dans le cas général, les éléments nouveaux dans notre théorème sont les suivants. D'une part a priori il n'existe pas (ou du moins on ne sait pas prouver l'existence) une fonction semi-algébrique (ou même seulement définissable dans une structure o-minimale) non triviale constante en restriction aux feuilles de $\mathcal{F}_\omega$: le feuilletage $\mathcal{F}_\omega$ n'admet peut être pas d'intégrale première définissable dans une structure o-minimale. C'est un problème ouvert. D'autre part, la courbe C n'est pas la trajectoire d'un champs de vecteurs et il il n'y a donc plus de distance. Par conséquent on ne peut pas rechercher le lieu des points où les feuilles voisines sont les plus proches. Dans la partie suivante nous allons expliquer comment les notions de dérivée de Lie et d'holonomie permettent de s'affranchir de ces difficultés.
Signalons que Khovanskii [14] et Moussu-Roche [18] parlent de feuilletage de Rolle s'il n'existe pas de courbe fermée $C \subset \Omega$ transverse aux feuilles du feuilletage.

feuilletage de Rolle



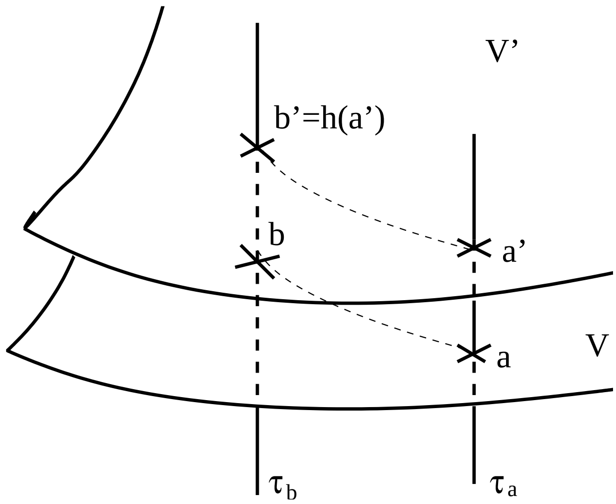
feuilletage non Rolle



Dans ce cas les feuilles sont toutes fermées et toute courbe C transverse au feuilletage coupe chaque feuille au plus une fois. Un résultat d'Haefliger [13] assure que si Ω est simplement connexe et si la restriction de ω à cet ouvert est analytique, le feuilletage $\mathcal{F}_\omega$ est de Rolle. Un résultat de Wilkie [21] qui repose sur [14] assure alors que les feuilles du feuilletage $\mathcal{F}_\omega$ sont définissables dans une structure o-minimale (voir aussi [15], [16], [20]). On verra dans la partie 5 l'exemple de Reeb d'un feuilletage de $\mathbb{R}^3$, qui n'est pas de Rolle mais qui est associé à une forme ω sans singularité et définissable dans la structure $\mathbb{R}_{\mathrm{an,\,exp}}$ [7].

3 Holonomie et dérivée de Lie

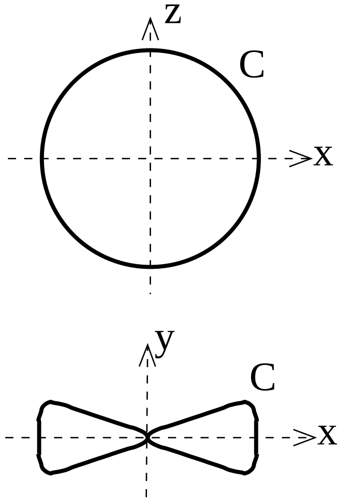
Soit V une feuille de $\mathcal{F}_\omega$, et a, b de points de celle-ci et τ_a, τ_b deux transversales à V en ces points. On suppose que τ_a, et τ_b coupent au plus une fois chaque feuille de $\mathcal{F}_\omega$. Soit V' une seconde feuille de $\mathcal{F}_\omega$ assez proche de V. On note a' le point d'intersection de τ_a avec V' et b' celui de τ_b avec V' (supposé existés). L'application qui à a' associe $b' = h(a')$ s'appelle l'holonomie du feuilletage entre τ_a et τ_b.



Si nous n'avions pas fait l'hypothèse que τ_a, et τ_b coupent au plus une fois chaque feuille de $\mathcal{F}_\omega$ le germe d'application ne h serait pas unique (voir [3] ou [12]). Supposons que h soit un difféomorphisme entre τ_a et τ_b (c'est possible quitte à racourcir les deux transversales). On a alors l'égalité $\int_{b'\in\tau_b}\eta(b')db' = \int_{a'\in\tau_a}\eta(h(a'))dh'$ qui est juste une formule de changement de variables. Supposons maintenant que τ_a et τ_b sont petites (on linéarise). Ca revient à considérer seulement des feuilles V' voisines de V. Supposons auusi que $|\int_{a'\in\tau_a}\eta(a')da'|$ majore $|\int_{b'\in\tau_b}\eta(b')db'|$ quelque soit le point b de V voisin de a. Le calcul différentiel de Lie combiné aux multiplicateurs de Lagrange nous dit qu'alors la dérivée de Lie $L_X\theta$ de la forme θ par rapport à tout champ de vecteurs X tangent à ω est nul en a [11]. De plus en raison de la colinéarité de ω et de θ on vérifie qu'il suffit de vérifier cette condition pour les champs $X_1(x), ..., X_n(x)$ obtenus par projection orthogonale de $\dfrac{\partial}{\partial x_1}$,..., $\dfrac{\partial}{\partial x_1}$ sur le noyau $\ker\omega(x)$. On déduit de la semi-algébricité de ces champs la semi-algébricité de la condition à vérifier (voir [2]). Tout ceci est la formalisation d'un exercice instructif qui consiste à résoudre le problème lorsqu'on est dans le plan, que la forme ω est la forme dy et que la forme η est de la forme $\eta = f(x,y)dy$. Dans ce cas $X_1 = \dfrac{\partial}{\partial x}$ et $L_{X_1}\theta = \dfrac{\partial f}{\partial x}(x,y)dy$. Finalement l'ensemble des points $a \in B$ possèdant la proriété d'extrémalité précédente lorsque V varie forme un semi-algébrique. Dans les bons cas c'est une courbe semi-algébrique $\Gamma_{\omega,\theta}$ (pour presque toute feuille les extrema locaux considérés forment un ensemble au plus dénombrable). Expliquons pourquoi c'est le bon candidat. Pour simplifier on suppose que $\Gamma_{\omega,\theta}$ coupe toute les feuilles et que sur chacune d'elles il existe un extremum global (contenu donc dans $\Gamma_{\omega,\theta}$). Considérons maintenant une courbe C qui coupe au plus une fois chaque feuille de $\mathcal{F}_\omega$. Par commodité on supposera que C est transverse au feuilletage. Soit b un point quelconque de C et $a(b)$ un extremum global sur la feuille V qui passe par b. On a $a(b) \in \Gamma_{\omega,\theta}$. On peut supposer que $\Gamma_{\omega,\theta}$ est transverse à V en $a(b)$. Maintenant le rôle de τ_a est tenu par un morceau de $\Gamma_{\omega,\theta}$ qui passe par $a(b)$ et celui de τ_b par un morceau de C qui passe par b. En faisant varier b le long de C, on déduit de la construction de $\Gamma_{\omega,\theta}$ et de celle de $a(b)$ que l'inégalité voulue est vraie c'est à dire $\int_C |\theta| \leq \int_{\Gamma_{\omega,\theta}} |\theta|$.

4 Forme de contact

On explique succintement ce qui se passe quand le champ de plans associés à la forme ω ne définit plus un feuilletage. On sait depuis Darboux que le modèle local d'une forme ω qui ne vérifie pas la condition de Frobénius est, si $n = 3$, la forme $\omega = dy - zdx$. On vérifie que la courbe $C = \{(\cos(\theta), (1/4)\sin(2\theta), \sin(\theta)),\ \theta \in \mathbb{R}\}$ est transverse à $\ker \omega$ (on a alors $\omega(C(\theta)).C'(\theta) = (1/2)$), reste dans la boule de rayon 2 et on a $\int_C |\omega| = \int_{\mathbb{R}} (1/2) d\theta = +\infty$.



De plus étant donner deux points A et B de $\mathbb{R}^3$ on peut construire une courbe γ tangente au noyau de ω en tout point et d'extrémités A et B. Ainsi s'il existait un feuilletage associé à ω il ne contiendrait qu'une feuille! Pour tout comprendre il suffit d'observer que si $f : I \to \mathbb{R}$ est une fonction C^1 alors la courbe $(x, f(x), f'(x))$ est tangente au noyau de ω en tout point. Pour plus de détails on peut consulter [1], [11] ou [9].

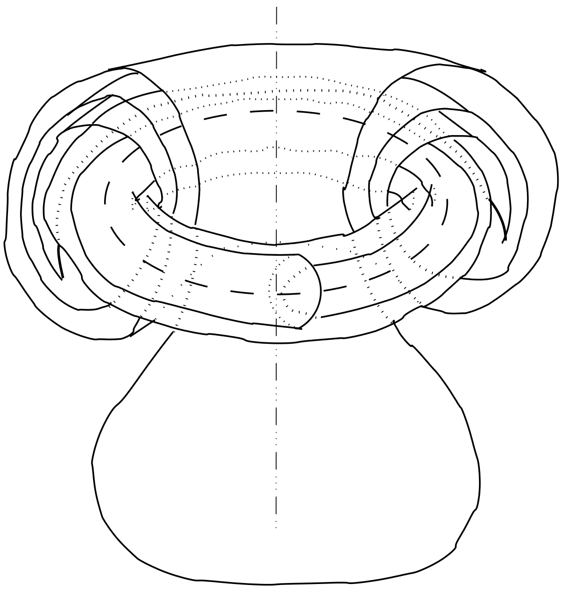
5 L'exemple de Reeb

L'exemple qu'on présente pour finir est dû à Reeb [8] (voir [12]). C'est un feuilletage de $\mathbb{R}^3$ qui n'est pas de Rolle et qui est associé à une une forme ω de classe C^∞ et dont les coefficients sont définissables dans la structure o-minimale $\mathbb{R}_{\mathrm{an,\,exp}}$. Soit $S^3 = \{|z_1|^2 + |z_2|^2 = 1\}$ la sphère unité de $\mathbb{C}^2$. On pose $z_1 = \rho_1 \exp(i\theta_1), z_2 = \rho_2 \exp(i\theta_2)$. On a donc $S^3 = \{\rho_1^2 + \rho_2^2 = 1\}$ et donc $d\rho_1^2 + d\rho_2^2 = 0$ en restriction à S^3 et $S^3 \subset \{\rho_1^2 \leq 1/2 \text{ ou } \rho_2^2 \leq 1/2\}$. On considère la fonction de recollement suivante $\mu(t) = \mathbf{1}_{]0,+\infty=(t)} \exp(-1/t)$. Elle est définissable dans $\mathbb{R}_{\mathrm{an,\,exp}}$. Soit alors $\widetilde{\omega}$ la 1-forme différentielle définie sur S^3 par

$$\widetilde{\omega} = \mu(1/2 - \rho_2^2)d\theta_1 + \mu(1/2 - \rho_1^2)d\theta_2 + d\rho_2^2.$$

Elle est aussi définissable dans $\mathbb{R}_{\mathrm{an,\,exp}}$. De plus

- si $\rho_2^2 \leq 1/2$ alors $\widetilde{\omega} = \mu(1/2 - \rho_2^2)d\theta_1 + d\rho_2^2$
et
- si $\rho_2^2 \leq 1/2$ alors $\widetilde{\omega} = \mu(1/2 - \rho_1^2)d\theta_2 - d\rho_1^2$.
On a donc bien $\widetilde{\omega} \wedge d\widetilde{\omega} \equiv 0$ et $\widetilde{\omega}$ définit un feuilletage $\mathcal{F}_\omega$ sur S^3. On remarque que le lacet $\widetilde{C} = \{\rho_2^2 = 0\}$ est transverse à ce feuilletage qui n'est donc pas un feuilletage de Rolle de S^3. Par projection stéréographique de pôle nord on obtient un feuilletage $\mathcal{F}_\omega$ de $\mathbb{R}^3$ défini à partir d'une forme ω de classe C^∞ et définissable dans la strucure o-minimale $\mathbb{R}_{\mathrm{an,\,exp}}$. Le feuilletage n'est pas de Rolle car le cercle C image par la projection stéréographique du lacet $\widetilde{C}$ est transverse au feuilletage $\mathcal{F}_\omega$. Décrivons un peu la géométrie de ce feuilletage. Il est invariant par rotation autour de l'axe $\{x = y = 0\}$. Les feuilles du feuilletage $\mathcal{F}_\omega$ sont de plusieurs natures. L'une d'elles est un tore T^2. Il est dans l'adhérence de toutes les autres feuilles. Les feuilles qui sont dans la composante bornée de $\mathbb{R}^3 \setminus T^2$ sont toutes homéomorphes à des plans et elles s'accumulent sur le tore en tournant autour de l'axe vertical. Les feuilles qui sont dans la composante non-bornée de $\mathbb{R}^3 \setminus T^2$ sont toutes homéomorphes à des plans sauf l'une d'elles qui visite l'infini et qui est homéomorphe à un cylindre. Elles s'accumulent sur le tore en suivant les méridiens. Le dessin suivant tente d'expliquer tout cela à homéomorphisme près. On représente de coupe, le tore, une feuille intérieure, une feuille extérieure, l'axe de rotation et le cercle C transverse au feuilletage. Avec un petit effort on devine comment l'espace se remplit.



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Limit sets in o-minimal structures

Lou van den Dries

Abstract

We show that taking Hausdorff limits of a definable family in an o-minimal expansion of the real field is a very tame operation: it preserves definability, cannot raise dimension, creates fewer limits than the family has members, and respects Lebesgue measure. We prove similar results for Gromov-Hausdorff limits and Tychonov limits of definable families of sets, and for pointwise limits of definable families of functions. The first part of these notes is purely geometric, and the second part uses some model theory. An appendix gives Gabrielov's geometric proof that a definable family has few Hausdorff limits.

1 Introduction

This article is a survey of what I have learned in the last twenty years about limit sets in the tame setting of o-minimal structures. Many results have been announced or published in some form, by myself and others, but it may be convenient to have it all in one place and treated in some detail. Theorem 4.2 on the Hausdorff measure of Hausdorff limits is perhaps the main new result.

Gromov introduced the operation of taking the Gromov-Hausdorff limit of a sequence of metric spaces as a tool in geometry and group theory; see [13]. In the general setting of metric spaces this operation has a highly infinitary character, and the limit space may differ wildly from the approximating spaces. We shall prove, however, that taking such limits is a very tame operation when restricted to *definable* families of (compact) subspaces of the euclidean space $\mathbb{R}^n$. Here "definable" is meant with respect to some o-minimal structure on the real field; "semialgebraic" is one instance of "definable".

We first consider the simpler operation of taking *Hausdorff limits* in a tame setting. (Hausdorff limits and their approximating spaces require a common ambient space that is fixed in advance, as opposed to Gromov-Hausdorff limits.) A key result is that the Hausdorff limits of a definable collection make up again a definable collection, see Theorem 3.1. (Hence each individual Hausdorff limit of the collection is a definable set.) Another striking fact is that *Hausdorff measure* behaves much better in this definable (tame) setting when taking Hausdorff limits than in the usual non-definable setting, see Theorem 4.2.

For the reader's convenience, Section 2 contains the basics on Hausdorff distance and Hausdorff limits; after these preparations we discuss properties of these Hausdorff limits in a tame setting in Section 3. Section 4 deals with the behaviour of Hausdorff measure in definable collections. In Section 5 we first summarize the definitions and basic facts concerning Gromov-Hausdorff limits, and then prove some results on these limits under tame conditions in Section 6. In Sections 7 and 8 we make a fresh start, and consider another kind of limits (*Tychonov limits*) from a model-theoretic point of view. In Section 9 and 10 we use (elementary) model theory to prove the geometric results stated without proof in Sections 3 and 4, and Section 11 treats *pointwise limits* of definable families of functions.

Some historical comments. These notes have been so long in coming that it demands an explanation. In the early eighties I found tame behaviour of Tychonov limits of semialgebraic families of functions (see [8], pp. 70–71), as part of a program to establish the o-minimality of the real field with Pfaffian functions. This program never worked out, and so these results on tame limits of semialgebraic families (and some Pfaffian families) went unpublished although I mentioned them in talks. (The o-minimality of the real field with Pfaffian functions was proved 15 years later by Wilkie [21]; his proof did involve taking limit sets, but in a different way.) In the semialgebraic case my proof used the work by Cherlin & Dickman [7] on real closed fields with a convex valuation. Next, Cherlin drew my attention to a connection with definability of types, stability, and nfcp; discussions with him and Martin Ziegler in 1984–85 on this issue led to Section 7 below. Answering a question in [8], Marker and Steinhorn [18] proved definability of types over o-minimal expansions of the real line; a simpler proof is in Pillay [19], and Section 8 below contains a third proof of a stronger result for expansions of the real field.

In the mean time Bröcker [4, 5] had used the Cherlin-Dickmann theorem to prove that Hausdorff limits of semialgebraic families are semialgebraic. Stimulated by this development and having further applications to limit sets in mind, Lewenberg and I investigated to what extent the Cherlin-Dickmann theorem goes through for o-minimal expansions of real closed fields. This led to [10, 11], some of which is used in Sections 8–11 below. A few years ago I asked myself some further questions, and this led to results like Theorem 3.1, part (3), and Theorems 4.2 and 6.2. See also [17] for another treatment of Theorem 3.1.

My thanks go to G. Cherlin, M. Ziegler, A. Lewenberg, C. Miller, M. Mazur, A. Gabrielov, M. Gabrilovich, and S. Solecki for helpful discussions on various aspects of these notes in the course of the years.

Notation. Throughout m, n range over $\mathbb{N} = \{0, 1, 2, \ldots\}$. "Space" means "non-empty metric space"; accordingly, a *subspace* of a space Z is a nonempty subset of Z with the induced metric. Unless specified otherwise each $\mathbb{R}^n$ is equipped with the standard euclidean metric, and for $X \subseteq \mathbb{R}^n$, we let $\overline{X}$ be the closure of X and $\partial X = \overline{X} \setminus X$ the frontier of X (in the ambient euclidean space $\mathbb{R}^n$). Given a space Z with metric d we put

$$\mathrm{diam}(Z) = \sup\{d(x, y) : x, y \in Z\},$$

and let $\mathrm{Isom}(Z)$ be the group of isometries of Z.
The cardinality of a set X is denoted by $|X|$, and its power set by $\mathcal{P}(X)$. Given $S \subseteq A \times B$ and $a \in A$ we put $S(a) = \{b \in B : (a, b) \in B\}$, and call it a *section* of S. We view S as defining the *family* of sets $(S(a))_{a \in A}$, as well as the *collection* of sets $\{S(a) : a \in A\}$.

