1 Basics - Superconductivity and Josephson effect

1.1 Short introduction to superconductivity

Electric resistivity of metals yields from the interaction of conduction electrons with lattice imperfections and phonons[1]. When lowering the temperature the number of thermal activated phonons is reduced. Therefore, one expects an increase of conductivity till it is limited only by impurities.

In 1911 Heike Kamerlingh Onnes discovered the "Disappearance of the resistance of mercury" at a temperature slightly above the boiling point of liquid Helium [69]. The effect he had found is superconductivity. It describes a phase transition at a critical temperature T_c that is found in several elements and materials. In table 1.1 the critical temperatures of metals used in this work are displayed.

Metal	T_c (K)	λ_L (nm)
Nb	9.2	32-44
Pb	7.2	32-39
Al	1.19	50

Table 1.1: Values of critical temperature T_c and London penetration depth $\lambda_L(T=0)$ for selected materials (from [70]).

The vanishing of the resistance at low temperatures yields from the pairing of electrons with opposite spin to so called Cooper pairs. It arises from a weak attractive coupling mechanism,

[1]Phonons are quantized lattice vibration. They act as quasiparticles in solid-state physics, since a quasi-momentum (no mass transport) and energy can be assigned to them.

which for conventional superconductors is given by electron phonon interaction. The energy gap for single-particle excitations is found in the theory of Bardeen, Cooper and Schrieffer (BCS) [71] as $2G = 3.5 k_\mathrm{B} T_\mathrm{c}$. Thus, low temperatures are needed to avoid breaking of the pairs by thermal excitation. Because the paired particles have an integral spin, they can be treated as Bosons. The total wave vector of a pair as sum of the electrons wave vectors $\vec{q} = \vec{k}_1 + \vec{k}_2$ is the same for all pairs. This quality enables the Cooper pairs to occupy the same quantum state. A description with only one wave function

$$\Psi(\vec{r}) = \Psi_0 \mathrm{e}^{-\mathrm{i}\chi(\vec{r})}, \tag{1.1}$$

where $\chi(\vec{r})$ is the coordinate dependent phase, becomes possible. This superconducting state is decoupled from the crystal lattice. Individual scattering of electrons cannot change the momentum $\vec{q}$, since it is common to all the Cooper pairs. With the momentum of the charge carriers being a conserved quantity ideal conductivity is achieved.

The BCS theory also gives explanations for various other phenomena connected with superconductivity. For example, the steep change in the specific heat and the Meissner effect are discussed. Latter was experimentally found in 1933 by Meissner and Ochsenfeld [72]. They observed that an external magnetic field is expelled completely from the bulk of a superconductor and, therefore, ideal diamagnetic properties are achieved.

An explanation was firstly given by London and London in 1935 [73]. They developed a phenomenological theory of the electromechanical properties from superconductors. By starting from the equation of motion of a single electron in the Drude model [74]

$$m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} + m\frac{\vec{v}_\mathrm{D}}{\tau} = -e\vec{E}, \tag{1.2}$$

where m is the mass and e the charge of a conduction electron, $\vec{E}$ the electric field, v the velocity , v_D the drift velocity, and τ the mean time to an interaction of an electron with the lattice, some general statements can be deduced. In the steady state $\mathrm{d}\vec{v}/\mathrm{d}t$ is equal to zero and one obtains Ohms law

$$\vec{j}_\mathrm{N} = -en_\mathrm{N}\vec{v}_\mathrm{D} = \frac{n_\mathrm{N} e^2 \tau}{m}\vec{E}, \tag{1.3}$$

with $\vec{j}_\mathrm{N}$ being the current density and n_N the density of charge carriers in a normal conducting metal. A normal conductance $\sigma = ne^2\tau/m$ may be introduced. By assuming the time to an interaction with the lattice τ to be infinite for a superconductor[2] equation (1.2) becomes an "acceleration equation"

[2]This assumption expresses that no interaction with the lattice occurs.

