Chapter 1

Introduction

For many physical problems the models derived by discretization are often composed of thousands, or even millions of differential equations. Despite the continuous increase of computational power, the complexity of these models generally makes it not feasible, or even downright impossible, to work with them. In that case it becomes inevitable to apply so-called model order reduction methods, which generate a lower-dimensional approximation of the original system. These methods have been developed for many years and allow handling of a wide variety of problems, both linear and nonlinear. They have proven themselves invaluable in practice.

In recent years parametric problems have come further into the focus of research. These systems are not fixed, but depend on a variety of parameters, e.g. material properties such as stiffness. In that case it is impractical, depending on the purpose infeasible, to apply model order reduction for every new parameter choice. Parametric model order reduction aims to provide a model of reduced dimension that retains the parametric dependencies and behaves similarly to the original model. In recent years various methods have been developed to handle parametric model reduction. While many of these methods have been applied with great success in practice, due to the wide variety of problems and objectives there are still numerous challenges to tackle.

The initial problem that inspired this thesis was such a parametric problem: a set of connected models with up to 32 different parameters of interest. Simplifications of the model led to a new one that depended on 4 different parameters, but for the desired small reduction size many parametric model reduction methods did not deliver acceptable results. The exception to this was parametric model reduction by matrix interpolation, where the system is reduced for sampling points and the interpolation of their system matrices yields the reduced parametric system. Because of this problem the focus for most of the doctorate was on studying the method (as well as the general framework) of matrix interpolation. For various problems, both academic and real-world, tests showed that preservation of stability is a pressing issue, for which only one procedure exists in literature. Unfortunately it is limited to interpolation with nonnegative interpolation weights and may lead to bad interpolation results.

The main focus of this thesis is twofold: to introduce criteria and ways to ensure better results for the existing stability procedure and to develop new approaches for stability preservation for parametric model reduction by matrix interpolation. The problem of meaningful stability preservation will be shown and two new approaches that preserve stability will be introduced. One approach of particular interest is based on the idea of adapting the locally reduced models in such way, that stability for matrix interpolation is preserved. In addition another new approach will be given, dubbed subspace extension approach, that uses extended local bases to obtain better interpolation results.

This thesis is organized as follows. The first chapter introduces various well-known definitions and results from system theory that are relevant for (the understanding of) this thesis, such as the definition of linear-time invariant systems, stability, or reachability and observability. It is followed by a brief introduction of maybe the two most common model reduction methods for linear time-invariant systems, balanced truncation and moment-matching. In addition to the necessary definitions and theorems, sketches of algorithms are given that allow the implementation of these procedures, as well as a simple example from literature that shows their usefulness. The fourth chapter then introduces existing methods from literature for parametric model order reduction (for linear time-invariant systems), such as multiparameter moment-matching, proper orthogonal decomposition or the eponymous parametric model reduction by matrix interpolation.

The fifth chapter is the core of this thesis and focuses on the problem of stability preservation for parametric model reduction (within the framework of matrix interpolation). In this chapter the only method from literature for stability preservation for matrix interpolation is considered. For this procedure, which is based on Lyapunov equations, examples are given to highlight some of its fundamental problems and new results are given that may yield better results. In addition, a new obvious procedure is introduced, where systems are brought into upper diagonal form. It is explained why said procedure is not feasible for practical problems. Further it is shown that under certain restrictions the reduced system obtained by model order reduction can be expressed in a feedback form. This expression is subsequently used to preserve stability under matrix interpolation. The chapter is closed by a brief look into stability preservation for matrix interpolation for descriptor systems, both the case of nonsingular and the case of singular descriptor matrices. The sixth chapter follows this by introducing two methods from literature that allow for a more meaningful interpolation if different local bases are used. Based on those two methods a new procedure is derived, called subspace extension approach, that extends local bases in an attempt to obtain better results for matrix interpolation. Furthermore interpolation over manifolds is introduced from literature and the general problem of interpolation weights and sampling points is briefly tackled. The seventh chapter examines the procedures for stability preservation for two parametric problems: a cantilever beam and an anemometer. It is shown that while the new stabilization procedure by feedback may generally lead to better results, the second example shows that in some cases its results may be limited by the approximation of the local systems. The second example also shows the potential use of the subspace extension approach. The thesis is closed by a conclusion.

Chapter 2

Mathematical Background

While system theory is a general term for a multitude of scientific fields, in maths it is usually used to describe the theory surrounding dynamical systems, i.e. mathematical constructs that are not static, but change over time. Of particular interest for this thesis are continuous dynamical systems in the context of control theory, where the term is used for systems whose dynamics are described by differential equations of a specific structure. Over the past decades both system and control theory have developed rapidly, to an extent where covering the theory even only for the systems of interest is beyond the scope of this thesis. Instead in the following a brief introduction to the numerous necessary definitions, as well as certain relevant results, is given. Readers interested in this field are referred e.g. to [42, 43, 71], whereas those interested in a focus on the directly related control theory are also recommended to examine e.g. [76, 35, 59].