2 Hausdorff distance and Hausdorff limits

We fix an ambient space Z, with metric d. For $x \in Z$ and subspace Y (non-empty by convention) of Z, put $d(x, Y) = \inf\{d(x, y) : y \in Y\}$.
The *Hausdorff distance* $d_H(X, Y)$ between compact subspaces X and Y of Z is given by

$$d_H(X, Y) = \min\{r \geq 0 : d(x, Y), d(y, X) \leq r \text{ for all } x \in X \text{ and all } y \in Y\}.$$

Then d_H is a metric on the set $\mathcal{K}(Z)$ of compact subspaces of Z, and we shall view $\mathcal{K}(Z)$ as a space with this metric. The topology on $\mathcal{K}(Z)$, called the *Vietoris topology*, depends only on the topology of Z, not on the metric d inducing this topology. If Z is compact, so is $\mathcal{K}(Z)$ with respect to d_H. The subspace $\mathcal{K}_{\text{fin}}(Z) = \{X \in \mathcal{K}(Z) : X \text{ is finite }\}$ of $\mathcal{K}(Z)$ is dense in $\mathcal{K}(Z)$. (See [15] for these facts.)
For example, with $Z = \mathbb{R}$, let $X_k = \{0, \frac{1}{k}, \frac{2}{k}, \dots, 1\}$ for $k = 1, 2, 3, \dots$. Then $d_H(X_k, [0,1]) = \frac{1}{2k}$, hence $\lim_{k\to\infty} X_k = [0,1]$. Thus we have here a semialgebraic Hausdorff limit whose dimension is larger than the dimensions of the approximating (semialgebraic) spaces. In the next section we show that this kind of pathology is impossible when the approximating spaces come from a single definable family.
Let $\mathcal{C} \subseteq \mathcal{K}(Z)$, that is, $\mathcal{C}$ is a collection of compact subspaces of Z. The "points" in the *closure* $\mathrm{cl}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{K}(Z)$ are the Hausdorff limits $\lim_{k\to\infty} X_k$ of sequences (X_k) in $\mathcal{C}$ that converge in $\mathcal{K}(Z)$.

Example. Let $\mathcal{C}$ be the collection of ellipses in the euclidean plane $\mathbb{R}^2$, so $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^2)$. Then

$$\mathrm{cl}(\mathcal{C}) = \mathcal{C} \cup \{\,\text{line segments}\,[p,q] \subseteq \mathbb{R}^2\}.$$

(We allow $p = q$, in other words, we include degenerate line segments.)

3 Hausdorff limits of definable collections

In this section, $Z = \mathbb{R}^n$ with its euclidean metric. We fix an o-minimal structure on the ordered field of reals $(\mathbb{R}, <, 0, 1, +, \cdot)$. *Thus all semialgebraic sets belong to this structure; a set $X \subseteq \mathbb{R}^m$ is said to be* definable *if it belongs to the structure; see [9].*

A collection $\mathcal{C}$ of subsets of Z is said to be *definable* if

$$\mathcal{C} = \{S(a) : a \in A\}$$

for some definable $A \subseteq \mathbb{R}^m$ and definable $S \subseteq A \times Z$. By *definable choice* [9], p. 94, we can then choose a definable $B \subseteq A$ such that $b \mapsto S(b) : B \to \mathcal{C}$ is a bijection, and we put $\dim \mathcal{C} = \dim B$ for such B. (This definition of $\dim \mathcal{C}$ is possible because $\dim B$ depends only on $\mathcal{C}$ and not on the choices of

S and B; this independence follows from the invariance of dimension under definable bijections.)

Example. Let $n = 2$, and $\mathcal{C} = \{$ ellipses in $\mathbb{R}^2\}$. Then $\mathcal{C}$ is definable (even semialgebraic), since $\mathcal{C} = \{S(p,q,r) : (p,q,r) \in A\}$ with $A = \{(p,q,r) \in \mathbb{R}^5 : p,q \in \mathbb{R}^2, r \in \mathbb{R}, r > d(p,q)\}$, $S = \{(p,q,r,x) \in A \times \mathbb{R}^2 : (p,q,r) \in A, x \in \mathbb{R}^2, d(x,p) + d(x,q) = r\}$. Note that the map $(p,q,r) \mapsto S(p,q,r) : A \to \mathcal{C}$ is not a bijection, since $S(p,q,r) = S(q,p,r)$ for $(p,q,r) \in A$. Put $B = \{(p,q,r) \in A : (p,q) \leq (q,p)\}$ with the lexicographic ordering on $\mathbb{R}^4$. Then $(p,q,r) \mapsto S(p,q,r) : B \to \mathcal{C}$ is a bijection, so $\dim \mathcal{C} = \dim B = 5$.
Also the collection $\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C} = \{$ line segments $[p,q] \subseteq \mathbb{R}^2\}$ is definable (even semialgebraic), and $\dim(\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C}) = 4$.

If $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{P}(Z)$ are definable collections, so are $\mathcal{C}_1 \cup \mathcal{C}_2$, $\mathcal{C}_1 \cap \mathcal{C}_2$ and $\mathcal{C}_1 \setminus \mathcal{C}_2$, and $\dim(\mathcal{C}_1 \cup \mathcal{C}_2) = \max(\dim \mathcal{C}_1, \dim \mathcal{C}_2)$, and $\mathcal{C}_1 \subseteq \mathcal{C}_2 \Longrightarrow \dim \mathcal{C}_1 \leq \dim \mathcal{C}_2$. Also, if $\mathcal{C} \subseteq \mathcal{P}(Z)$ is definable, so is $\mathcal{C}_{\mathrm{compact}} = \{X \in \mathcal{C} : X \neq \emptyset, X \mathrm{compact}\}$. The next result says that the facts observed about the collection of ellipses in the euclidean plane extend to arbitrary definable collections in $\mathcal{K}(Z)$.

Theorem 3.1 *Suppose $\mathcal{C} \subseteq \mathcal{K}(Z)$ is definable. Then*

(1) $\mathrm{cl}(\mathcal{C})$ *is definable;*

(2) *if (X_k) is a sequence in $\mathcal{C}$ and $X = \lim_{k\to\infty} X_k$, then*

$$\dim X \leq \liminf_{k\to\infty} \dim X_k;$$

(3) *if $\mathcal{C} \neq \emptyset$, then* $\dim(\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C}) < \dim \mathcal{C}$.

Part (1) implies in particular that each Hausdorff limit of a sequence in $\mathcal{C}$ is definable. Part (3) says that in a certain sense $\mathcal{C}$ has very few Hausdorff limits that are not already in $\mathcal{C}$. We postpone the proof of the theorem to Section 9 in the model-theoretic part of the paper.
Our tame setting suggests that the Hausdorff limits of $\mathcal{C}$ as in the theorem should be limits along definable curves. The next result makes this precise; its proof uses only part (1) of Theorem 3.1.

Proposition 3.2 *Let $A \subseteq \mathbb{R}^m$ and $S \subseteq A \times Z$ be definable such that $\mathcal{C} = \{S(a) : a \in A\} \subseteq \mathcal{K}(Z)$. Then*

(1) *If* $\gamma : (0,1] \to A$ *is definable and the subspace* $\{S(\gamma(t)) : 0 < t \le 1\}$ *of* $\mathcal{K}(Z)$ *is bounded, then* $\lim_{t\to 0} S(\gamma(t))$ *exists in* $\mathcal{K}(Z)$*.*

(2) *Let* $\mathrm{cl}(\mathcal{C}) = \{T(b) : b \in B\}$ *where* $B \subseteq \mathbb{R}^M$ *and* $T \subseteq B \times Z$ *are definable. Then there is a definable map* $\Gamma : B \times (0,1] \to A$ *such that for each* $b \in B$*,*

$$T(b) = \lim_{t\to 0} S(\Gamma(b,t)).$$

Proof. Let γ be as in the hypothesis of (1), and put $X_k = S(\gamma(\frac{1}{k}))$ for $k = 1, 2, 3, \ldots$. By the boundedness assumption, (X_k) has a subsequence that converges in $\mathcal{K}(Z)$, say to $X \in \mathcal{K}(Z)$. Then $\lim_{t\to 0} S(\gamma(t)) = X$:

$$t \mapsto d_H(S(\gamma(t)), X) : (0,1] \to \mathbb{R}$$

is definable and $\lim_{k\to\infty} d_H(S(\gamma(\frac{1}{k})), X) = 0$, hence

$$\lim_{t\to 0} d_H(S(\gamma(t)), X) = 0.$$

For (2), note that, given any $b \in B$ and $0 < t \le 1$, there exists $a \in A$ such that $d_H(T(b), S(a)) \le t$. By definable choice this yields a definable map $\Gamma : B \times (0,1] \to A$ such that $d_H(T(b), S(\Gamma(b,t))) \le t$ for all $(b,t) \in B \times (0,1]$.
□

4 Hausdorff measure of definable sets

In this section the conventions of the previous section remain in force. We shall refine part (2) of theorem 3.1 in terms of Hausdorff measure, using [20] as our reference for geometric measure theory.

For $e \in \{0, \ldots, n\}$, let $\mathcal{H}^e$ denote e*-dimensional Hausdorff measure* on the n-dimensional euclidean space Z, as defined in [20], p.6. For $e = 0$ this is the counting measure on Z, that is, $\mathcal{H}^0(X) = |X|$ for finite $X \subseteq Z$. For $e = 1$ one can view it as measuring the *length* of subsets of Z. We also note that $\mathcal{H}^n$ is Lebesgue measure on Z.

Suppose $X \subseteq Z$ is definable. It is easy to see that the dimension of X as a definable set [9] agrees with its Hausdorff dimension as a subset of the euclidean space Z. If in addition $\dim X \le e$ and X is bounded, then X is $\mathcal{H}^e$-measurable with $\mathcal{H}^e(X) < \infty$. Here is a uniform version of this fact.

Proposition 4.1 *Let $\mathcal{C}$ be a definable collection of subsets of Z, all of dimension $\leq e$, such that $\{\mathrm{diam}(X) : X \in \mathcal{C}\}$ is a bounded subset of $\mathbb{R}$. Then $\{\mathcal{H}^e(X) : X \in \mathcal{C}\}$ is a bounded subset of $\mathbb{R}$.*

In the proof we need that the e-dimensional Hausdorff measure of a bounded definable $X \subseteq Z$ equals its e-dimensional integral geometric measure. We first explain the meaning of this equality. Equip each linear subspace E of $Z = \mathbb{R}^n$ with the Lebesgue measure $\mathcal{L}^E$ that corresponds to the usual Lebesgue measure on $\mathbb{R}^e$ via a linear isometry $E \cong \mathbb{R}^e$, where $e = \dim E$. (Note: $\mathcal{L}^E$ does not depend on the choice of linear isometry.)
Given any linear subspace E of Z we have a corresponding orthogonal projection map $p : Z \to Z$, that is, for each vector $x \in Z$ we have $p(x) \in E$ and $x - p(x) \perp E$. Let $O(n, e)$ be the set of orthogonal projection maps $p : Z \to Z$ such that $\dim p(Z) = e$. The map that assigns to each $p \in O(n, e)$ its matrix with respect to the standard basis of $Z = \mathbb{R}^n$ is a bijection of $O(n, e)$ onto a compact subset of the euclidean space $\mathbb{R}^{n \times n}$. We equip $O(n, e)$ with the topology that makes this map a homeomorphism. Let $O(n)$ be the compact group of orthogonal linear transformations of Z. We have a continuous transitive action

$$(s, p) \mapsto sps^{-1} : O(n) \times O(n, e) \to O(n, e)$$

of $O(n)$ on $O(n, e)$. It follows that there is a unique Borel probability measure $\mu_{n,e}$ on $O(n, e)$ that is invariant under this action.
Suppose $X \subseteq Z$ is bounded and definable with $\dim X \leq e$. Then the function $f_X : O(n, e) \to [0, +\infty]$ given by

$$f_X(p) = \int_{x \in E} \mathcal{H}^0(p^{-1}(x) \cap X) \, d\mathcal{L}^E, \quad E = p(Z)$$

is integrable with respect to $\mu_{n,e}$. The fact that the e-dimensional Hausdorff measure of X equals its e-dimensional integral-geometric measure can now be expressed as

$$\mathcal{H}^e(X) = \frac{1}{\beta} \int f_X \, d\mu_{n,e},$$

where $\beta = \beta(n, e)$ is a positive normalizing constant:

$$\beta = \Gamma(\frac{e+1}{2})\Gamma(\frac{n-e+1}{2})\Gamma(\frac{n+1}{2})^{-1}\pi^{(-1/2)}.$$

Note that given any $p \in O(n,e)$ and $E = p(Z)$, the definable set

$$I_p = \{x \in E : p^{-1}(x) \cap X \text{ is infinite}\}$$

has dimension $< e$, so $\mathcal{L}^E(I_p) = 0$.

Proof of 4.1. Let $X \in \mathcal{C}$. We have $\mathcal{H}^e(X) = \frac{1}{\beta}\int f_X \, d\mu_{n,e}$ with

$$f_X(p) = \int_{x \in E} \mathcal{H}^0(p^{-1}(x) \cap X)\, d\mathcal{L}^E, \quad E = p(\mathbb{R}^n).$$

Take $N = N(n,e,\mathcal{C}) \in \mathbb{N}$ such that $|p^{-1}(x) \cap X| \le N$ for all $p \in O(n,e)$ and all $x \in E \setminus I_p$ where $E = p(Z)$ and $I_p \subseteq E$ is a definable set of dimension $< e$. Clearly $f_X(p) \le N \cdot \mathcal{L}^E(pX) \le NC$ where $C = C(n,e,\mathcal{C})$ is a positive constant independent of $X \in \mathcal{C}$ and $p \in O(n,e)$. Hence $\mathcal{H}^e(X) \le \frac{NC}{\beta}$. This finishes the proof of proposition 4.1.

Here is the main result of this section (and of the paper):

Theorem 4.2 *Let $\mathcal{C} \subseteq \mathcal{K}(Z)$ be definable. Then there is a positive constant $c = c(n,e,\mathcal{C})$ such that if (X_k) is any sequence in $\mathcal{C}$ with $\dim X_k \le e$ for all k, and (X_k) converges in the Hausdorff distance to $X \in \mathcal{K}(Z)$, then*

$$c \cdot \limsup_{k\to\infty} \mathcal{H}^e(X_k) \le \mathcal{H}^e(X) \le \liminf_{k\to\infty} \mathcal{H}^e(X_k) < \infty.$$

For $e = n$ this holds with $c = 1$, that is, $\lim_{k\to\infty} \mathcal{L}(X_k) = \mathcal{L}(X)$, where $\mathcal{L}$ is the Lebesgue measure on Z.

Remarks.
Let $\mathcal{C}$ be a definable collection of nonempty bounded subsets of Z. Then the theorem applies to the definable collection $\overline{\mathcal{C}} = \{\overline{X} : X \in \mathcal{C}\} \subseteq \mathcal{K}(Z)$. Note in this connection that if $X \subseteq Z$ is definable and bounded, then $\dim X = \dim \overline{X}$ and $\mathcal{H}^e(X) = \mathcal{H}^e(\overline{X})$.

With $\mathcal{C}$, (X_k), X and e as in the theorem, here are two consequences:

(1) $\limsup_{k\to\infty} \mathcal{H}^e(X_k) > 0 \implies \dim X = e$;

(2) $\liminf_{k\to\infty} \mathcal{H}^e(X_k) = 0 \implies \dim X < e$.

Example. Let $n = 1$ and $e = 0$, and let $\mathcal{C}$ be the collection of subsets of $\mathbb{R}$ of cardinality 1 or 2. Then the conclusion of the theorem holds with $c = 1/2$ and for no larger c. To see this, put $X_k = \{0, \frac{1}{k}\}$ for odd $k > 0$ and $X_k = \{0\}$ for even $k > 0$. Then $X_k \to_H X = \{0\}$, $\liminf_{k\to\infty} \mathcal{H}^0(X_k) = 1$ and $\limsup_{k\to\infty} \mathcal{H}^0(X_k) = 2$, so the two inequalities of the theorem become equalities for $c = 1/2$.
The proof of Theorem 4.2 uses non-standard methods and is given in Section 10. The constant c of the theorem is provided by the next lemma.

Lemma 4.3 *Let $\mathcal{C}$ be a definable collection of bounded subsets of Z. Then there is a $c > 0$ with the following property: if $X \in \mathcal{C}$ and* $\dim X \le e$, *then $\mathcal{L}^E(pX) \ge c \cdot \mathcal{H}^e(X)$ for some $p \in O(n, e)$, with $E = p(Z)$.*

Proof. Suppose $X \in \mathcal{C}$ and $\dim X \le e$. Then $\int f_X d\mu_{n,e} \ge \beta \cdot \mathcal{H}^e(X)$, so there exists $p \in O(n, e)$ such that

$$f_X(p) = \int_{x\in E} \mathcal{H}^0(p^{-1}(x) \cap X)\, d\mathcal{L}^E \ge \beta \cdot \mathcal{H}^e(X), \quad E = p(Z).$$

Fix such a p and set $E = p(Z)$. The definable set

$$I_p = \{x \in E : p^{-1}(x) \cap X \text{ is infinite}\}$$

has dimension $< e$, so $\mathcal{L}^E(I_p) = 0$. Take a positive integer $N = N(n, e, \mathcal{C})$ such that $|p^{-1}(x) \cap X| \le N$ for all $x \in E \setminus I_p$. Then

$$\int_{x\in E} \mathcal{H}^0(p^{-1}(x) \cap X)\, d\mathcal{L}^E \le N \cdot \mathcal{L}^E(pX),$$

hence $N \cdot \mathcal{L}^E(pX) \ge \beta \cdot \mathcal{H}^e(X)$. This proves the lemma with $c = \beta/N$. □

5 Gromov-Hausdorff limits

We refer to [3] and [13] as general background for this Section. Given any compact spaces X and Y their Gromov-Hausdorff distance $d_{GH}(X, Y)$ is given by $d_{GH}(X, Y) = \inf d_H(iX, jY)$ where the infimum is over all isometric embeddings $i : X \to Z$ and $j : Y \to Z$ in all spaces Z.
In this definition it suffices to consider spaces $Z = iX \cup jY$ such that $iX \cap jY = \emptyset$. Here are some basic facts (with X, Y, X_1, X_2, X_3 compact spaces):

- $d_{GH}(X, Y) = 0 \iff X$ and Y are isometric;
- $d_{GH}(X_1, X_3) \le d_{GH}(X_1, X_2) + d_{GH}(X_2, X_3)$ (triangle inequality).