$$\dot{\vec{j}}_{\mathrm{S}} = \frac{n_{\mathrm{S}} q_{\mathrm{S}}^2}{m} \vec{E}. \tag{1.4}$$

Here, the electron charge is replaced by the charge of a cooper pair q_S. A stationary current can be found in a superconducting material even if $\vec{E}$ =0 immediately by integration. After taking the curl, one can substitute the current density by the magnetic field $\vec{H}$ as $\vec{j}_{\mathrm{S}} = \mathrm{curl}\vec{H}$ and $\mathrm{curl}\vec{E} = -\mu_0 \dot{\vec{H}}$ as found from the Maxwell equations[3]. Integration with respect to the time together with the identity $\mathrm{curl}\left(\mathrm{curl}\vec{Y}\right) = \mathrm{grad}(\mathrm{div}\vec{Y}) - \Delta\vec{Y}$ and again one Maxwell equation, $\mathrm{div}\vec{H} = 0$, yields the homogeneous screening relation for the magnetic field

$$\Delta\vec{H} = \frac{1}{\lambda_{\mathrm{L}}^2} \vec{H}. \tag{1.5}$$

This equation includes the Meissner effect. The general solution gives an exponential decay of the magnetic field in a region of size $\lambda_{\mathrm{L}} = \sqrt{m/n_{\mathrm{S}} q_{\mathrm{S}}^2 \mu_0}$, the London penetration depth, from the surface of the superconductor. The supercurrent follows the same exponential decay. A list of λ_{L} for different materials can be found in table 1.1.

The London theory has several limitations. For example, it gives no explanation for the dependence of the London penetration depth on temperature nor on the thickness of a superconducting film. The theory developed by Ginzburg and Landau in 1950 [75] to overcome these problems marked the first complete theoretical explanation of superconductivity and is still commonly used for describing inhomogeneous superconductors. Starting from the basic theory of phase transitions of the second kind[4], they introduced an ordering Parameter Ψ that is zero above the critical temperature. It can be identified with the common wave function for the superconducting charge carriers. The normalization is selected such that the ordering parameter will be connected to the density of superconducting charge carriers $|\Psi|^2 = n_{\mathrm{S}}$. The phase $\chi(\vec{r})$ of this "effective" wave function (1.1) depends on the applied magnetic field due to the vector potential $\vec{A}$. The magnetic field is connected to the superconducting currents by the gradient of Ψ and, therefore, by the phase of the wave function.

1.2 Flux quantization

One important effect for the development and the understanding of superconducting electronics is the quantization of magnetic flux in a closed superconducting loop. As mentioned,

[3] The displacement current is neglected.

[4] Phase transitions without latent heat.

the Ginzburg-Landau theory connects the supercurrent $\vec{j}_S$ to the gradient of the phase $\phi(\vec{r})$ and the vector potential $\vec{A}$. This statement can be expressed by the equation [75]

$$\vec{j}_S = -\frac{iq_S\hbar}{2m}\left(\Psi^* \mathrm{grad}\Psi - \Psi \mathrm{grad}\Psi^*\right) - \frac{q_S^2}{m}|\Psi|^2\vec{A}, \tag{1.6}$$

where q_s denotes the charge and m the mass of the "superconducting electrons".

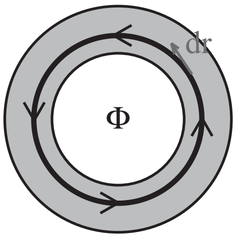

Figure 1-1: Sketch of a thin superconducting ring. A supercurrent represented by the current density j_S will create a magnetic flux in the loop. The line element for the integration in the text is always parallel to the vector of the current density.