2.1 Linear Time-Invariant System

There are numerous concepts for and different approaches towards dynamical systems in general. The focus of this thesis however lies on linear time-invariant systems, defined as follows.

Definition 2.1.
The linear time-invariant (LTI) descriptor system Σ is defined as the system governed by the differential equations

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) && (2.1) \\ y(t) &= Cx(t) + Du(t), && (2.2) \end{aligned}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{k\times n}$ and $D \in \mathbb{R}^{k\times l}$.

$x(t) \in \mathbb{R}^n$ is called the state of the system, $y(t) \in \mathbb{R}^k$ the output of the system and $u(t) \in \mathbb{R}^l$ the input of the system. In addition A is called the state matrix, B the input matrix, C the output matrix, D the input-output matrix and E the descriptor matrix. Equation (2.1) *is also called the state equation.*

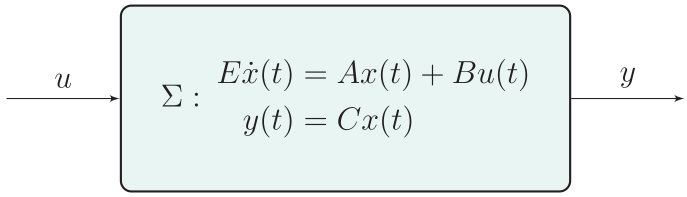

Figure 2.1: The schematic of a LTI descriptor system.

This important class of systems, in the following simply called LTI systems, allows the mathematical modeling and simulation of a plethora of different dynamics and has significant practical relevance. The state $x \in \mathbb{R}^n$ governed by (2.1) is influenced by the input u, which e.g. could be a force applied on the system. The output y usually is the interaction of the system with the outside, governed by (2.2), depends both on x as well as the input u. For many systems, one may encounter that $D = 0_{k \times l}$; since in addition D is unaffected by model order reduction, as will be seen later, in the following it is assumed w.l.o.g. that $D = 0$. While this thesis generally assumes real matrices, the results and procedures given in this thesis can usually be extended to complex systems without difficulty.

Oftentimes systems only have a single input and a single output, i.e. $k = l = 1$, in which case they are called single-input-single-output (SISO) systems, else multiple-input-multiple-output (MIMO) systems. For ease of use, in the following $\Sigma = (E, A, B, C)$ is used as an abbreviation for systems as in Definition 2.1.

These systems are particularly interesting for $rank(E) < n$, as then there are algebraic constraints that need to be satisfied. Unless mentioned otherwise, this thesis restricts itself to the case $rank(E) = n$, which is easier to handle. It should be noted that many results and definitions from this chapter can be extended to arbitrary linear descriptor systems.

To avoid confusion, $\phi(t, t_0, x_0, u)$ is used in the following to describe the solution of the state equation, i.e. the state of the system at time t under the input u and with initial state x_0 at time t_0. For a system as in Definition 2.1 it can be given as

$$\phi(t, t_0, x_0, u) = e^{E^{-1}A(t-t_0)}x_0 + \int_{t_0}^{t} e^{E^{-1}A(t-\tau)}E^{-1}Bu(\tau)d\tau, \quad t \geq t_0$$

and the corresponding output $\mu(t, t_0, x_0, u) = C\phi(t, t_0, x_0, u)$ follows accordingly.

Of specific interest for these systems, schematically given in Figure 2.1, is their input-output behavior, e.g. one would rather be interested in a good approximation of the output $\mu(t, t_0, x_0, u) = C\phi(t, t_0, x_0, u)$, than of $\phi(t, t_0, x_0, u)$. For the general examination of this input-output behavior it has proven to be very useful to investigate the system for a frequency s instead of time t. This requires the switch from the time domain to the frequency domain, which is done via the Laplace transform.

Definition 2.2 ([35]).
Let $f \in \mathcal{C}^0([0,\infty))$ be such that for $m \in \mathbb{R}$ and $0 < k < \infty$ holds

$$|f(t)| < ke^{mt}, \;\; \forall\, t \geq 0,$$

then for $s \in \mathbb{C}$ its Laplace transform $F(s)$ is defined as

$$\mathcal{L}[t \mapsto f(t)](s) = F(s) := \int_{0^-}^{\infty} e^{-st} f(t) dt.$$

Remark 2.3.
It is easy to see for the Laplace transform that for $s \in \mathbb{C}$ and f as in Definition 2.2 holds

$$\mathcal{L}\left[t \mapsto \frac{df(t)}{dt}\right](s) = sF(s) - f(0).$$

If applied to systems from Definition 2.1, the input-output behavior over the frequency domain can be given in a rather simple form as follows.

Definition 2.4 ([35]).
Let $\Sigma = (E, A, B, C)$ be an LTI system with $x(0) = 0$, then for $s \in \mathbb{C}\setminus\{\Lambda(E, A)\}$, i.e. it is not a generalized eigenvalue of the pair (E, A), the transfer function H of the system for zero initial value is defined as

$$H(s) := C(sE - A)^{-1}B,$$

which maps the frequency domain input $U(s)$ onto the frequency domain output $Y(s)$, i.e.

$$H(s)U(s) = Y(s).$$

The transfer function is a very simple, yet elegant way to obtain the input-output behavior in the frequency space and therefore useful to examine the general behavior of the system. In addition, it is of great importance for model order reduction.