Note that $d_{GH}(X, \{p\}) = \frac{1}{2}\mathrm{diam}(X)$ (X a compact space, $\{p\}$ a one-point space). We shall write $X_k \to_{GH} X$ to indicate that (X_k) is a sequence of compact spaces, X is a compact space, and $\lim_{k\to\infty} d_{GH}(X_k, X) = 0$.
We denote the isometry class of a compact space X by $[X]$, and let $\mathcal{G}$ be the set of isometry classes of compact spaces, equipped with the metric $d_{GH}([X], [Y]) = d_{GH}(X, Y)$. The *Gromov space* $\mathcal{G}$ is complete and connected. The closure of a set $\mathcal{S} \subseteq \mathcal{G}$ in $\mathcal{G}$ is denoted by $\mathrm{cl}(\mathcal{S})$.
Let Z be a space. Note that if $X, Y \in \mathcal{K}(Z)$, then $d_{GH}(X, Y) \le d_H(X, Y)$, in other words, the map $X \mapsto [X] : \mathcal{K}(Z) \to \mathcal{G}$ is continuous with Lipschitz constant 1. For $\mathcal{C} \subseteq \mathcal{K}(Z)$ we denote its image in $\mathcal{G}$ as follows:

$$\mathcal{C}_{\mathcal{G}} = \{[X] : X \in \mathcal{C}\} \qquad (\text{a subset of } \mathcal{G}).$$

A space Z with metric d is said to be *proper* if each closed ball

$$\overline{B}(a, r) = \{z \in Z : d(a, z) \le r\}, \quad (a \in Z, r > 0)$$

is compact.

Lemma 5.1 *Suppose Z is proper and* $\mathrm{Isom}(Z)$ *acts transitively on Z. Then $\mathcal{K}(Z)_{\mathcal{G}}$ is closed in $\mathcal{G}$.*

Proof. Suppose $X_k \to_{GH} X$ where (X_k) is a sequence in $\mathcal{K}(Z)$. We have to show that then X is isometric to some $Y \in \mathcal{K}(Z)$. Take any $R > \mathrm{diam}(X)$. Then $\mathrm{diam}(X_k) < R$ for all sufficiently large k. Fix some point $a \in Z$. Applying suitable isometries of Z we can assume that $a \in X_k$ for all k, hence $X_k \subseteq \overline{B}(a, R)$ for all sufficiently large k. Since $\mathcal{K}(\overline{B}(a, R))$ is compact, some subsequence of (X_k) converges with respect to the Hausdorff metric to some $Y \in \mathcal{K}(\overline{B}(a, R)) \subseteq \mathcal{K}(Z)$. This subsequence also converges in the Gromov-Hausdorff metric to Y, hence Y is isometric to X, as desired. $\square$

The hypothesis of the lemma is clearly satisfied for $Z = \mathbb{R}^n$.

6 Gromov-Hausdorff limits in a tame setting

In this section $Z = \mathbb{R}^n$, unless specified otherwise. Definability is with respect to some fixed o-minimal structure on the ordered field of real numbers.
We focus on compact subspaces of Z and their Gromov-Hausdorff limits. The main point is that then the Gromov-Hausdorff distance can be replaced by another distance, which for lack of a better term we call the isometric Hausdorff distance. In contrast to the Gromov-Hausdorff distance it has a finitary "definable" character, which is an advantage when dealing with definable collections of spaces. Note that for $\mathcal{S} \subseteq \mathcal{K}(Z)_{\mathcal{G}}$ the closure $\mathrm{cl}(\mathcal{S})$ of $\mathcal{S}$ in $\mathcal{G}$ is also the closure of $\mathcal{S}$ in $\mathcal{K}(Z)_{\mathcal{G}}$, by Lemma 5.1.

Given any $X, Y \in \mathcal{K}(Z)$ their *isometric Hausdorff distance* $d_{iH}(X, Y)$ is given by
$$d_{iH}(X, Y) = \inf\{d_H(X, \sigma Y) : \sigma \in \mathrm{Isom}(Z)\}.$$
Here are some basic facts (for $X, Y, X_1, X_2, X_3 \in \mathcal{K}(Z)$):

- the infimum in this definition is a minimum;
- $d_{iH}(X, Y) = 0 \iff [X] = [Y]$;
- $d_{iH}(X_1, X_3) \leq d_{iH}(X_1, X_2) + d_{iH}(X_2, X_3)$ (triangle inequality).

The second fact follows from Proposition 2.20 in [3]. Thus d_{iH} induces a metric (also denoted by d_{iH}) on the set $\mathcal{K}(Z)_{\mathcal{G}}$ given by
$$d_{iH}([X], [Y]) = d_{iH}(X, Y).$$

Proposition 6.1 *d_{iH} and d_{GH} induce the same topology on the set $\mathcal{K}(Z)_{\mathcal{G}}$.*

Proof. Let $\overline{B}$ be a closed ball in Z. It suffices to show that d_{iH} and d_{GH} define the same topology on $\mathcal{K}(\overline{B})_{\mathcal{G}} \subseteq \mathcal{K}(Z)_{\mathcal{G}}$. Since $d_H \geq d_{iH} \geq d_{GH}$ on $\mathcal{K}(\overline{B})$, we have continuous maps
$$(\mathcal{K}(\overline{B}), d_H) \to (\mathcal{K}(\overline{B})_{\mathcal{G}}, d_{iH}) \to (\mathcal{K}(\overline{B})_{\mathcal{G}}, d_{GH})$$
where the map $X \mapsto [X]$ on the left is surjective, and the map on the right is the identity on the underlying sets. Since the leftmost space is compact, so is the middle space. Thus the right hand map, being a continuous bijection between compact spaces, is a homeomorphism. $\square$

This proof, due to Marcin Mazur, is much shorter than my original proof.

We now turn to the analogue of 3.1 for Gromov-Hausdorff limits. Let $\mathcal{C} \subseteq \mathcal{K}(Z)$ be a definable collection, say $\mathcal{C} = \{S(a) : a \in A\}$ where $A \subseteq \mathbb{R}^m$ and $S \subseteq A \times Z$ are definable. It is not clear (to me) if the map

$$(a, b) \mapsto d_{GH}(S(a), S(b)) : A \times A \to \mathbb{R}$$

is always definable, but in any case, the map

$$(a, b) \mapsto d_{iH}(S(a), S(b)) : A \times A \to \mathbb{R}$$

is definable. Thus definable choice [9], p.94, yields a definable $B \subseteq A$ such that the map $b \mapsto [S(b)] : B \to \mathcal{C}_{\mathcal{G}}$ is a bijection. Put $\dim \mathcal{C}_{\mathcal{G}} = \dim B$ for such B. (This definition of $\dim \mathcal{C}_{\mathcal{G}}$ is possible because $\dim B$ depends only on $\mathcal{C}_{\mathcal{G}}$, not on $\mathcal{C}$, S and B.)
A *definable subset* of $\mathcal{K}(Z)_{\mathcal{G}}$ is by definition a subset of the form $\mathcal{C}_{\mathcal{G}}$ for some definable collection $\mathcal{C} \subseteq \mathcal{K}(Z)$.

Example. Let $n = 2$, and $\mathcal{C} = \{$ ellipses in $\mathbb{R}^2\}$, so $\mathcal{C}_{\mathcal{G}}$ is a definable subset of $\mathcal{K}(Z)_{\mathcal{G}}$. Let A and S be as in the example in Section 3, and put $E = \{(a, r) \in \mathbb{R}^2 : 0 \leq a < r/2\}$. Then the map

$$(a, r) \mapsto [S((-a, 0), (a, 0), r)] : E \to \mathcal{C}_{\mathcal{G}}$$

is a bijection, hence $\dim \mathcal{C}_{\mathcal{G}} = \dim E = 2$.

If $\mathcal{S}_1, \mathcal{S}_2$ are definable subsets of $\mathcal{K}(Z)_{\mathcal{G}}$, so are $\mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2$ and $\mathcal{S}_1 \setminus \mathcal{S}_2$, and $\dim(\mathcal{S}_1 \cup \mathcal{S}_2) = \max(\dim \mathcal{S}_1, \dim \mathcal{S}_2)$, and we have the implication

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \Longrightarrow \dim \mathcal{S}_1 \leq \dim \mathcal{S}_2.$$

Theorem 6.2 *Let $\mathcal{S}$ be a definable subset of $\mathcal{K}(Z)_{\mathcal{G}}$. Then $\mathrm{cl}(\mathcal{S})$ is also a definable subset of $\mathcal{K}(Z)_{\mathcal{G}}$, and $\dim(\mathrm{cl}(\mathcal{S}) \setminus \mathcal{S}) < \dim \mathcal{S}$ if $\mathcal{S} \neq \emptyset$.*

Proof. Let $\mathcal{C} \subseteq \mathcal{K}(Z)$ be a definable collection such that $\mathcal{S} = \mathcal{C}_{\mathcal{G}}$. By definable choice we can arrange that the map $X \mapsto [X] : \mathcal{C} \to \mathcal{S}$ is a bijection, and (using also translations in Z) that $0 \in X$ for each $X \in \mathcal{C}$, where 0 is the origin of $Z = \mathbb{R}^n$. The proof of Proposition 6.1 shows that then

$\mathrm{cl}(\mathcal{S}) = \mathrm{cl}(\mathcal{C})_{\mathcal{G}}$. By Theorem 3.1 the collection $\mathrm{cl}(\mathcal{C})$ is definable. Hence $\mathrm{cl}(\mathcal{S})$ is a definable subset of $\mathcal{K}(Z)_{\mathcal{G}}$. The dimension inequality follows from

$$\mathrm{cl}(\mathcal{S}) \setminus \mathcal{S} \ = \ \mathrm{cl}(\mathcal{C})_{\mathcal{G}} \setminus \mathcal{C}_{\mathcal{G}} \ \subseteq \ (\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C})_{\mathcal{G}},$$

in view of $\dim \mathcal{S} = \dim \mathcal{C}$ and the dimension inequality of Theorem 3.1. □

Mimicking the proof of Proposition 3.2 we obtain:

Proposition 6.3 *Let $A \subseteq \mathbb{R}^m$ and $S \subseteq A \times Z$ be definable such that $\mathcal{C} = \{S(a) : a \in A\} \subseteq \mathcal{K}(Z)$. Then*

(1) *If $\gamma : (0,1] \to A$ is definable and the subspace $\{[S(\gamma(t))] : 0 < t \leq 1\}$ of $\mathcal{G}$ is bounded, then $\lim_{t\to 0}\,[S(\gamma(t))]$ exists in $\mathcal{G}$.*

(2) *Let $\mathrm{cl}(\mathcal{C}_{\mathcal{G}}) = \{[T(b)] : b \in B\}$ where $B \subseteq \mathbb{R}^M$ and $T \subseteq B \times Z$ are definable. Then there is a definable map $\Gamma : B \times (0,1] \to A$ such that for each $b \in B$,*
$$[T(b)] = \lim_{t\to 0}\,[S(\Gamma(b,t))].$$

A remark on the inner metric. Until now we considered a nonempty compact set $X \subseteq \mathbb{R}^n$ as a space by restricting the euclidean metric to X. But if such X is definable and connected, we can also make X into a space by the *inner* metric where the distance between two points is the infimum of the lengths of the rectifiable paths in X that connect these points. This is arguably a more natural way to proceed in connection with the Gromov-Hausdorff distance, and immediately suggests the well-known open problem whether this inner metric is definable in some o-minimal expansion of the given o-minimal structure. See [1] and [2] for results on Gromov-Hausdorff limits of definable families of sets with this inner metric.

Improvements and generalizations. Proposition 6.1 raises the question whether there is a constant $C(n) > 0$ such that $d_{iH} \leq C(n) \cdot d_{GH}$ on $\mathcal{K}(Z)$? The answer is "yes" if $n = 1$, and "no" for $n \geq 2$. Perhaps $d_{iH} = d_{GH}$ on $\mathcal{K}(\mathbb{R})$, but I can only prove a weaker statement:

Proposition 6.4 *Let $X, Y \in \mathcal{K}(\mathbb{R})$. Then $d_{iH}(X,Y) \leq 4d_{GH}(X,Y)$.*

Proof. Put $a = \min X$ and $b = \max X$. Suppose $d_{GH}(X,Y) < \delta$. It will suffice to show that then $d_{iH}(X,Y) < 4\delta$. Take a metric space M with metric d and isometric embeddings $i : X \to M$ and $j : Y \to M$ such that $d_H(iX, jY) < \delta$. Take points $y_a, y_b \in Y$ such that $d(i(a), j(y_a)) < \delta$ and $d(i(b), j(y_b)) < \delta$. After a translation we may assume that $y_a = a$. Let $x \in X$ and $y \in Y$ such that $d(ix, jy) < \delta$. Then $|y - a| = d(j(y), j(y_a)) \leq d(j(y), i(x)) + d(i(x), i(a)) + d(i(a), j(y_a)) < |x - a| + 2\delta$. For $x = b$ and $y = y_b$ this gives $|y_b - a| < |b - a| + 2\delta$. After reflecting Y in the point $y_a = a$ we may assume that $a \leq y_b$, so the previous inequality becomes $y_b - a < b - a + 2\delta$, that is, $y_b < b + 2\delta$. Also $b - a = d(i(b), i(a)) \leq d(i(b), j(y_b)) + d(j(y_b), j(y_a)) + d(j(y_a), i(a)) < (y_b - a) + 2\delta$, so $b < y_b + 2\delta$. Thus $|y_b - b| < 2\delta$. Next, with x and y as before:

$$|y - b| \leq |y - y_b| + |y_b - b| < (|x - b| + 2\delta) + 2\delta = |x - b| + 4\delta.$$

We now have the inequality $y - a < (x - a) + 2\delta$, so $y - x < 2\delta$, and the inequality $x - b - 4\delta < y - b$, so $-4\delta < y - x$. Hence $|y - x| < 4\delta$. $\square$

The following example is due to M. Gabrilovich and shows that there is no constant $C > 0$ such that $d_{iH} \leq C \cdot d_{GH}$ on $\mathcal{K}(\mathbb{R}^2)$. This is easily adapted to show that for no $n \geq 2$ is there is a constant $C > 0$ such that $d_{iH} \leq C \cdot d_{GH}$ on $\mathcal{K}(\mathbb{R}^n)$. Let $X = \{(-1,0), (0,0), (1,0)\} \subseteq \mathbb{R}^2$, and put

$$X_\epsilon = \{(-1,0), (0,\epsilon), (1,0)\} \subseteq \mathbb{R}^2, \quad \epsilon > 0.$$

Then $d_{iH}(X_\epsilon, X) = \frac{1}{2}\epsilon$ for all sufficiently small $\epsilon > 0$, but $d_{GH}(X_\epsilon, X) \leq \epsilon^2$ for all sufficiently small $\epsilon > 0$.

We can generalize some of this section as follows. Relax the assumption that $Z = \mathbb{R}^n$ (with euclidean metric) to the assumption that Z is a proper space such that any isometry between compact subspaces of Z extends to an isometry of Z, that is, to an element of $\mathrm{Isom}(Z)$. (In particular, $\mathrm{Isom}(Z)$ acts transitively on Z.) Then the definition of "isometric Hausdorff distance" and the basic facts about it, including Proposition 6.1, go through. Here are two situations where these assumptions are satisfied (according to [3], 2.20). We shall argue that Theorem 6.2 also goes through in these two cases.

(1) Let $Z = \mathbb{S}^n$, the n-sphere. Its underlying set is the unit sphere

$$\{x \in \mathbb{R}^{n+1} : \quad \sum_{i=1}^{n+1} x_i^2 = 1\}$$

in $\mathbb{R}^{n+1}$. The metric is given by

$$d(x,y) \in [0,\pi], \quad \cos d(x,y) = \sum_{i=1}^{n+1} x_i y_i.$$

(2) Let $Z = \mathbb{H}^n$, hyperbolic n-space. Its underlying set is

$$\{x \in \mathbb{R}^{n+1} : \ x_{n+1} > 0, \quad x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1\},$$

the upper sheet of a hyperboloid in $\mathbb{R}^{n+1}$. The metric is given by

$$d(x,y) \geq 0, \quad \cosh d(x,y) = -(x_1y_1 + \ldots + x_ny_n) + x_{n+1}y_{n+1}.$$

Let $n > 0$ and let Z be the space in (1) or (2). The metric of Z may or may not be definable with respect to the given o-minimal structure on $\mathbb{R}$: the metric of $\mathbb{S}^n$ is definable iff the cosine function restricted to $[0,\pi]$ is definable, and the metric of $\mathbb{H}^n$ is definable iff the exponential function exp is definable. Whatever the case may be, the isometry group $\mathrm{Isom}(Z)$ *is* definable, that is, the set of graphs of isometries of Z is a definable collection of subsets of $\mathbb{R}^{2(n+1)}$; see [3], 2.24.
Let $\mathcal{C} \subseteq \mathcal{K}(Z)$ be definable, say $\mathcal{C} = \{S(a) : a \in A\}$ where $A \subseteq \mathbb{R}^m$ and $S \subseteq A \times Z \subseteq \mathbb{R}^{m+n+1}$ are definable. Since the topology of Z induced by its metric equals its topology as a subspace of the euclidean space $\mathbb{R}^{n+1}$, we have an obvious inclusion $\mathcal{K}(Z) \hookrightarrow \mathcal{K}(\mathbb{R}^{n+1})$, and this inclusion is a homeomorphism onto its image (Vietoris). The closure $\mathrm{cl}(\mathcal{C})$ of $\mathcal{C}$ in $\mathcal{K}(Z)$ coincides with its closure in $\mathcal{K}(\mathbb{R}^{n+1})$ under this inclusion, so Theorem 3.1 on Hausdorff limits goes through for the present Z.
Turning to Gromov-Hausdorff limits, the equivalence relation $\sim$ on A,

$$a \sim b \ :\Longleftrightarrow \ S(a) = \sigma S(b) \, \text{for some} \, \sigma \in \mathrm{Isom}(Z),$$

is definable. Definable choice then yields a definable $B \subseteq A$ that intersects each $\sim$-class in exactly one point, so the map $b \mapsto [S(b)] : B \to \mathcal{C}_{\mathcal{G}}$ is a

bijection. Put $\dim \mathcal{C}_{\mathcal{G}} = \dim B$ for such B. (This definition of $\dim \mathcal{C}_{\mathcal{G}}$ is possible because $\dim B$ depends only on $\mathcal{C}_{\mathcal{G}}$, not on $\mathcal{C}$, S and B.)
A *definable subset* of $\mathcal{K}(Z)_{\mathcal{G}}$ is by definition a subset of the form $\mathcal{C}_{\mathcal{G}}$ for some definable collection $\mathcal{C} \subseteq \mathcal{K}(Z)$. The statement right before Theorem 6.2 goes through for our present Z, and so does the theorem itself, with almost the same proof. (For $Z = \mathbb{S}^n$, this is actually contained in Theorem 6.2 because two subsets of $\mathbb{S}^n$ are isometric with respect to the metric of $\mathbb{S}^n$ if and only if they are isometric with respect to the euclidean metric of the ambient space $\mathbb{R}^{n+1}$.)