Assuming the geometry shown in Fig. 1-1, integration of (1.6) along the closed superconducting ring together with (1.1) yields

$$\oint_{\partial D} \vec{j}_S \,\mathrm{d}\vec{r} = -\oint_{\partial D} \frac{q_S\hbar}{2m}|\Psi_0|^2 \,\mathrm{grad}\chi(r)\,\mathrm{d}\vec{r} - \frac{q_S^2}{m}|\Psi_0|^2 \oint_{\partial D} \vec{A}\,\mathrm{d}\vec{r}.$$

Here, D is the sphere enclosed by the circular integration path. The current density $\vec{j}_S$ can be set to zero, if the integration path is shifted away from the surface of the superconductor, because the supercurrents are located only in a small layer of thickness λ_L. A simplification to

$$\frac{q_S\hbar}{m}\oint \mathrm{grad}\chi\,\mathrm{d}\vec{r} = \frac{q_S^2}{m}\int_D \vec{B}\,\mathrm{d}\vec{F} \tag{1.7}$$

can be found by the use of Stoke's theorem and $|\Psi_0|^2 = n_S$. The integral on the right is equal to the magnetic flux in the loop Φ. The integral on the left side gives the phase difference between the wave function at the start and the end of the integration path. Because both points coincide and the wave function should be single valued, the integral necessarily has to be a multiple of 2π. Therefore, the total flux Φ enclosed by the loop has to be quantized. This quantization is expressed by

$$\Phi = n\frac{h}{q_s}. \tag{1.8}$$

The first experimental observations of quantized flux were reported independently from Doll and Näbauer [76] as well as from Deaver and Fairbank [77] in 1961. The value both groups found for the flux quantum is $\Phi_0 = h/2e$. A comparison with (1.8) shows that the charge of the supercurrent carriers is given by $2e$ and indicates the pairing of electrons.

1.3 The Josephson effect

The main building block of superconducting electronics, and therewith superconducting quantum bits, is the Josephson junction. It is named after B.D. Josephson. In 1962 he predicted "possible new effects" [12] on coupled superconductors by a general perturbation theory, today summarized as Josephson effect. It is found for superconductors separated by a region of weakened superconductivity or by thin layers of conducting or isolating materials. The latter type is sketched in Fig. 1-2 and called tunnel junction. Its non-superconducting layers have a typical thickness of several nanometers. All junctions considered in this work are tunnel junctions with an isolating barrier of aluminum oxide.

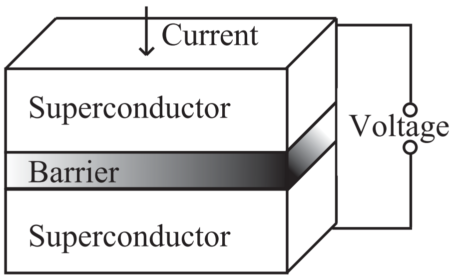

Figure 1-2: Schematic of a Josephson tunnel junction. Two superconductors are connected via an isolating barrier. The current through and the voltage across the junction are defined by its properties.

The electronic properties of a Josephson junction are found by simple considerations [78] assuming two superconductors with wave functions Ψ_1 and Ψ_2 and corresponding eigenenergies $E_{1,2}$. Their dynamic is given by the Schrödinger equation

$$\frac{\partial \Psi_k}{\partial t} = -\frac{\mathrm{i}}{\hbar}(E_k \Psi_k + K\Psi_l). \tag{1.9}$$

Here K is a weak coupling coefficient and the indices $k, l \in [1,2]; k \neq l$. A solution is given by (1.1) for each of the superconductors. Also the normalization of the Ginzburg-Landau theory $|\Psi_k| = \sqrt{n_k}$ can be used. Inserting $\Psi_k = \sqrt{n_k}\mathrm{e}^{\mathrm{i}\chi_k}$ into (1.9) yields

$$\frac{1}{2\sqrt{n_k}}\dot{n}_k + \mathrm{i}\dot{\chi}_k\sqrt{n_k} = -\frac{\mathrm{i}}{\hbar}\left(E_k\sqrt{n_k} + K\sqrt{n_l}\mathrm{e}^{\mathrm{i}[\chi_l - \chi_k]}\right). \tag{1.10}$$