2.2 Stability

The concept of stability of a system is at the very core of this thesis and while there are many different concepts of stability for systems in general, the two most common and important concepts for LTI systems are stability and asymptotic stability.

Definition 2.5 ([75]).
Let $\Sigma = (E, A, B, C)$, then the system is stable if for any two states x_1, x_2 belonging to the same input u there exists a value $M \in \mathbb{R}_+$ s.t.

$$\|x_1(t) - x_2(t)\| < M, \forall\ t \in \mathbb{R}_+.$$

It is called asymptotically stable if additionally

$$lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0.$$

While different norms can be chosen for this definition, if not specified otherwise, the Euclidean norm is commonly chosen.

Definition 2.5 states that a system is stable if for any two states (under the same input) the distance of the state trajectories remains bounded as $t \to \infty$, whereas asymptotic stability demands that under the same input the system, no matter the initial state, converges to the same state as $t \to \infty$, which in case of $u \equiv 0$ would be the zero state $0 \in \mathbb{R}^n$. While these definitions are reasonable, they are not very handy to investigate the system. The following result however gives another criterion for stability and asymptotic stability, that is often used as an alternative definition.

Proposition 2.6 ([75]).
Let $\Sigma = (E, A, B, C)$ with $rank(E) = n$, then the following holds:

1. *Σ is stable if and only if for any generalized eigenvalue $\lambda \in \Lambda(E, A)$ it holds that $Re(\lambda) \leq 0$ and if $Re(\lambda) = 0$ then for λ the geometric multiplicity is equal to its algebraic multiplicity*

2. *Σ is asymptotically stable if and only if for any $\lambda \in \Lambda(E, A)$ holds $Re(\lambda) < 0$.*

This relation between the state matrix and (asymptotic) stability gives an algebraic, simple way to handle stability. In addition to this, in practice one may encounter systems that are unstable for physical reasons, or even as a result of the discretization. In that case part of control theory is to find an appropriate input u that stabilizes the system.

Definition 2.7 ([75]).
The system $\Sigma = (E, A, B, C)$ (or just (A, B) if $E = I$) is called stabilizable (by state feedback) if there exists a matrix $F \in \mathbb{R}^{l\times n}$ such that $\Lambda(E, A + BF) \subset \mathbb{C}_-$.

While stabilization may be very difficult for nonlinear systems, both theoretically and practically, at least in theory it is not problematic for linear systems; interested readers are referred e.g. to [76].

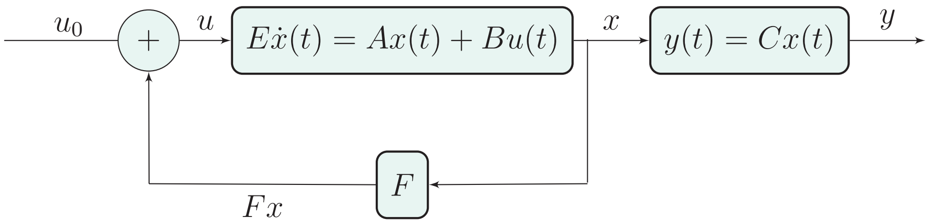

Figure 2.2: Sketch of a state-feedback loop.

2.3 Observability and Reachability

Aside from stability, two other important properties that are corner stones of (linear) system theory are reachability and observability, the former is also important for stabilization by state-feedback as in Figure 2.2.

Definition 2.8 ([75]).
Let $\Sigma = (E, A, B, C)$, t_0 be fixed, $X \subset \mathbb{R}^n$ be the subspace of feasible initial states and U the set of feasible inputs. The state $x_1 \in \mathbb{R}^n$ is called reachable from $x_0 \in X$ if there exists $u \in U, \tau > 0$ s.t.

$$\phi(t_0 + \tau, t_0, x_0, u) = x_1.$$

Equivalently, one says that x_0 can be controlled to x_1.

The state x_1 can be distinguished from x_0 if there exists $u \in U, \tau > 0$ and $t_0 \leq t \leq t_0 + \tau$ s.t.

$$\mu(t, t_0, x_0, u) \neq \mu(t, t_0, x_1, u).$$

The set of feasible (or admissible) initial states and input functions may depend on the specific problem, else a common choice is $\mathbb{R}^n$ for the initial states and the set of continuous functions $\mathcal{C}^1(\mathbb{R})$ (or commonly also piecewise continuous functions) for the input functions. With these definitions, one can give the definition of controllability (or reachability) and observability.

Definition 2.9 ([75]).
Let $\Sigma = (E, A, B, C)$, then the system is called reachable if for any $x_0, x_1 \in \mathbb{R}^n$ there exists $u \in U$ s.t. $\phi(t_0 + \tau, t_0, x_0, u) = x_1$. For E nonsingular, with $\tilde{A} := E^{-1}A$ and $\tilde{B} := E^{-1}B$, the reachability matrix of Σ is defined as