7 Tychonov limits, elementary pairs, stability, and nfcp

With this section we begin the model-theoretic part of these notes; in particular we prove Theorem 3.1 in Section 9 and Theorem 4.2 in Section 10. We begin by considering Tychonov limits because they are a little easier to handle than Hausdorff limits, and make sense in a wider setting. As usual, "definable" will mean "definable with parameters" unless specified otherwise.
Let Z be a set, and identify each subset of Z with its characteristic function $Z \to 2$ where $2 = \{0, 1\}$. Thus the power set of Z gets identified with 2^Z. We make 2^Z into a topological space by giving $\{0, 1\}$ the discrete topology and 2^Z the corresponding product topology. With this identification a collection $\mathcal{C}$ of subsets of Z is a subset of 2^Z and thus has a closure in the topological space 2^Z which we shall call the *Tychonov closure* of $\mathcal{C}$ and denote by $\mathrm{cl}_t(\mathcal{C})$. A set $X \subseteq Z$ belongs to $\mathrm{cl}_t(\mathcal{C})$ iff X agrees on each finite set $G \subseteq Z$ with some set in $\mathcal{C}$, that is, $X \cap G = Y \cap G$ for some $Y \in \mathcal{C}$. (We also say that then X is a *Tychonov limit* of $\mathcal{C}$.) So if $(X_i)_{i \in I}$ is a family of sets in $\mathcal{C}$ such that for any two indices i and j, either $X_i \subseteq X_j$ or $X_j \subseteq X_i$, then the union $\cup_i X_i$ and the intersection $\cap_i X_i$ are Tychonov limits of $\mathcal{C}$. Note that the topological space 2^Z is compact Hausdorff; it is not metrizable if Z is uncountable.

Examples.

(1) Let Z be a finite-dimensional vector space over some infinite field, and $\mathcal{C}$ the collection of all linear subspaces of codimension 1 of Z. Then $\mathrm{cl}_t(\mathcal{C})$ is the collection of all linear subspaces of Z of codimension ≥ 1.

(2) Let Z be a real closed field, and $\mathcal{C}$ the collection of open intervals in Z. Then $\mathrm{cl}_t(\mathcal{C})$ is the collection of all convex subsets of Z.

In the rest of this section we fix a (one-sorted) L-structure $\mathcal{M} = (\mathcal{M}, \ldots)$, put $Z = M^n$, and let $\mathcal{C}$ be a definable collection of subsets of Z, that is,

$$\mathcal{C} = \{S(a) : a \in A\}$$

for sets $A \subseteq M^m$ and $S \subseteq A \times Z \subseteq M^{m+n}$ that are definable in $\mathcal{M}$.

Model-theoretic characterization of Tychonov limits.
Let ${}^*\mathcal{M} = ({}^*\mathcal{M}, \ldots)$ be an elementary extension of $\mathcal{M}$, put ${}^*Z = {}^*M^n$, and let ${}^*A \subseteq {}^*M^m$ and ${}^*S \subseteq {}^*A \times {}^*Z \subseteq {}^*M^{m+n}$ be the sets defined in ${}^*\mathcal{M}$ by the same formulas that define A and S in $\mathcal{M}$. Put ${}^*\mathcal{C} = \{{}^*S(a) : a \in {}^*A\}$ a collection of subsets of ${}^*Z$), ${}^*\mathcal{C} \cap Z = \{Y \cap Z : Y \in {}^*\mathcal{C}\}$ a collection of subsets of Z). Then we have the following results as is easily verified.

Lemma 7.1 *${}^*\mathcal{C} \cap Z \subseteq \mathrm{cl}_t(\mathcal{C})$, with equality if ${}^*\mathcal{M}$ is $|M|^+$-saturated.*

To exploit this we need to know more about $\mathcal{M}$. We first consider the case of stable $\mathcal{M}$, and in the next section the case that $\mathcal{M}$ is an o-minimal expansion of the ordered field of real numbers.

Connection to stability and nfcp. Recall that $\mathcal{M}$ is said to be *stable* if for every L-formula $\phi(x, y)$ with $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, there exists an integer $d \geq 1$ such that for all $a_1, \ldots, a_d \in M^m$ and $b_1, \ldots, b_d \in M^n$ there are $i, j \in \{1, \ldots, d\}$ for which *either* $i \leq j$ and $\mathcal{M} \not\models \phi(a_i, b_j)$, *or* $i > j$ and $\mathcal{M} \models \phi(a_i, b_j)$. (It suffices that this holds for $m = 1$; see [14], 6.7.)
A stronger property than stability is "nfcp" (non-finite-cover-property): $\mathcal{M}$ is said to have *nfcp* if for every L-formula $\phi(x, y)$ as before there exists an integer $d \geq 1$ such that for every finite set $G \subseteq M^n$,

$$\mathcal{M} \models [\bigwedge_{F \subseteq G, |F| = d} \exists x \bigwedge_{b \in F} \phi(x, b)] \longrightarrow \exists x \bigwedge_{b \in G} \phi(x, b).$$

(It suffices that this holds for $m = 1$.)
These properties really pertain to the theory $T = \mathrm{Th}(\mathcal{M})$ of $\mathcal{M}$: if $\mathcal{M}$ is stable, then every model of T is stable, and if $\mathcal{M}$ has nfcp, then every model of T has nfcp.

Now, if $\mathcal{M}$ is stable and $Y \subseteq {}^*Z$ is definable in ${}^*\mathcal{M}$, then its trace $Y \cap Z$ is definable in $\mathcal{M}$. (This is a form of *definability of types in stable theories*; see [14], 6.7) In view of Lemma 7.1 we conclude that if $\mathcal{M}$ is stable, then every set in $\mathrm{cl}_t(\mathcal{C})$ is definable in $\mathcal{M}$. More is true:

Proposition 7.2 *If $\mathcal{M}$ is stable, then $\mathrm{cl}_t(\mathcal{C}) \subseteq \mathcal{C}'$ for some definable collection $\mathcal{C}'$ of subsets of Z. If $\mathcal{M}$ has nfcp, then $\mathrm{cl}_t(\mathcal{C})$ is itself a definable collection of subsets of Z.*

Proof. Let T' be the theory of elementary pairs of models of $T = \mathrm{Th}(\mathcal{M})$. In more detail, the language of T' is $L(U) = L \cup \{U\}$ with U a new unary relation symbol, and the models of T' are the $L(U)$-structures $({}^*N, N, \ldots)$ where ${}^*\mathcal{N} = ({}^*\mathcal{N}, \ldots) \models \mathcal{T}$, and N (the interpretation of U) is the underlying set of an elementary substructure $\mathcal{N}$ of ${}^*\mathcal{N}$; we shall denote this model $({}^*N, N, \ldots)$ by $({}^*\mathcal{N}, \mathcal{N})$. Let $\phi(x, y)$ with $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, be an $L(M)$-formula that defines the set $S \subseteq M^{m+n}$ in $\mathcal{M}$. Suppose now that $\mathcal{M}$ is stable. Then for any model $({}^*\mathcal{N}, \mathcal{N})$ of T' with $\mathcal{M} \preceq \mathcal{N}$ and any $a \in {}^*N^m$, the set

$$\{b \in N^n : {}^*\mathcal{N} \models \phi(a, b)\}$$

is definable in $\mathcal{N}$. Model-theoretic compactness then yields an L-formula $\phi'(x', y)$, with $x' = (x'_1, \ldots, x'_k)$ a tuple of new variables, such that for any $({}^*\mathcal{N}, \mathcal{N})$ as above and any $a \in {}^*N^m$ there exists $a' \in N^k$ with

$$\{b \in N^n : {}^*\mathcal{N} \models \phi(a, b)\} = \{b \in N^n : \mathcal{N} \models \phi'(a', b)\}.$$

Thus $\mathrm{cl}_t(\mathcal{C}) \subseteq \mathcal{C}'$, where $\mathcal{C}'$ is the (definable) collection of subsets of M^n of the form $\{b \in M^n : \mathcal{M} \models \phi'(a', b)\}$ for some $a' \in M^k$.
Suppose next that $\mathcal{M}$ has nfcp. With the notations above, let $\theta(x, x', y)$ be the L-formula $\phi(x, y) \longleftrightarrow \phi'(x', y)$. Then, given any $a' \in M^k$ we have: $B = \{b \in M^n : \mathcal{M} \models \psi'(a', b)\} \in \mathrm{cl}_t(\mathcal{C})$ if and only if the above set B agrees on each finite subset of M^n with some set in $\mathcal{C}$ if and only if for each finite set $G \subseteq M^n$ there is $a \in A$ with $\mathcal{M} \models \bigwedge_{b \in G} \theta(a, a', b)$. The nfcp assumption gives an integer $d \geq 1$ such that in the line above we can restrict to sets G of size d; this yields the desired result. $\square$

The following partial converses are left to the reader:

(1) If for each $\mathcal{N} \models T$ and each n, every Tychonov limit of every definable collection of subsets of N^n is definable in $\mathcal{N}$, then $\mathcal{M}$ is stable.

(2) If for each $\mathcal{N} \models T$ and each n, the Tychonov closure $\mathrm{cl}_t(\mathcal{C})$ of every definable collection $\mathcal{C}$ of subsets of N^n is itself a definable collection of subsets of N^n, then $\mathcal{M}$ has nfcp.

8 Tychonov limits in the o-minimal setting

Much of the previous section can be adapted to the case that $\mathcal{M}$ is an o-minimal expansion of the real field. The role of elementary pairs in the characterization of Tychonov limits is taken over by *tame* elementary pairs. Let T be a complete o-minimal L-theory extending the theory of real closed ordered fields; the language L extends the language $\{0, 1, +, -, \cdot, <\}$ of ordered rings. Also, $\mathcal{R} = (R, \dots)$ and $\mathcal{N} = (N, \dots)$ will denote models of T. As in [10] we say that $\mathcal{N}$ is a *tame elementary substructure of* $\mathcal{R}$ (notation: $\mathcal{N} \preceq_{\text{tame}} \mathcal{R}$) if $\mathcal{N} \preceq \mathcal{R}$ and for each $\mathcal{N}$-bounded $r \in R$ there is a (necessarily unique) $a \in N$ such that $r - a$ is $\mathcal{N}$-infinitesimal. We then call a the *standard part* of r in $\mathcal{N}$ and write $a = \mathrm{st}_{\mathcal{N}}(r)$. Let T_{tame} be the theory of *tame elementary pairs*: these are the structures $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ with $\mathcal{N} \preceq_{\text{tame}} \mathcal{R}$, $N \neq R$, and with $\mathrm{st} : R \to N$ given by $\mathrm{st}(r) = \mathrm{st}_{\mathcal{N}}(r)$ if r is $\mathcal{N}$-bounded, and $\mathrm{st}(r) = 0$ otherwise. The language L_{tame} of T_{tame} consists of L augmented by a unary relation symbol U, interpreted in $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ as the underlying set N of $\mathcal{N}$, and by a unary function symbol st.

The following result of Marker and Steinhorn [18] takes the place of definability of types in stable structures as used in the previous section:

If $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ is a model of T_{tame} and the set $Y \subseteq R^n$ is definable in $\mathcal{R}$, then its trace $Y \cap N^n$ is definable in $\mathcal{N}$.

This yields the definability of Tychonov limits of definable collections $\mathcal{C}$ in o-minimal expansions of the real field, but for definability of $\mathrm{cl}_t(\mathcal{C})$ we shall need an improvement:

Proposition 8.1 *Suppose $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ is a model of T_{tame}. Then each subset of N^n that is definable in $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ is definable in $\mathcal{N}$.*

Proof. (Without using the Marker–Steinhorn theorem.) We have to show, given any $L_{\text{tame}}(R)$-formula $\phi(y)$, $y = (y_1, \dots, y_n)$, that $U(y) \wedge \phi(y)$ is equivalent in $(\mathcal{R}, \mathcal{N}, \text{st})$ to $U(y) \wedge \phi'(y)$ for some $L(N)$-formula $\phi'(y)$, where $U(y) = U(y_1) \wedge \dots \wedge U(y_n)$. By the Stone duality between formulas and types, this reduces to proving the following statement (*):

Let $(\mathcal{R}_1, \mathcal{N}_1, \text{st}_1)$ *and* $(\mathcal{R}_2, \mathcal{N}_2, \text{st}_2)$ *be elementary extensions of* $(\mathcal{R}, \mathcal{N}, \text{st})$ *and suppose that* $b_1 \in N_1^n$ *and* $b_2 \in N_2^n$ *realize the same* n*-type over* N *in* $\mathcal{N}_1$ *and* $\mathcal{N}_2$*, respectively. Then* b_1 *and* b_2 *realize the same* n*-type over* R *in* $(\mathcal{R}_1, \mathcal{N}_1, \text{st}_1)$ *and* $(\mathcal{R}_2, \mathcal{N}_2, \text{st}_2)$, respectively.

In order to prove (*) we may extend T by definitions, and thus assume that T is universally axiomatized and has QE.

The hypothesis of (*) means that we have an isomorphism $\mathcal{N}\langle b_1 \rangle \cong \mathcal{N}\langle b_2 \rangle$ over $\mathcal{N}$ sending b_1 to b_2. The conclusion of (*) will hold if we can extend this isomorphism to an isomorphism $\mathcal{R}\langle b_1 \rangle \cong \mathcal{R}\langle b_2 \rangle$ over $\mathcal{R}$, and show that $\text{st}_i \mathcal{R}\langle b_i \rangle = \mathcal{N}\langle b_i \rangle$ for $i = 1, 2$. By induction on n we may as well assume that $n = 1$. Then there are two cases.

Case 1. $b_1 \in N$. This case is trivial.

Case 2. $b_1 \notin N$. Then also $b_2 \notin N$. Clearly b_1 and b_2 realize the same cut in $\mathcal{N}$, and therefore the same cut in $\mathcal{R}$. It follows that we have an an isomorphism $\mathcal{R}\langle b_1 \rangle \cong \mathcal{R}\langle b_2 \rangle$ over $\mathcal{R}$ sending b_1 to b_2. Moreover, Lemma (5.3) in [10] shows that $\text{st}_i \mathcal{R}\langle b_i \rangle = \mathcal{N}\langle b_i \rangle$ for $i = 1, 2$. □

The *dimension* of Tychonov limits is controlled by the next result. To formulate it, let $(\mathcal{R}, \mathcal{N}, \text{st})$ be a model of T_{tame}, and let V be the convex hull of N in $\mathcal{R}$, in other words, V is the set of $\mathcal{N}$-bounded elements of $\mathcal{R}$. If $Y \subseteq R^n$ is definable in $\mathcal{R}$, then we indicate its dimension by $\dim_{\mathcal{R}} Y$ if we wish to be explicit about the ambient model $\mathcal{R}$.

Proposition 8.2 *Suppose* $Y \subseteq R^n$ *is definable in* $\mathcal{R}$*. Then*

$$\dim_{\mathcal{R}} Y \geq \dim_{\mathcal{N}}(Y \cap N^n).$$

Proof. Use that $Y \cap N^n \subseteq \text{st}(Y \cap V^n)$, and apply Proposition 1.10 of [11]. □

Application to Tychonov limits. Let some o-minimal expansion of the ordered field of real numbers be given; to keep notations simple, just write

$\mathbb{R}$ for this expansion; let $L = \{0, 1, +, -, \cdot, <, \dots\}$ be the language of this expansion, and $T = \mathrm{Th}(\mathbb{R})$ its L-theory. Definability is with parameters and with respect to $\mathbb{R}$.

Proposition 8.3 *Let $\mathcal{C}$ be a definable collection of subsets of $\mathbb{R}^n$. Then*

(1) $\mathrm{cl}_t(\mathcal{C})$ *is a definable collection of subsets of Z;*

(2) *if $X \in \mathrm{cl}_t(\mathcal{C})$, then* $\dim X \leq \max\{\dim Y : Y \in \mathcal{C}\}$.

Proof. Note first that $\mathbb{R} \preceq_{\mathrm{tame}} {}^*\mathbb{R}$ for each elementary extension ${}^*\mathbb{R}$ of $\mathbb{R}$. By taking ${}^*\mathbb{R}$ sufficiently saturated, the Marker–Steinhorn theorem with Lemma 7.1 yields that each Tychonov limit of $\mathcal{C}$ is definable. Using model-theoretic compactness as in the proof of Proposition 7.2 (with T_{tame} instead of T') the Marker–Steinhorn theorem also yields an L-formula $\phi'(x', y)$, with $x' = (x'_1, \dots, x'_k), y = (y_1, \dots, y_n)$, such that each set in $\mathrm{cl}_t(\mathcal{C})$ is of the form

$$\{b \in \mathbb{R}^n : \mathbb{R} \models \phi'(a', b)\}$$

for some $a' \in \mathbb{R}^k$. Taking a sufficiently saturated elementary extension ${}^*\mathbb{R}$ of $\mathbb{R}$, the model $({}^*\mathbb{R}, \mathbb{R}, \mathrm{st})$ of T_{tame} satisfies

$$\mathrm{cl}_t(\mathcal{C}) = \{Y \cap \mathbb{R}^n : Y \in {}^*\mathcal{C}\}.$$

The set of parameters $a' \in \mathbb{R}^k$ such that for some $Y \in {}^*\mathcal{C}$,

$$\{b \in \mathbb{R}^n : \mathbb{R} \models \phi'(a', b)\} = Y \cap \mathbb{R}^n,$$

is clearly definable in $({}^*\mathbb{R}, \mathbb{R}, \mathrm{st})$, and thus definable in $\mathbb{R}$ by Proposition 8.1. This proves item (1). Item (2) follows from Proposition 8.2. $\square$

A difference with Hausdorff limits is that in this proposition we can have $\dim \mathrm{cl}_t(\mathcal{C}) > \dim \mathcal{C}$: let $\mathcal{C}$ be the collection of $\mathbb{R}$-linear subspaces of codimension 1 of $\mathbb{R}^4$; then $\dim \mathcal{C} = 3$, and $\dim \mathrm{cl}_t(\mathcal{C}) > 3$, since all $\mathbb{R}$-linear subspaces of $\mathbb{R}^4$ of codimension 2 belong to $\mathrm{cl}_t(\mathcal{C})$.
Also, the proposition fails when we replace $\mathbb{R}$ by any real closed field not isomorphic to the real field; see Example (2) of the previous section.
Two comments on the proof of the proposition:

(1) At the end of the proof we took ${}^*\mathbb{R}$ sufficiently saturated, but T_{tame} is complete, so in *every* model $({}^*\mathbb{R}, \mathbb{R}, \text{st})$ of T_{tame} we have

$$\text{cl}_t(\mathcal{C}) = \{Y \cap \mathbb{R}^n : Y \in {}^*\mathcal{C}\}.$$

(2) Extending T suitably by definitions, it has QE and a universal axiomatization. By [10] the theory T_{tame} then also has QE and, after adding a constant symbol c to the language and the axiom $\neg U(c)$, a universal axiomatization, and thus definable Skolem functions.

This fact has the following consequence: Let $\mathcal{C} = \{S(a) : a \in A\}$ where $A \subseteq \mathbb{R}^m$ and $S \subseteq \mathbb{R}^{m+n}$ are definable in $\mathbb{R}$; using notations in the proof of the proposition, let ${}^*A$ and ${}^*S$ be the subsets of ${}^*\mathbb{R}^m$ and ${}^*\mathbb{R}^{m+n}$ defined in ${}^*\mathbb{R}$ by the formulas that define A and S in $\mathbb{R}$, respectively, and let $S' \subseteq \mathbb{R}^{k+n}$ be defined by the formula $\phi'(x', y)$ in $\mathbb{R}$. Then there is a map $a \mapsto a' : {}^*A \to \mathbb{R}^k$ that is definable in $({}^*\mathbb{R}, \mathbb{R}, \text{st})$ such that for all $a \in {}^*A$:

$${}^*S(a) \cap \mathbb{R}^n = S'(a').$$

In other words, the map that assigns to each set in ${}^*\mathcal{C}$ its trace in $\mathcal{C}$ is itself definable in a certain sense.

Proposition 3.2 for Hausdorff limits also has an analogue for Tychonov limits; this will be treated in a more general setting in Section 11.

9 Hausdorff limits by model theory

Here we prove Theorem 3.1 by treating Hausdorff limits in much the same way as Tychonov limits in the previous two sections. We begin by mimicking the description of Tychonov limits as *traces* by describing Hausdorff limits as *standard parts*.

Hausdorff limits as standard parts. Suppose the space Z is proper, that is, each closed ball in Z is compact. Let $\mathcal{C} \subseteq \mathcal{K}(Z)$ be a collection of compact subspaces of Z. In order to give a useful non-standard characterization of the collection $\text{cl}(\mathcal{C})$ of Hausdorff limits of $\mathcal{C}$, we introduce the 3-sorted structure $\mathcal{Z} = (Z, \mathcal{C}, \mathbb{R}; E, d)$, where E denotes the membership relation between elements of Z and sets in $\mathcal{C}$, (that is, $E = \{(z, X) \in Z \times \mathcal{C} : z \in X\}$), and

$d : Z \times Z \to \mathbb{R}$ is the metric of Z; in addition $\mathbb{R}$ is equipped with its usual field structure. We now take an elementary embedding

$$i = (i_Z, i_{\mathcal{C}}, i_{\mathbb{R}}) : \mathcal{Z} \to {}^*\mathcal{Z} = ({}^*Z, {}^*\mathcal{C}, {}^*\mathbb{R}; {}^*E, {}^*d)$$

where ${}^*\mathcal{Z}$ is $\aleph_1$-saturated. We identify Z with a subset of ${}^*Z$ via the map i_Z, and $\mathbb{R}$ with a subfield of ${}^*\mathbb{R}$ via the map $i_{\mathbb{R}}$. We can also assume that ${}^*\mathcal{C}$ is a collection of subsets of ${}^*Z$ by replacing every "abstract" element $X \in {}^*\mathcal{C}$ by the set

$$\{z \in {}^*Z : (z, X) \in {}^*E\}.$$

With this replacement ${}^*E$ becomes the usual membership relation between elements of ${}^*Z$ and members of ${}^*\mathcal{C}$: ${}^*E = \{(z, X) \in {}^*Z \times {}^*\mathcal{C} : z \in X\}$.
Let $Z \cup \{\infty\}$ = one-point compactification of Z; the *standard part* map

$$\mathrm{st} : {}^*Z \to Z \cup \{\infty\}$$

assigns to each point of ${}^*Z$ infinitely close to $z \in Z$ the value z; to each point of ${}^*Z$ not infinitely close to any point of Z it assigns the value ∞.

Lemma 9.1 *Let $X \subseteq Z$. Then:*

$$X \in \mathrm{cl}(\mathcal{C}) \iff X = \mathrm{st}(Y) \text{ for some } Y \in {}^*\mathcal{C}.$$

Proof. Suppose $X \in \mathrm{cl}(\mathcal{C})$. Take for each $p \in \mathbb{N}$ a finite $2^{-(p+1)}$-net X_p in X, that is, X_p is a finite subset of X such that $d_H(X, X_p) \le 2^{-(p+1)}$. We also arrange that $X_p \subseteq X_{p+1}$. Put $\mathcal{C}_p = \{Y \in \mathcal{C} : d_H(X_p, Y) < 2^{-p}\}$, a subset of $\mathcal{C}$ which is definable in the 3-sorted structure $\mathcal{Z}$. Then $\mathcal{C}_p \neq \emptyset$: take any $Y \in \mathcal{C}$ with $d_H(X, Y) < 2^{-(p+1)}$, and note that then $Y \in \mathcal{C}_p$. Also $\mathcal{C}_{p+1} \subseteq \mathcal{C}_p$: let $Y \in \mathcal{C}_{p+1}$; then $d_H(X_{p+1}, Y) < 2^{-(p+1)}$, so

$$d_H(X_p, Y) \le d_H(X_p, X_{p+1}) + d_H(X_{p+1}, Y) < 2^{-(p+1)} + 2^{-(p+1)} = 2^{-p}.$$

Thus by $\aleph_1$-saturation there is $Y \in \bigcap_{p \in \mathbb{N}} {}^*\mathcal{C}_p$. Then $X = \mathrm{st}\, Y$: for each $p \in \mathbb{N}$ we have ${}^*d_H({}^*X_p, Y) < 2^{-p}$, so ${}^*d_H({}^*X, Y)$ is infinitesimal, in particular, each point of X is at infinitesimal distance from some point of Y, and every point of Y is at infinitesimal distance from some point of ${}^*X$, hence from some point of X (since X is compact). Now assume that $X = \mathrm{st}\, Y$, with $Y \in {}^*\mathcal{C}$. Then X is bounded, since otherwise Y would have for each $R > 0$ in $\mathbb{R}$ a point at distance $> R$ from a fixed point $z_0 \in Z$, and hence by saturation Y would contain a point at infinite distance from z_0, so $\infty \in \mathrm{st}Y$, contradicting $X = \mathrm{st}\ Y$. A similar argument shows that X is closed: if $x \in \overline{X} \setminus X$, then saturation gives a point $y \in Y$ with infinitesimal ${}^*d(x, y)$, so $x = \mathrm{st}\ y$, contradiction. Thus $X \in \mathcal{K}(Z)$. Let $\epsilon > 0$ in $\mathbb{R}$. Take a finite $\epsilon/2$-net E in X, and note that then each point of Y is at distance $\leq \epsilon/2$ from a point in E, and that each point of E is at infinitesimal distance, hence distance $\leq \epsilon/2$ from a point of Y. Since E is finite, there exists $Y_\epsilon \in \mathcal{C}$ with $d_H(E, Y_\epsilon) \leq \epsilon/2$, which implies $d_H(X, Y_\epsilon) \leq \epsilon$. Since ϵ was arbitrary, we conclude that $X \in \mathrm{cl}(\mathcal{C})$. $\square$

Definability of Hausdorff limits. Fix an o-minimal structure on the ordered field of real numbers. Then part (1) of Theorem 3.1 says:

the closure $\mathrm{cl}(\mathcal{C})$ *of a definable collection* $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ *is definable.*

The proof is almost the same as that of Proposition 8.3: standard parts take over the role of traces, and Proposition 8.1 is used instead of the Marker-Steinhorn theorem. This proof easily extends to give a more general result: Let $Z \subseteq \mathbb{R}^n$ be a *definable* metric space, that is, the definable set Z is equipped with a definable metric $d : Z \times Z \to \mathbb{R}$. (We do not assume continuity of d with respect to the euclidean topology on Z.) So $\mathcal{K}(Z)$ is the set of compact subspaces of this definable metric space, and $\mathcal{K}(Z)$ is itself a space with respect to the Hausdorff metric that corresponds to d. For a definable collection $\mathcal{C}$ of subsets of Z, we define $\dim \mathcal{C}$ as in Section 3.

Proposition 9.2 *Suppose the space* Z *is proper and* $\mathcal{C} \subseteq \mathcal{K}(Z)$ *is definable. Then its closure* $\mathrm{cl}(\mathcal{C})$ *in* $\mathcal{K}(Z)$ *is definable.*

Proof. For simplicity, write $\mathbb{R}$ for the expansion of the ordered field of real numbers by the sets of the given o-minimal structure. Let ${}^*\mathbb{R}$ be a proper elementary extension of $\mathbb{R}$, and let ${}^*Z$ be the subset of ${}^*\mathbb{R}^n$ defined in ${}^*\mathbb{R}$ by the same formula that defines Z in $\mathbb{R}$. (All prefixed asterisks are used in this way in the proof.) We have a map $\mathrm{st}_Z : {}^*Z \to Z \cup \{\infty\}$ that is definable in the tame pair $({}^*\mathbb{R}, \mathbb{R}, \mathrm{st})$: fix a point $z_0 \in Z$, and for $z \in {}^*Z$, put $\mathrm{st}_Z(z) = \infty$ if ${}^*d(z, z_0) > \mathbb{R}$, and otherwise, let $\mathrm{st}_Z(z)$ be the unique point $z' \in Z$ such that ${}^*d(z, z')$ is infinitesimal. Define the *bounded* part Z^{b} of ${}^*Z$ by

$$Z^{\mathrm{b}} = \{z \in {}^*Z : {}^*d(z, z_0) < r \,\text{for some}\, r \in \mathbb{R}\}.$$

Let $A \subseteq \mathbb{R}^m$ and $S \subseteq A \times Z$ be definable such that $\mathcal{C} = \{S(a) : a \in A\}$, and let $^*A$ and $^*S$ be the corresponding subsets of $^*\mathbb{R}^m$ and $^*\mathbb{R}^{m+n}$. If $^*\mathbb{R}$ is sufficiently saturated, then by Lemma 9.1 we have

$$\mathrm{cl}(\mathcal{C}) = \{\mathrm{st}_Z(^*S(a)) : a \in {}^*A,\ {}^*S(a) \subseteq Z^{\mathrm{b}}\}.$$

Hence by Proposition 8.1, the sets in $\mathrm{cl}(\mathcal{C})$ are definable. Arguing as in the proof of Proposition 8.3 we conclude that $\mathrm{cl}(\mathcal{C})$ is a definable collection. (Since T_{tame} is complete, where $T = \mathrm{Th}(\mathbb{R})$, the description of $\mathrm{cl}(\mathcal{C})$ just displayed even holds without any saturation assumption on $^*\mathbb{R}$.) $\square$

Part (2) of Theorem 3.1 follows from Proposition 1.10 in [11].

A useful two-sorted theory. It remains to prove part (3) of Theorem 3.1, the dimension inequality. Towards this goal we adopt the notations from Section 8 concerning T, L, $\mathcal{R} = (R, \ldots)$ and $\mathcal{N} = (N, \ldots)$. Sometimes a model $(\mathcal{R}, \mathcal{N}, \mathrm{st}_{\mathcal{N}})$ of T_{tame} is better viewed as a model of a less expressive two-sorted theory T_{c}, with one sort of variables ranging over R and the other sort over N. In detail: the models of T_{c} are the two-sorted structures $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathrm{st})$ where $\mathcal{M}_1 = (M_1, \ldots)$ and $\mathcal{M}_2 = (M_2, \ldots)$ are models of T, and $\mathrm{st} : M_1 \to M_2$ is such that the convex hull V of $\{x \in M_1 : \mathrm{st}x \neq 0\}$ is a proper T-convex subring V of $\mathcal{R}$ and there is a (necessarily unique) isomorphism $\overline{V} \cong \mathcal{M}_2$ of models of T that for each $x \in V$ maps its residue class $\overline{x} \in \overline{V}$ to $\mathrm{st}x$. The language L_{c} of T_{c} consists of two disjoint copies L_1 and L_2 of the language L of T, plus a function symbol st relating the two sorts; in the model $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2, \mathrm{st})$ above, $\mathcal{M}_1$ is viewed as L_1-structure, $\mathcal{M}_2$ as L_2-structure, and the interpretation of the symbol st is the function $\mathrm{st} : M_1 \to M_2$. This theory T_{c} is naturally bi-interpretable with T_{convex}. The next result is a good example of a fact whose statement in terms of T_{convex} would be convoluted, and whose T_{tame}-analogue is false.
In the rest of this section $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ is a model of T_{tame}, but is construed as a model of T_{c} when so indicated; V denotes the convex hull of N in $\mathcal{R}$.

Proposition 9.3 *Suppose $X \subseteq R^m$ is definable in $\mathcal{R}$ and $g : X \to N^k$ is definable in the T_{c}-model $(\mathcal{R}, \mathcal{N}, \mathrm{st})$. Then $\dim_{\mathcal{R}} X \geq \dim_{\mathcal{N}} g(X)$.*

Proof. Let $(\mathcal{R}', \mathcal{N}', \mathrm{st}')$ be a sufficiently saturated elementary extension of the T_{tame}-model $(\mathcal{R}, \mathcal{N}, \mathrm{st})$, let $g' : X' \to (N')^k$ be the corresponding extension of g, and let V' be the convex hull of N' in $\mathcal{R}'$. We can assume that $X \neq \emptyset$. The desired inequality will follow from

$$\mathrm{rk}(\mathcal{R}\langle x\rangle|\mathcal{R}) \geq \mathrm{rk}(\mathcal{N}\langle g(x)\rangle|\mathcal{N})\,\text{for all}\, x \in X',$$

since $\dim_{\mathcal{R}} X$ is the maximum of the ranks on the left and $\dim_{\mathcal{N}} g(X)$ is the maximum of the ranks on the right, with x ranging over X'. To prove the rank inequality, let $x \in X'$ and note that

$$(\mathcal{R},\mathcal{N},\mathrm{st}) \;\preceq\; (\mathcal{R}\langle x\rangle, \mathrm{st}'(V' \cap R\langle x\rangle), \mathrm{st}'|R\langle x\rangle) \quad (\text{as}\, L_{\mathrm{c}} - \text{structures}).$$

Hence $g(x)$ must have its coordinates in $\mathrm{st}'(V' \cap R\langle x\rangle)$. Thus the rank inequality follows from Lemma (5.3) in [10]. □

This proof requires the two-sorted L_{c}-setting, since $\mathrm{st}'(V' \cap R\langle x\rangle)$ might not be a subset of $R\langle x\rangle$. Also, the T_{tame}-analogue of the result is false: take some $b > N$ in R, and define $g : R \to N^2$ by $g(x) = (\mathrm{st}x, \mathrm{st}(b(x - \mathrm{st}x)))$. Let $s, t \in N$ and put $a = s + \frac{t}{b}$, so that $g(a) = (s,t)$; since s and t were arbitrary, this gives $g(R) = N^2$.

If $Y \subseteq R^n$ is definable in $\mathcal{R}$, then $\mathrm{st}(Y \cap V^n) \subseteq N^n$ is definable in $\mathcal{N}$ (by Proposition 8.1). We make this more precise by specifying parameters sufficient for defining $\mathrm{st}(Y \cap V^n)$ in $\mathcal{N}$ in terms of parameters used in defining Y in $\mathcal{R}$. As in [10] we consider the prime model $\mathcal{P} = (P, \ldots)$ of T as an elementary submodel of any model of T that gets mentioned.

Lemma 9.4 *Suppose $Y \subseteq R^n$ is definable in $\mathcal{R}$ over $P\langle a\rangle$, $a \in R^m$. Then* $\mathrm{st}(Y \cap V^n)$ *is definable in $\mathcal{N}$ over* $\mathrm{st}(V \cap P\langle a\rangle)$.

Proof. By considering a defining formula for Y it is easy to check that the subset $\mathrm{st}(Y \cap V^n)$ of N^n is defined in the T_{c}-model $(\mathcal{R},\mathcal{N},\mathrm{st})$ by some $L_{\mathrm{c}}(a)$-formula. We now distinguish two cases:

Case 1. $P\langle a\rangle$ is not contained in V. Then $(\mathcal{P}\langle a\rangle, \mathrm{st}(V \cap P\langle a\rangle), \mathrm{st}|P\langle a\rangle)$ is an elementary substructure of the T_{c}-model $(\mathcal{R},\mathcal{N},\mathrm{st})$. Now use that $\mathrm{st}(Y \cap V^n)$ is definable in the latter model using only parameters from the first model.

Case 2. $P\langle a\rangle \subseteq V$. Then we take some $b > V$ in R, and apply Case 1 with $P\langle a, b\rangle$ instead of $P\langle a\rangle$, using also that $\mathrm{st}(V \cap P\langle a, b\rangle) = \mathrm{st}(P\langle a\rangle)$. □

Proof of the dimension inequality. Let some o-minimal structure be given on the ordered field of real numbers; to keep notations simple, just write $\mathcal{R}$ for the expansion of the ordered field of real numbers by all the sets belonging to this structure; let $L = \{0, 1, +, -, \cdot, <, \ldots\}$ be the language of this expansion, and $T = \mathrm{Th}(\mathcal{R})$ its L-theory. So $\mathcal{R}$ is now the prime model of T. Definability is with respect to $\mathcal{R}$ unless specified otherwise.
Next, let $\mathcal{C} \subseteq \mathcal{K}(\mathcal{R}^n)$ be a nonempty definable collection, and let $A \subseteq \mathcal{R}^m$ and $S \subseteq \mathcal{R}^{m+n}$ be definable such that $\mathcal{C} = \{S(a) : a \in A\}$.

Lemma 9.5 *Let $a \in A_{\mathcal{R}}$ be such that $\mathcal{R}\langle a \rangle \subseteq V$ and $S_{\mathcal{R}}(a) \subseteq V^n$. Then* $\mathrm{st}a \in A_{\mathcal{N}}$ *and* $\mathrm{st}(S_{\mathcal{R}}(a)) = S_{\mathcal{N}}(\mathrm{st}a)$.

Proof. Since $\mathcal{R}\langle a \rangle \subseteq V$, the map $\mathrm{st}|\mathcal{R}\langle a \rangle : \mathcal{R}\langle a \rangle \to \mathcal{N}$ is an elementary embedding, hence $\mathrm{st}a \in A_{\mathcal{N}}$. Consider the definable function

$$f : A \times \mathcal{R}^n \to \mathcal{R}, \quad f(x, y) = d(S(x), y),$$

so for each $x \in A$ the function

$$f_x : \mathcal{R}^n \to \mathcal{R}, \quad f_x(y) = f(x, y)$$

is continuous. Hence by [9], Chapter 6, \$ 2, we can partition A into definable subsets $A_1, \ldots, A_k$ such that each restriction

$$f|A_i \times \mathcal{R}^n : A_i \times \mathcal{R}^n \to \mathcal{R}$$

is continuous. Replacing A by the unique A_i for which $a \in (A_i)_{\mathcal{R}}$ and S by $S \cap (A_i \times \mathcal{R}^n)$ we reduce to the case that f is continuous. Let now $b \in S_{\mathcal{R}}(a)$, that is, $f_{\mathcal{R}}(a, b) = 0$. The hypothesis $S_{\mathcal{R}}(a) \subseteq V^n$ together with $a \in V^m$ yields that $(a, b) \in V^{m+n}$. Since $\mathrm{st}(a, b) = (\mathrm{st}a, \mathrm{st}b) \in A_{\mathcal{N}} \times N^n$, we can apply (1.13) of [10] to conclude that $f_{\mathcal{N}}(\mathrm{st}a, \mathrm{st}b) = \mathrm{st}(f_{\mathcal{R}}(a, b)) = \mathrm{st}(0) = 0$, that is, $\mathrm{st}b \in S_{\mathcal{N}}(\mathrm{st}a)$. Conversely, let $y \in S_{\mathcal{N}}(\mathrm{st}a)$. Since $\mathrm{st}y = y$ we have $f_{\mathcal{N}}(\mathrm{st}a, \mathrm{st}y) = 0$. Reversing the arguments above we obtain that $f_{\mathcal{R}}(a, y)$ is $\mathcal{N}$-infinitesimal. By the definition of f this means that for some $z \in S_{\mathcal{R}}(a)$ the distance $d_{\mathcal{R}}(y, z)$ is $\mathcal{N}$-infinitesimal. Thus $y = \mathrm{st}z \in \mathrm{st}(S_{\mathcal{R}}(a))$ as desired.
□

We can now finish the proof that $\dim(\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C}) < \dim \mathrm{cl}(\mathcal{C})$. Take a sufficiently saturated model $(\mathcal{R}, \mathcal{N}, \mathrm{st})$ of T_{tame}. Besides this model we also have the model $(\mathcal{R}, \mathbb{R}, \mathrm{st}_{\mathbb{R}})$ of T_{tame}. Let $V_{\mathbb{R}}$ be the convex hull of $\mathbb{R}$ in $\mathcal{R}$. From $\mathrm{cl}(\mathcal{C}) = \{\mathrm{st}_{\mathbb{R}}(S_{\mathcal{R}}(a)) : a \in A_{\mathcal{R}},\ S_{\mathcal{R}}(a) \subseteq V_{\mathbb{R}}^n\}$, we obtain

$$\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C} \ = \ \{\mathrm{st}_{\mathbb{R}}(S_{\mathcal{R}}(a)) : a \in A_{\mathcal{R}},\ S_{\mathcal{R}}(a) \subseteq V_{\mathbb{R}}^n\} \ \setminus \ \{S(a) : a \in A\}.$$

Proposition 9.2 yields definable sets $A' \subseteq \mathbb{R}^k$ and $S' \subseteq \mathbb{R}^{k+n}$ such that $b \mapsto S'(b) : A' \to \mathcal{K}(\mathbb{R}^n)$ is injective and $\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C} = \{S'(b) : b \in A'\}$, hence

$$\{\mathrm{st}_{\mathbb{R}}(S_{\mathcal{R}}(a)) : a \in A_{\mathcal{R}},\ S_{\mathcal{R}}(a) \subseteq V_{\mathbb{R}}^n\} \setminus \{S(a) : a \in A\} \ = \ \{S'(b) : b \in A'\}.$$

It remains to show that $\dim A' < \dim A$. The last displayed equality can be expressed as an elementary property of the structure $(\mathcal{R}, \mathbb{R}, \mathrm{st}_{\mathbb{R}})$. Since $(\mathcal{R}, \mathcal{N}, \mathrm{st}) \equiv (\mathcal{R}, \mathbb{R}, \mathrm{st}_{\mathbb{R}})$ it follows that $\{\mathrm{st}(S_{\mathcal{R}}(a)) :\ a \in A_{\mathcal{R}},\, S_{\mathcal{R}}(a) \subseteq V^n\}$ $\setminus \{S_{\mathcal{N}}(a) : a \in A_{\mathcal{N}}\} = \{S'_{\mathcal{N}}(b) : b \in A'_{\mathcal{N}}\}$. Let $b \in A'_{\mathcal{N}}$ and take $a \in A_{\mathcal{R}}$ such that $S_{\mathcal{R}}(a) \subseteq V^n$ and $\mathrm{st}(S_{\mathcal{R}}(a)) = S'_{\mathcal{N}}(b)$. By Lemma 9.5 and the fact that $A_{\mathcal{N}} \subseteq \mathrm{st}A_{\mathcal{R}}$, it follows that

$$\mathrm{rk}(\mathrm{st}(V \cap \mathbb{R}\langle a \rangle)) \ < \ \mathrm{rk}(\mathbb{R}\langle a \rangle).$$

Thus by Lemma 9.4 the set $S'_{\mathcal{N}}(b)$ is definable in $\mathcal{N}$ over $\mathbb{R}\langle c \rangle$ where c is a tuple from $\mathcal{N}$ with $\mathrm{rk}(\mathbb{R}\langle c \rangle) < \dim A$. In particular, $S'_{\mathcal{N}}(b) = S'_{\mathcal{N}}(d)$ for some $d \in A'_{\mathbb{R}\langle c \rangle}$, and hence $b = d$, so $\mathrm{rk}(\mathbb{R}\langle b \rangle) < \dim A$. It follows that $\dim A' < \dim A$, as desired.

10 Hausdorff measure: a nonstandard approach

In this section we shall prove Theorem 4.2 by nonstandard methods, the basics of which can be found for example in [6], 4.4.

Let Z be a proper space. In the nonstandard setting we have $Z \subseteq {}^*Z$ with the standard map $\mathrm{st} : {}^*Z \to Z \cup \{\infty\}$, and for each $X \subseteq Z$ a corresponding ${}^*X \subseteq {}^*Z$ with ${}^*X \cap Z = X$. We shall use that for $K \subseteq Z$,

$$K = \mathrm{st}({}^*K) \iff K \,\text{is compact}.$$

The metric $d : Z \times Z \to \mathbb{R}$ extends to an *internal metric* $d : {}^*Z \times {}^*Z \to {}^*\mathbb{R}$. To keep notation simple this internal metric is indicated by d instead of the more correct ${}^*d$. To each set $X \subseteq Z$ is assigned its boundary bd$X$ (a closed subset of Z) consisting of the points $a \in Z$ such that for each $\epsilon \in \mathbb{R}^{>0}$ there exist $x \in X$ and $z \in Z \setminus X$ with $d(a,x) < \epsilon$ and $d(a,z) < \epsilon$. Similarly, to each *internal* set $Y \subseteq {}^*Z$ is assigned its internal boundary bd$Y$ (an *internally closed* subset of ${}^*Z$) consisting of the points $a \in {}^*Z$ such that for each $\epsilon \in {}^*\mathbb{R}^{>0}$ there exist $y \in Y$ and $z \in {}^*Z \setminus Y$ with $d(a,y) < \epsilon$ and $d(a,z) < \epsilon$. (The notation *bd$Y$ for this set would have been more correct, but we want to cut down on asterisks; ambiguity does not arise because if Y is both an internal subset of ${}^*Z$ and a subset of Z, then Y is finite, so bd$Y = Y$ in both readings of bd$Y$.) We let $A \triangle B$ denote the symmetric difference $(A \setminus B) \cup (B \setminus A)$ of sets A and B.

Lemma 10.1 *Let Y be an internal subset of ${}^*Z$ contained in ${}^*B(a,r)$ for some $a \in Z$ and real $r > 0$. Then* $\mathrm{st}(Y \triangle {}^*\mathrm{st}Y) \subseteq \mathrm{st}(\mathrm{bd}Y)$.

Here we write ${}^*\mathrm{st}Y$ for ${}^*(\mathrm{st}Y)$ in order to cut down on parentheses.
Proof. First, consider a point $p \in Y \setminus {}^*\mathrm{st}Y$; we have to show that $\mathrm{st}p = \mathrm{st}q$ for some $q \in \mathrm{bd}Y$. Suppose there is a positive real ϵ such that

$$ {}^*\overline{B}(\mathrm{st}p, \epsilon) \subseteq Y. $$

Take such an ϵ, and note that then

$$ \overline{B}(\mathrm{st}p, \epsilon) = \mathrm{st}({}^*\overline{B}(\mathrm{st}p, \epsilon)) \subseteq \mathrm{st}Y, $$

and thus $p \in {}^*\overline{B}(\mathrm{st}p, \epsilon) \subseteq {}^*\mathrm{st}Y$, a contradiction. The non-existence of such an ϵ implies that the set

$$ D = \{\delta \in {}^*\mathbb{R} : d(p,x) = \delta \,\text{for some}\, x \in {}^*Z \setminus Y\} $$

contains for each positive real ϵ an element $\delta \leq \epsilon$. Hence the infimum δ_0 of D in ${}^*\mathbb{R}$ is infinitesimal. By the internal properness of ${}^*Z$ there is $q \in \mathrm{bd}Y$ such that $d(p,q) = \delta_0$, and then $\mathrm{st}p = \mathrm{st}q$ as desired.
Next, consider a point $p \in {}^*\mathrm{st}Y \setminus Y$; we have to find $q \in \mathrm{bd}Y$ such that $\mathrm{st}p = \mathrm{st}q$. We have $\mathrm{st}p \in \mathrm{st}({}^*\mathrm{st}Y) = \mathrm{st}Y$, so $\mathrm{st}p = \mathrm{st}y$ with $y \in Y$, that is, $d(p,y)$ is infinitesimal. Since $p \notin Y$, the same argument as before with s instead of p yields a $q \in \mathrm{bd}S$ such that $d(q,y)$ is infinitesimal. Then $\mathrm{st}p = \mathrm{st}y = \mathrm{st}q$ as desired. $\square$

Conventions for the rest of this section. Each $\mathbb{R}^m$ is given its usual euclidean norm $|\cdot|$ and euclidean metric d; the corresponding internal norm and metric on $^*\mathbb{R}^m$ are also denoted by $|\cdot|$ and d; the standard map from $^*\mathbb{R}^m$ onto $\mathbb{R}^m \cup \{\infty\}$ is denoted by st. For $k \in \{0, \ldots, m\}$ the k-dimensional Hausdorff measure $\mathcal{H}^k$ on $\mathbb{R}^m$ assigns to each Borel set $X \subseteq \mathbb{R}^m$ a number $\mathcal{H}^k(X) \in \mathbb{R} \cup \{\infty\}$. Its nonstandard extension to $^*\mathbb{R}^m$ assigns to each *internally Borel* set $Y \subseteq {}^*\mathbb{R}^m$ an element $\mathcal{H}^k(Y) \in {}^*\mathbb{R} \cup \{\infty\}$; the notation $^*\mathcal{H}^k(Y)$ would be more correct, but for the usual reasons we omit the asterisk here. For $k = m$ this is just the Lebesgue measure $\mathcal{L}$, written $\mathcal{L}^m$ if we need to indicate the dependence on m. Throughout, $e \in \{0, \ldots, n\}$.
We assume that some o-minimal structure is given on the ordered field $\mathbb{R}$, and for simplicity we just write $\mathbb{R}$ for the expansion of $\mathbb{R}$ by the sets in this structure. We also work in the elementary extension $^*\mathbb{R}$ of $\mathbb{R}$ that the nonstandard setting provides to us. Definability is with parameters and either in the structure $\mathbb{R}$ or in the structure $^*\mathbb{R}$, which ever fits the context.

In the next two results $\mathcal{L}$ denotes Lebesgue measure on $\mathbb{R}^n$.

Lemma 10.2 *Let $Y \subseteq {}^*\mathbb{R}^n$ be definable and suppose that $Y \subseteq {}^*B(0, r)$ for some positive real r. Then $\mathcal{L}(Y \triangle {}^*\mathrm{st}Y))$ is infinitesimal, and thus*

$$\mathrm{st}(\mathcal{L}Y) = \mathcal{L}(\mathrm{st}Y).$$

Proof. Put $B = \mathrm{st}(\mathrm{bd}Y)$, so B is compact, definable, and $\dim B < n$, and hence $\mathcal{L}B = 0$. Let $B_k = \{x \in \mathbb{R}^n : d(x, B) \leq 1/k\}$ for $k = 1, 2, \ldots$, so $B_1, B_2, \ldots$ is a descending sequence of compact subsets of $\mathbb{R}^n$ with intersection B. Hence $\mathcal{L}(B_k) \to 0$ as $k \to \infty$. By the previous lemma $Y \triangle {}^*\mathrm{st}Y$ is a subset of $^*B_k$ for $k = 1, 2 \ldots$, so $\mathcal{L}(Y \triangle {}^*\mathrm{st}Y) \leq \mathcal{L}({}^*B_k) = \mathcal{L}(B_k)$ for $k = 1, 2, \ldots$. Letting k tend to ∞ shows that $\mathcal{L}(Y \triangle {}^*\mathrm{st}Y)$ is infinitesimal. $\square$

We can now prove the special case $e = n$ of Theorem 4.2:

Proposition 10.3 *Let $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ be definable, and let (X_k) be a sequence in $\mathcal{C}$ converging in the Hausdorff metric to $X \in \mathcal{K}(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty} \mathcal{L}(X_k) = \mathcal{L}(X).$$

Proof. Take a real $r > 0$ such that $X_k \subseteq B(0,r)$ for all k and $X \subseteq B(0,r)$. The sequence $(\mathcal{L}(X_k))$ is bounded, so it suffices to show that the limit of each convergent subsequence is equal to $\mathcal{L}(X)$. Let $(\mathcal{L}(X_{k(i)}))$ be a convergent subsequence. By familiar nonstandard principles this yields $Y \in {}^*\mathcal{C}$ such that $Y \subseteq {}^*B(0,r)$, $\lim_{i\to\infty} \mathcal{L}(X_{k(i)}) = \mathrm{st}(\mathcal{L}Y)$, and $\mathrm{st}Y = X$. The previous lemma then gives $\lim_{i\to\infty} \mathcal{L}(X_{k(i)}) = \mathcal{L}(X)$. $\square$

Here is another part of Theorem 4.2.

Proposition 10.4 *Let $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ be definable, and let (X_k) be a sequence in $\mathcal{C}$ with $\dim X_k \le e$ for all k, such that (X_k) converges in the Hausdorff metric to $X \in \mathcal{K}(\mathbb{R}^n)$. Then*

$$\mathcal{H}^e(X) \ge c \cdot \limsup_{k\to\infty} \mathcal{H}^e(X_k),$$

where $c = c(n,e,\mathcal{C})$ is the positive real constant of Lemma 4.3.

Proof. By Proposition 4.1 the sequence $(\mathcal{H}^e(X_k))$ is bounded, so by passing to a subsequence we reduce to the case that this sequence converges. Take a real $r > 0$ such that $X_k \subseteq B(0,r)$ for all k and $X \subseteq B(0,r)$. By familiar nonstandard principles we obtain $Y \in {}^*\mathcal{C}$ such that

$$Y \subseteq {}^*B(0,r), \quad \lim_{k\to\infty} \mathcal{H}^e(X_k) = \mathrm{st}(\mathcal{H}^e(Y)), \quad \mathrm{st}Y = X.$$

Lemma 4.3 yields $p \in {}^*O(n,e) \subseteq {}^*\mathbb{R}^{n\times n}$ such that

$$\mathcal{L}^E(pY) \ge c \cdot \mathcal{H}^d(Y), \quad E = p({}^*\mathbb{R}^n).$$

Since $O(n,e) \subseteq \mathbb{R}^{n\times n}$ is compact, we obtain $q \in O(n,e)$ at infinitesimal distance to p under the natural inclusion $O(n,e) \hookrightarrow {}^*O(n,e)$. Then $\mathrm{st}(pY) = q(\mathrm{st}Y) = qX$, and thus $\mathrm{st}(\mathcal{L}^E(pY)) = \mathcal{L}^F(qX)$ by Lemma 10.2, where $F = q(\mathbb{R}^n)$. In view of the last inequality, this gives $\mathcal{L}^F(qX) \ge c \cdot \mathrm{st}(\mathcal{H}^e(Y))$. To finish, use that $\mathcal{H}^e(X) \ge \mathcal{L}^F(qX)$. $\square$

We now turn to the other inequality of Theorem 4.2:

Proposition 10.5 *Let $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ be definable, and let (X_k) be a sequence in $\mathcal{C}$ with $\dim X_k \le e$ for all k, such that (X_k) converges in the Hausdorff metric to $X \in \mathcal{K}(\mathbb{R}^n)$. Then*

$$\mathcal{H}^e(X) \le \liminf_{k\to\infty} \mathcal{H}^d(X_k).$$

This will be derived from the following analogue of Lemma 10.2:

Lemma 10.6 *If $Y \subseteq {}^*\mathbb{R}^n$ is definable,* $\dim_{^*\mathbb{R}} Y \le e$ *and* $Y \subseteq {}^*B(0,r)$ *for some real* $r > 0$*, then* $\mathcal{H}^e(\mathrm{st}Y) \le \mathrm{st}(\mathcal{H}^e(Y))$*.*

In general we do not have equality here, in contrast to Lemma 10.2: take $e = 1$, $n = 2$, and $Y = ({}^*[0,1]) \times \{0, \delta\}$ where $\delta \in {}^*\mathbb{R}^{>0}$ is infinitesimal. Then $\mathrm{st}Y = [0,1] \times \{0\}$, so $\mathcal{H}^1(\mathrm{st}Y) = 1 < 2 = \mathcal{H}^1(Y)$.

Proof of Proposition 10.5 from Lemma 10.6. Take a real $r > 0$ such that $X_k \subseteq B(0,r)$ for all k and $X \subseteq B(0,r)$. Passing to a subsequence of (X_k) we can assume that $\lim_{k\to\infty} \mathcal{H}^e(X_k)$ exists. By familiar nonstandard principles this yields $Y \in {}^*\mathcal{C}$ such that $Y \subseteq {}^*B(0,r)$, $\lim_{k\to\infty} \mathcal{H}^e(X_k) = \mathrm{st}(\mathcal{H}^e(Y))$, and $\mathrm{st}Y = X$. The desired inequality now follows from Lemma 10.6.

Proof of Lemma 10.6. This proof will take some effort, and for the rest of this section n and $e \in \{0, \dots, n\}$ are fixed, and Y is as in the hypothesis of Lemma 10.6. We shall use the following obvious fact to reduce to a case where Y has a simple form.

Lemma 10.7 *If Y is the disjoint union of definable subsets $Y_1, \dots, Y_k$, and* $\mathcal{H}^e(\mathrm{st}Y_i) \le \mathrm{st}(\mathcal{H}^e(Y_i))$ *for* $i = 1, \dots, k \in \mathbb{N}$*, then* $\mathcal{H}^e(\mathrm{st}Y) \le \mathrm{st}(\mathcal{H}^e(Y))$*.*

The idea is now to reduce to the case where Y is the graph of a nice map ϕ and to express its e-dimensional Hausdorff measure as the Lebesgue measure of an internal subset of ${}^*\mathbb{R}^{e+1}$, so that we can use Lemma 10.2. We first discuss this reduction to Lebesgue measure in the standard setting.

Let $U \subseteq \mathbb{R}^e$ be open and nonempty, and let $\phi : U \to \mathbb{R}^{n-e}$ be a C^1-map. Then $\mathrm{graph}(\phi) \subseteq \mathbb{R}^n$ is an embedded C^1-submanifold of $\mathbb{R}^n$ of dimension e. Note that $\mathrm{graph}(\phi) = \psi(U)$, where

$$\psi : U \to \mathbb{R}^n, \quad \psi(u) = (u, \phi(u)), \text{ with } u = (u_1, \dots, u_e).$$

Hence by [20], p. 48:

$$g(u) \ge 0 \text{ for all } u \in U, \qquad \mathcal{H}^e(\mathrm{graph}(\phi)) = \int_U \sqrt{g}\, d\mathcal{L}^e$$

where $g = \det(g_{ij})$ and the matrix $(g_{ij})_{i,j=1,\dots,e}$ has entries $g_{ij} : U \to \mathbb{R}$ given by $g_{ij}(u) = \frac{\partial \psi}{\partial u_i}(u) \cdot \frac{\partial \psi}{\partial u_j}(u)$ (dot product of vectors in $\mathbb{R}^n$). Thus,

$$\mathcal{H}^e(\text{graph}(\phi)) = \mathcal{L}^{e+1}(\{(u,t) \in U \times \mathbb{R} : 0 \le t \le \sqrt{g(u)}\}).$$

Because of the boundedness assumption in Lemma 10.2 we make a further reduction to *flat* graphs. Given $\epsilon \in \mathbb{R}^{>0}$ and $M \subseteq \mathbb{R}^n$, we say that M is (e,ϵ)-*flat*, if $M = \text{graph}(\phi)$ with $\phi : U \to \mathbb{R}^{n-e}$ a (necessarily unique) C^1-map on open $U \subseteq \mathbb{R}^e$ such that $|\frac{\partial \phi_i}{\partial u_j}(u)| < \epsilon$ for all $i, j = 1, \dots, e$ and $u \in U$. To arrange flat graphs we use a uniform version of the next lemma:

Lemma 10.8 *Let $\epsilon \in \mathbb{R}^{>0}$. Then there are $s_1, \dots, s_q \in O(n)$, $q \in \mathbb{N}^{>0}$, with the following property: if $X \subseteq \mathbb{R}^n$ is definable and $\dim X \le e$, then X is the disjoint union of definable subsets $X_0, X_1, \dots, X_k$, $k \in \mathbb{N}$, such that $\dim X_0 < e$, and such that for each $i \in \{1, \dots, k\}$ there is $j \in \{1, \dots, q\}$ with (e,ϵ)-flat $s_j(X_i)$.*

This is close to the subanalytic Proposition 1.4 of [16]. The proof there is easily adapted to give Lemma 10.8, and also works in ${}^*\mathbb{R}$:

Let $\epsilon \in \mathbb{R}^{>0}$. Then there are $s_1, \dots, s_q \in O(n)$ as in Lemma 10.8, with the following additional property: Y is the disjoint union of definable subsets $Y_0, Y_1, \dots, Y_k$, $k \in \mathbb{N}$, with $\dim Y_0 < e$, and such that for each $i \in \{1, \dots, k\}$ there is $j \in \{1, \dots, q\}$ with internally (e,ϵ)-flat $s_j(Y_i)$.

(This yields a uniformity in Lemma 10.8, namely, when X in that lemma varies in a definable family, then k can be taken to depend only on that family and not on the particular member X.) In view of Lemma 10.7 we can now assume that we have a real $\epsilon > 0$ such that Y is internally (e,ϵ)-flat. We claim that then

$$(\triangle) \qquad \text{st}(\mathcal{H}^e(Y)) = \mathcal{H}^e(\text{st}Y).$$

It is clear that $(\triangle)$ yields Lemma 10.6 and adds further precision to it. We have $Y = \text{graph}(\Phi)$ with $\Phi : U \to {}^*\mathbb{R}^{n-d}$ a definable C^1-map on definable open $U \subseteq {}^*\mathbb{R}^e$ such that $|\frac{\partial \Phi_i}{\partial u_j}(u)| < \epsilon$ for all $i, j = 1, \dots, e$ and $u \in U$.

To prove ($\triangle$) we need four more lemmas. The first two involve path length. A *path in* $\mathbb{R}^m$ is a continuous map $\gamma : [a,b] \to \mathbb{R}^m$, $(a, b \in \mathbb{R}, a < b)$, and its *length*, denoted $\mathrm{length}(\gamma)$, is the supremum of the sums

$$\sum_{i=1}^{k} d(\gamma(a_{i-1}), \gamma(a_i))$$

taken over all finite sequences $a = a_0 < a_1 < \ldots < a_k = b$. A path γ is said to be *rectifiable* if $\mathrm{length}(\gamma) < \infty$. Note that for γ as above we have $\mathrm{length}(\gamma) \geq d(\gamma(a), \gamma(b))$. We also define the *diameter* of a path $\gamma : [a,b] \to \mathbb{R}^m$ to be the diameter of the subspace $\gamma([a,b])$ of $\mathbb{R}^m$.

Lemma 10.9 *Given a real $r > 0$ there is a real $c > 0$ depending only on n and r, such that if $M \subseteq \mathbb{R}^n$ is (e,r)-flat and $\Gamma : [a,b] \to M$ is a path, then*

$$\mathrm{length}(\gamma) \leq c \cdot \mathrm{length}(p \circ \Gamma)$$

where $p : \mathbb{R}^n \to \mathbb{R}^e$ is given by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_e)$.

It was shown in [12] that definable paths in $\mathbb{R}^m$ are rectifiable. A variant of this result says, roughly speaking, that definable paths of infinitesimal diameter have infinitesimal length:

Lemma 10.10 *Suppose the internal path $\gamma : {}^*[a,b] \to {}^*\mathbb{R}^m$ is definable and its internal diameter is infinitesimal. Then the internal length of γ is infinitesimal.*

Proof. Let $\gamma = (\gamma_1, \ldots, \gamma_m)$, and take $a = a_0 < a_1 < \ldots < a_k = b$ such that $\gamma_1, \ldots, \gamma_m$ are monotone on ${}^*[a_{i-1}, a_i]$ for $i = 1, \ldots, k \in \mathbb{N}$. Now use [12], C.16, part (1). $\square$

For open $C \subseteq \mathbb{R}^e$, put

$$C^{\mathrm{h}} = \mathrm{st}^{-1} C = \{x \in {}^*\mathbb{R}^e : \mathrm{st} x \in C\}, \quad (\text{the hull of } C),$$

so $C \subseteq C^{\mathrm{h}} \subseteq {}^*C$.

Lemma 10.11 *Let $P \subseteq {}^*\mathbb{R}^e$ be definable such that $P \subseteq {}^*B(0,r)$ for some real $r > 0$. Then we have a finite disjoint union*

$$\mathrm{st}P = C_1 \cup \ldots \cup C_k \cup D$$

where $C_1, \ldots, C_k, D \subseteq \mathbb{R}^e$ are definable, each C_j is an open cell in $\mathbb{R}^e$ with $C_j^{\mathrm{h}} \subseteq P$, and $\dim D < e$.

Proof. By Propositions 8.1 and 9.3, the set $\mathrm{st}P$ and its subset $\mathrm{st}(\mathrm{bd}P)$ are definable in $\mathbb{R}$, and $\dim(\mathrm{st}(\mathrm{bd}P)) < e$, so we have a disjoint union $\mathrm{st}P = C_1 \cup \ldots \cup C_k \cup D$ where $C_1, \ldots, C_k, D$ are definable, each C_j is an open cell in $\mathbb{R}^e$ and $\dim D < e$, $\mathrm{st}(\mathrm{bd}P) \subseteq D$. We claim that then $C_j^{\mathrm{h}} \subseteq P$ for each j. To see why, first note that $C_j^{\mathrm{h}} \subseteq {}^*C_j \subseteq {}^*\mathrm{st}P$. Let $x \in C_j^{\mathrm{h}}$, and suppose that $x \notin P$. Then $x \in ({}^*\mathrm{st}P) \setminus P$, so $\mathrm{st}x \in \mathrm{st}(\mathrm{bd}P)$ by Lemma 10.1, contradicting $\mathrm{st}x \in C_j$. □

Lemma 10.12 *Let $r \in \mathbb{R}^{>0}$, and let $P \subseteq {}^*\mathbb{R}^e$ and $f : P \to {}^*\mathbb{R}$ be definable such that P is open in ${}^*\mathbb{R}^e$, $P \subseteq {}^*B(0,r)$ and $|f(u)| < r$ for all $u \in P$. Then we have a finite disjoint union*

$$\mathrm{st}P = C_1 \cup \ldots \cup C_k \cup D$$

as in Lemma 10.11, and for each cell C_j a continuous definable function $f_j : C_j \to \mathbb{R}$, such that

$$\mathrm{st}(f(u)) = f_j(\mathrm{st}u), \quad \text{for all}\, u \in C_j^{\mathrm{h}}.$$

If in addition f is (internally) of class C^1 and $|\frac{\partial f}{\partial u_i}(u)| < r$ for $i = 1, \ldots, e$ and all $u \in P$, then we can arrange also that each f_j is of class C^1, and

$$\mathrm{st}(\frac{\partial f}{\partial u_i}(u)) = \frac{\partial f_j}{\partial u_i}(\mathrm{st}u), \quad \text{for all}\, u \in C_j^{\mathrm{h}}.$$

Proof. The first part of this lemma combines Proposition 1.7 in [11] and Lemma 10.11. Suppose now that f is of class C^1 and $|\frac{\partial f}{\partial u_i}(u)| < r$ for $i = 1, \ldots, e$ and all $u \in P$. After suitably subdividing $C_1, \ldots, C_k$ and increasing

D, Proposition 1.7 in [11] also yields for each open C_j and each $i = 1, \ldots, e$ a continuous definable function $h_{ij} : C_j \to \mathbb{R}$, such that

$$\mathrm{st}(\frac{\partial f}{\partial u_i}(u)) = h_{ij}(\mathrm{st}u), \quad \text{for all } u \in C_j^{\mathrm{h}}.$$

Let $i \in \{1, \ldots, e\}$ and $j \in \{1, \ldots, k\}$, and let f_j be as in the first part of the lemma; we claim that then f_j is of class C^1 and $h_{ij} = \frac{\partial f_j}{\partial u_i}$ on C_j. To see this, let the vector $v_i \in \mathbb{R}^e$ have ith component 1 and the other components equal to zero. Let $a \in C_j$ and take a real $\delta > 0$ such that $a + \lambda v_i \in C_j$ for all real λ with $|\lambda| \leq \delta$. Then by the Mean Value Theorem there is for each such λ a $\mu \in {}^*\mathbb{R}$ with $|\mu| \leq \lambda$ such that $f_j(a + \lambda v_i) - f_j(a) = \mathrm{st}(f(a + \lambda v_i) - f(a)) = \mathrm{st}(\lambda \cdot \frac{\partial f}{\partial u_i}(a + \mu v_i)) = \lambda \cdot \mathrm{st}(\frac{\partial f}{\partial u_i}(a + \mu v_i)) = \lambda \cdot h_{ij}(a + \mathrm{st}(\mu) v_i)$. Letting λ tend to 0, this yields $\frac{\partial f_j}{\partial u_i}(a) = h_{ij}(a)$. Since the h_{ij} are continuous, it follows that f_j is of class C^1. $\square$

Proof of $(\triangle)$**.** Recall that $Y = \mathrm{graph}(\Phi) = \Psi(U)$, where

$$\Psi : U \to {}^*\mathbb{R}^n, \quad \Psi(u) = (u, \Phi(u)), \quad u = (u_1, \ldots, u_e),$$

and thus $G(u) \geq 0$ for all $u \in U$ where $G = \det(G_{ij})$ and the matrix $(G_{ij})_{i,j=1,\ldots,e}$ has entries $G_{ij} : U \to {}^*\mathbb{R}$ given by $G_{ij}(u) = \frac{\partial \Psi}{\partial u_i}(u) \cdot \frac{\partial \Psi}{\partial u_j}(u)$ (dot product of vectors in ${}^*\mathbb{R}^n$). By our earlier considerations,

$$\mathcal{H}^e(Y) = \mathcal{L}^{e+1}(\{(u,t) \in U \times {}^*\mathbb{R} : 0 \leq t \leq \sqrt{G(u)}\}).$$

Hence by Lemma 10.2,

$$\mathrm{st}(\mathcal{H}^d(Y)) = \mathcal{L}^{d+1}(\{(u,t) \in \mathrm{st}U \times \mathbb{R} : u \in U, 0 \leq t \leq h(u)\},$$

where $h : \mathrm{st}U \to \mathbb{R}$ is given by $h(u) =$least real number $\geq \mathrm{st}(\sqrt{G(u')})$ for all $u' \in U$ with $\mathrm{st}u' = u$. By Lemma 10.12 we have a finite disjoint union $\mathrm{st}U = C_1 \cup \ldots \cup C_m \cup D$ where $C_1, \ldots, C_m, D$ are definable, each C_k is an open cell in $\mathbb{R}^e$ such that $C_k^{\mathrm{h}} \subseteq U$, $\dim D < e$, and for each cell C_k we have a a definable map $\phi_k : C_k \to \mathbb{R}^{n-e}$, of class C^1, such that $\mathrm{st}(\Phi(u)) = \phi_k(\mathrm{st}u)$ for all $u \in C_k^{\mathrm{h}}$, $\mathrm{st}(\frac{\partial \Phi}{\partial u_i}(u)) = \frac{\partial \phi_k}{\partial u_i}(\mathrm{st}u)$, for all $u \in C_k^{\mathrm{h}}$ and $i = 1, \ldots, e$. Hence $\mathcal{H}^e(\mathrm{graph}(\phi_k)) = \mathcal{L}^{e+1}(\{(u,t) \in C_k \times \mathbb{R} : 0 \leq t \leq h(u)\})$ for $k = 1, \ldots, m$.

Also $\mathcal{L}^{e+1}(\{(u,t) \in D \times \mathbb{R} : 0 \le t \le h(u)\}) = 0$ since $\dim D < e$. In combination with the earlier expression for $\mathrm{st}(\mathcal{H}^e(Y))$ this gives

$$\mathrm{st}(\mathcal{H}^e(Y)) = \sum_{k=1}^{m} \mathcal{H}^e(\mathrm{graph}(\phi_k)).$$

Put $X = \mathrm{st}Y$, so by the above we have the disjoint union

$$X = \mathrm{graph}(\phi_1) \cup \ldots \cup \mathrm{graph}(\phi_m) \cup E$$

where $E := \{x \in X : p(x) \in D\}$, with $p : \mathbb{R}^n \to \mathbb{R}^e$ the projection map given by $p(x_1, \ldots, x_n) = (x_1, \ldots, x_e)$. Hence the claim ($\triangle$) reduces to showing that $\mathcal{H}^e(E) = 0$, that is, $\dim E < e$. We shall prove the following stronger claim:

$$(\triangle\triangle) \quad \text{for each } u \in \mathbb{R}^e \text{ the set } p^{-1}(u) \cap E \text{ is finite.}$$

Towards a contradiction, assume we have a point $u \in \mathbb{R}^e$ for which $p^{-1}(u) \cap E$ is infinite. Let $P = {}^*p : {}^*\mathbb{R}^n \to {}^*\mathbb{R}^e$, $P(x_1, \ldots, x_n) = (x_1, \ldots, x_e)$. By saturation we have an infinitesimal ball $B = {}^*B(u, \delta) \subseteq {}^*\mathbb{R}^e$ (with infinitesimal radius $\delta \in {}^*\mathbb{R}^{>0}$) such that each point of $p^{-1}(u) \cap E$ is at infinitesimal distance from some point in $P^{-1}(B) \cap Y$. Then one of the finitely many definably connected components of $P^{-1}(B) \cap Y$ contains points x and y such that $d(x, y)$ is not infinitesimal. Therefore we have a definable internal path $\Gamma : {}^*[0,1] \to p^{-1}(B) \cap Y$, such that $d(\Gamma(0), \Gamma(1))$ is not infinitesimal, and thus the internal length of Γ is not infinitesimal. But the projection $P \circ \Gamma : {}^*[0,1] \to B$ of this path has infinitesimal internal diameter, hence infinitesimal internal length by Lemma 10.10, contradicting Lemma 10.9.
This concludes the proof of ($\triangle\triangle$), and hence of ($\triangle$), and therefore of Lemma 10.6, and thus of Theorem 4.2.

11 Pointwise limits

Recall the standard example showing the difference between pointwise and uniform convergence: for $0 < \epsilon \le 1$, let $f_\epsilon : \mathbb{R} \to [0,1]$ be given by $f_\epsilon(x) = |x|/\epsilon$ if $|x| \le \epsilon$ and $f_\epsilon = 1$ if $|x| \ge \epsilon$. Let $g : \mathbb{R} \to [0,1]$ be the function defined by $g(0) = 0$ and $g(x) = 1$ for $x \neq 0$. Then, as $\epsilon \to 0$, f_ϵ converges to g pointwise but not uniformly. On the "tame" side, the function $(\epsilon, x) \mapsto$

$f_\epsilon(x) : (0, 1] \times \mathbb{R} \to [0, 1]$ is continuous and semialgebraic, the limit function g is semialgebraic (though not continuous), and the convergence is uniform on compact subsets of $\mathbb{R} \setminus \{0\}$. In this section we show that in the o-minimal setting all definable families of functions behave like this family (f_ϵ) when considering pointwise limit functions.

Let B be a set and K a compact space. We make the set K^B of all functions $f : B \to K$ into a topological space by giving it the product topology. (This topological space is compact Hausdorff, but not metrizable if B is uncountable and $|K| > 1$.) The closure of a set $\mathcal{F} \subseteq K^B$ in K^B is denoted by $\mathrm{cl}_t(\mathcal{F})$ and called its *Tychonov closure*. Note that if $f_k : B \to K$ for $k = 1, 2, 3, \ldots$ and $f : B \to K$, then $\lim_{k\to\infty} f_k = f$ iff $\lim_{k\to\infty} f_k(b) = f(b)$ for each $b \in B$.
In what follows we adopt the nonstandard setting of the previous section; in particular, we have a standard map $\mathrm{st} : {}^*K \to K$. This allows us to give a useful characterization of the Tychonov closure:

Lemma 11.1 *Let $\mathcal{F} \subseteq K^B$ and $g : B \to K$. Then*

$$g \in \mathrm{cl}_t(\mathcal{F}) \iff \text{there exists } f \in {}^*\mathcal{F} \text{ such that } g(b) = \mathrm{st} f(b) \text{ for all } b \in B.$$

Proof. Let $g \in \mathrm{cl}_t(\mathcal{F})$. Then there exists for any positive real ϵ and finite $B_0 \subseteq B$ a function $f \in \mathcal{F}$ such that $d(f(b), g(b)) < \epsilon$ for all $b \in B_0$. Thus by saturation there is a function $f \in {}^*\mathcal{F}$ as required. The converse is equally straighforward. $\square$

For the rest of this section we also adopt the *conventions* of the previous section, fixing in particular an o-minimal expansion of the ordered field of real numbers, and writing $\mathbb{R}$ for this expansion and ${}^*\mathbb{R}$ for the corresponding elementary extension provided by the nonstandard setting.
We shall consider pointwise limits of functions that vary in a definable family. For the rest of this section $B \subseteq \mathbb{R}^n$ is definable and K is a definable compact subspace of some euclidean space. A collection $\mathcal{F} \subseteq K^B$ of functions is said to be *definable* if $\mathcal{F} = \{f_a : a \in A\}$ for some definable set $A \subseteq R^m$ and some definable map $f : A \times B \to K$, where for $a \in A$ the function $f_a : B \to K$ is given by $f_a(y) = f(a, y)$. (Section 8 dealt with the case $B = \mathbb{R}^n$ and $K = \{0, 1\} \subseteq \mathbb{R}$.)

Theorem 11.2 *Let $\mathcal{F} \subseteq K^B$ be a definable collection of functions. Then*

(1) $\mathrm{cl}_t(\mathcal{F})$ *is a definable collection of functions* $B \to K$.

(2) *Each function in* $\mathrm{cl}_t(\mathcal{F})$ *is the limit of a sequence* (f_k) *in* $\mathcal{F}$.

Proof. Let $\mathcal{F} = \{f_a : a \in A\}$ with definable $f : A \times B \to K$ as above. Then ${}^*f : {}^*A \times {}^*B \to {}^*K$ yields $\mathrm{cl}_t(\mathcal{F})$ as follows: to each $x \in {}^*A$ we associate the function

$$f_x : B \to K, \quad f_x(b) = \mathrm{st}({}^*f(x, b)),$$

so f_x is definable (in $\mathbb{R}$) by Proposition 8.1. Also, $\mathrm{cl}_t(\mathcal{F}) = \{f_x : x \in {}^*A\}$ by Lemma 11.1, so (1) follows as in the proof of part (1) of Proposition 8.3. Item (2) follows from a sharper result in Proposition 11.4 below. □

Of course, (2) would follow if $\mathrm{cl}_t(\mathcal{F})$ with the topology induced by K^B were metrizable. Solecki showed me the following non-metrizable example.

Example. Take $B = [0, 1]$, $K = \{0, 1\}$, and define $f : [0, 1] \times B \to K$ by $f(x, y) = 1$ if $y \le x$ and $f(x, y) = 0$ if $y > x$. The corresponding $\mathcal{F}$ consists of the characteristic functions of the segments $[0, x]$ with $0 \le x \le 1$, so

$$\mathrm{cl}_t(\mathcal{F}) = \mathcal{F} \cup \{\text{characteristic functions of the sets } [0, x) \text{ with } 0 < x \le 1\}.$$

Then $\mathrm{cl}_t(\mathcal{F})$ with the topology induced by K^B is not metrizable.

For (2) we shall need the next lemma, which is basically [11], Proposition 1.7, suitably reformulated to fit our purpose. (For the semialgebraic case, see also [8], p. 70.) For $C \subseteq \mathbb{R}^n$ we put [8]

$$C^{\mathrm{h}} = \{y \in {}^*C : \mathrm{st}(y) \in C\}, \quad (\text{the hull of } C).$$

Lemma 11.3 *Let* $\phi : {}^*B \to {}^*K$ *be definable. Then we can partition* B *into (definable) cells* $C_1, \ldots, C_k$, $k \in \mathbb{N}$, *and choose for each cell* C_i *a definable continuous function* $\phi_i : C_i \to K$ *such that*

$$\mathrm{st}(\phi(y)) = {}^*\phi_i(\mathrm{st}(y)), \quad \text{for all } y \in C_i^{\mathrm{h}}.$$

Proposition 1.7 in [11] deals only with open cells; a non-open cell can be dealt with inductively by means of a definable homeomorphism with an open cell. Lemma 11.3 is the key to:

[8] When C is open in $\mathbb{R}^n$, this agrees with the notations in [8] and [11], but for non-open C there can be a difference.

Proposition 11.4 *Let $\mathcal{F} = \{f_a : a \in A\}$ be a definable collection of functions $B \to K$, where $f : A \times B \to K$ is as before. Then*

(1) *For each $g \in \mathrm{cl}_t(\mathcal{F})$ there is a definable curve $\gamma : (0,1] \to A$ such that $g = \lim_{t\to 0} f_{\gamma(t)}$ in K^B.*

(2) *If $\gamma : (0,1] \to A$, is definable, then $g = \lim_{t\to 0} f_{\gamma(t)}$ exists in K^B, and there is a partition of B into finitely many (definable) cells such that on each compact subset of each of the cells the convergence of $f_{\gamma(t)}$ to g as $t \to 0$ is uniform.*

Proof. Let $g \in \mathrm{cl}_t(\mathcal{F})$, and take ${}^*a \in {}^*A$ such that $g(y) = \mathrm{st}({}^*f({}^*a, y))$ for all $y \in B$. By Lemma 11.3 we can partition B into (definable) cells $C_1, \ldots, C_k$, $k \in \mathbb{N}$, and take a definable continuous function $\phi_i : C_i \to K$ for $i = 1, \ldots, k$ such that $\mathrm{st}({}^*f({}^*a, y)) = \phi_i(\mathrm{st}(y))$ for all $y \in C_i^{\mathrm{h}}$. Thus $\phi_i = g|_{C_i}$ for $i = 1, \ldots, k$.

Claim. Given any compact sets $D_1 \subseteq C_1, \ldots, D_k \subseteq C_k$ and any positive real number ϵ, there is $a \in A$ such that $|f(a,y) - g(y)| < \epsilon$ for all $y \in D_1 \cup \ldots \cup D_k$.

To see why this claim holds, let $i \in \{1, \ldots, k\}$, and note that ${}^*D_i \subseteq C_i^{\mathrm{h}}$. By the continuity of $\phi_i = g|_{C_i}$ we have for all $y \in {}^*D_i$,

$$\mathrm{st}({}^*f({}^*a, y)) = \phi_i(\mathrm{st} y) = \mathrm{st}({}^*\phi_i(y)) = \mathrm{st}({}^*g(y)),$$

hence $|{}^*f({}^*a, y) - {}^*g(y)| < \epsilon$ for $y \in {}^*D_i$. This establishes the claim.
Using a definable homeomorphism of C_i with some euclidean space, it is easy to construct for each $i \in \{1, \ldots, k\}$ a definable curve of compacta that exhausts C_i, that is, a definable set $D_i \subseteq (0,1] \times C_i$ such that

- for each $\epsilon \in (0,1]$ the section $D_i(\epsilon) \subseteq C_i$ is compact,
- whenever $0 < \epsilon < \delta \leq 1$, then $D_i(\epsilon) \supseteq D_i(\delta)$,
- each compact subset of C_i is contained in some $D_i(\epsilon)$.

By the claim, and by definable choice, there is a definable map $\gamma : (0,1] \to A$ such that for each $\epsilon \in (0,1]$,

$$|f(\gamma(\epsilon), y) - g(y)| < \epsilon, \text{ for all } y \in D_1(\epsilon) \cup \ldots \cup D_k(\epsilon).$$

In particular, $g = \lim_{t\to 0} f_{\gamma(t)}$ as promised. This finishes the proof of (1).
For (2), let $\gamma : (0,1] \to A$ be definable, and define $g : B \to K$ by $g(y) = \lim_{t\to 0} f(\gamma(t), y)$, so $g = \lim_{t\to 0} f_{\gamma(t)}$ in K^B. Take some positive infinitesimal $\tau \in {}^*\mathbb{R}$ and put ${}^*a = {}^*\gamma(\tau)$. As in the proof of (1) and using its notations we obtain a partition of B into cells $C_1, \ldots, C_k$ such that the claim above holds with $a = \gamma(t)$, where we can take $t \in (0,1]$ less than any preassigned positive real. Without loss of generality we can take $D_1, \ldots, D_k$ in the claim to be definable, hence by by o-minimality the claim holds with $a = \gamma(t)$ for all sufficiently small $t \in (0,1]$. This establishes (2). $\square$

There is also a uniform version of this proposition, similar to part (2) of Proposition 3.2. We leave this to the reader.

12 Appendix

Let an o-minimal structure be given on the ordered field of real numbers, and let $\mathcal{C} \subseteq \mathcal{K}(\mathbb{R}^n)$ be a nonempty definable collection. Part (3) of Theorem 3.1 says that

$$\dim(\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C}) < \dim(\mathcal{C}).$$

After I mentioned this inequality in a talk in Luminy, Andrei Gabrielov indicated to me the *geometric* proof below; some readers may prefer it over the model-theoretic treatment in Section 9. See also [17] for a geometric proof of the full Theorem 3.1.

Geometric proof of the dimension inequality. Take definable $A \subseteq \mathbb{R}^m$ and definable $S \subseteq A \times \mathbb{R}^n$ such that

$$\mathcal{C} = \{S(a) : a \in A\} \subseteq \mathcal{K}(\mathbb{R}^n).$$

By shrinking A we may assume that $a \mapsto S(a) : A \to \mathcal{C}$ is bijective. We may also assume that $A \subseteq \mathbb{R}^m$ is bounded. Choose definable $T \subseteq B \times \mathbb{R}^n$ with definable $B \subseteq \mathbb{R}^k$ such that $\mathrm{cl}(\mathcal{C}) \setminus \mathcal{C} = \{T(b) : b \in B\}$. Again, we do this in such a way that $B \subseteq \mathbb{R}^k$ is bounded and

$$b \mapsto T(b) : B \to \mathrm{cl}(\mathcal{C}) \setminus \mathcal{C}$$

is bijective. By [9], Chapter 9, we have a partition of B into definable locally closed nonempty subsets $B_1, \ldots, B_N$ ($B_i \neq B_j$ for $i \neq j$), such that each of

the projection maps $T \cap (B_i \times \mathbb{R}^n) \to B_i$ is definably trivial. In particular, for each $b_0 \in B_i$ we have $\lim_{b \to b_0, b \in B_i} T(b) = T(b_0)$ in $\mathcal{K}(\mathbb{R}^n)$. Let $\epsilon > 0$, $1 \leq i \leq N$, and put $B_i(\epsilon) = \{b \in B_i : d(b, \partial B_i) \geq \epsilon\}$ if $\partial B_i \neq \emptyset$, and otherwise, put $B_i(\epsilon) = B_i$; then $B_i(\epsilon)$ is compact. We now focus attention on one of those sets $B_i(\epsilon)$, and call it C for simplicity. It suffices to show that then $\dim C < \dim A$: fixing i and taking the union over $\epsilon = 1/p$ with $p = 1, 2, 3, \ldots$ (and appealing to, say, Baire's theorem) it would follow that then $\dim(B_i \setminus \partial B_i) < \dim A$, and hence $\dim(B_i) < \dim A$, and thus taking the union over $i \in \{1, \ldots, N\}$ we obtain $\dim B < \dim A$.
Define $\delta : A \to \mathbb{R}$ by

$$\delta(a) = \min_{c \in C} d_H(S(a), T(c))$$

and choose a definable function $\gamma : A \to C$ such that for all $a \in A$,

$$\delta(a) = d_H(S(a), T(\gamma(a))).$$

Next, let $S' \subseteq A \times C \times \mathbb{R} \times \mathbb{R}^n$ be the set of all points $(a, \gamma(a), \delta(a), x)$ with $(a, x) \in S$, and let $\Gamma \subseteq A \times C \times \mathbb{R}$ be the graph of the function $a \mapsto (\gamma(a), \delta(a)) : A \to C \times \mathbb{R}^{>0}$. Let $\overline{S'}$ and $\overline{\Gamma}$ be the closures of S' and Γ in $\mathbb{R}^{m+k+1+n}$ and $\mathbb{R}^{m+k+1}$ respectively, and let Δ be the subset of $\overline{\Gamma}$ consisting of the points with last coordinate 0, so $\Delta \subseteq \partial\Gamma \subseteq \overline{A} \times C \times \mathbb{R}$. We claim:

(1) Let (a_p, c_p, δ_p) $(p \in \mathbb{N})$ be a sequence in Γ converging to $(a, c, 0) \in \Delta$; then $S(a_p) \to T(c)$ as $p \to \infty$.

(2) Let $(a, c, 0) \in \Delta$; then $\overline{S'}(a, c, 0) = T(c)$.

(3) $\{T(c) : c \in C\} \subseteq \{\overline{S'}(\beta) : \beta \in \Delta\}$.

Item (1) follows by noting that as $p \to \infty$ we have

$$d_H(S(a_p), T(c_p)) = \delta_p \to 0 \text{ and } d_H(T(c_p), T(c)) \to 0.$$

For (2), let $x \in \overline{S'}(a, c, 0)$. Then we have a sequence $(a_p, c_p, \delta_p, x_p)$ $(p \in \mathbb{N})$ in S' converging to $(a, c, 0, x)$. Hence $x_p \in S(a_p)$ and $x_p \to x$ as $p \to \infty$. Hence $x \in T(c)$ by item (1). Conversely, let $x \in T(c)$. Take a sequence (a_p, c_p, δ_p) $(p \in \mathbb{N})$ in Γ converging to to $(a, c, 0)$. Then (1) gives us a sequence (x_p) with each $x_p \in S(a_p)$ such that $x_p \to x$ as $p \to \infty$. Hence the sequence $(a_p, c_p, \delta_p, x_p)$ $(p \in \mathbb{N})$ in S' converges to $(a, c, 0, x) \in \overline{S'}$, and thus $x \in$

$\overline{S'}(a,c,0)$. For item (3), let $c \in C$. Then there is a sequence (a_p) $(p \in \mathbb{N})$ in A such that $S(a_p) \to T(c)$ as $p \to \infty$. Put $c_p = \gamma(a_p)$ and $\delta_p = \delta(a_p)$. Note that then $0 < \delta_p \le d_H(S(a_p), T(c))$, so $\delta_p \to 0$ and $T(c_p) \to T(c)$ as $p \to \infty$. Compactness of C and the continuity properties of the family $(T(c))_{c \in C}$ implies that then $c_p \to c$ as $p \to \infty$. Passing to a subsequence if necessary we may assume that also $a_p \to a$ for some $a \in \overline{A}$ as $p \to \infty$. Hence $(a_p, c_p, \delta_p) \to (a, c, 0) \in \Delta$, and by (2) we have $T(c) = \overline{S'}(a,c,0)$ as desired.

From (3) above we obtain that $\dim C \le \dim \Delta$. Since $\dim \Delta < \dim \Gamma = \dim A$, this gives $\dim C < \dim A$ as promised.

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