Here the dot indicates a partial time derivative. Under the assumption that two superconductors of the same kind are used ($n_1 = n_2 = n_S$) the real part of the equation multiplied with charge $2e$ gives

$$j_S = 2e\dot{n} = \frac{4en_SK}{\hbar}\sin\varphi = j_c\sin\varphi, \tag{1.11}$$

where $\varphi = \chi_2 - \chi_1$ is the phase difference across the junction and j_c the critical current density. This equation describes the DC-Josephson effect and is known as the first Josephson equation. From it follows that a Josephson junction can carry a superconducting current that is created by the tunneling of Cooper pairs through the barrier. Its value depends on the phase difference across the junction and is limited to a maximum value of j_c. Another effect is found by considering the imaginary parts of (1.10) and subtracting them[5]

$$\dot{\varphi} = \frac{E_2 - E_1}{\hbar} = \frac{2eV}{\hbar}. \tag{1.12}$$

Here, V denotes the voltage across the junction. This equation explains the AC-Josephson effect, which states that a voltage drop at a Josephson junction is connected to a time varying phase difference. Furthermore, by integration of (1.12) and inserting into (1.11) the corresponding AC-current can be identified. Its frequency is given by $\nu = 2eV/h$.

Together with the voltage drop a discussion of further current channels, besides the supercurrent explained by (1.11), becomes necessary at the Josephson junction. It is summarized in the so-called RCSJ (Resistive and Capacitive Shunted Junction)-model, as illustrated in Fig. 1-3. There are two main additional channels to consider for a tunnel junction. On the

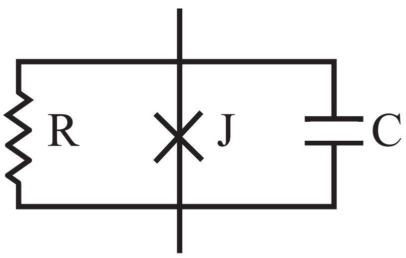

Figure 1-3: Circuit diagram of a Josephson junction in the RCSJ-model.

one hand, the superconducting electrodes together with the isolating barrier form a capacitor and therewith make a displacement current possible. The value of capacitance is given by the material and the size of the junction. A typical value for aluminum oxide barriers is

[5]To clarify, subtracting the equation for $\dot{\chi}_1$ from similar one for $\dot{\chi}_2$.

about 50 fF/μm^2. On the other hand, besides the tunneling of Cooper pairs also quasiparticles can give a contribution to the current flow. Because the propagation of these electrons is connected to losses, one can introduce a normal resistance R_N. This current channel can be neglected in most cases connected to superconducting quantum circuits because the junctions are usually kept in the superconducting state at temperatures well below T_c. Therefore, the quasi particle density can be neglected. In summary, by expressing the voltage with the derivative of the phase at the junction the sum of the currents is given by

$$I = I_\mathrm{c}\sin\varphi + \frac{\Phi_0}{2\pi R_\mathrm{N}}\dot{\varphi} + \frac{\Phi_0}{2\pi}C_\mathrm{J}\ddot{\varphi} \tag{1.13}$$

as firstly proposed in the works by Stewart [79] and McCumber [80].

1.4 Quantum mechanics of a Josephson junction

As described before, superconductivity as well as the Josephson effect are quantum phenomena. But in general also a quantum theory has to be considered for the observables (current and voltage or phase and charge) at the junction [81].

A first step is to find the Hamiltonian and, therefore, start with the energy conservation law on the Josephson junction. It can be found by multiplying (1.13) with the voltage (1.12). Neglecting the dissipative current channels yields

$$IV = \frac{\mathrm{d}}{\mathrm{d}t}\left(E_\mathrm{J}(1-\cos\varphi) + \frac{1}{2}\left(\frac{\Phi_0}{2\pi}\right)^2 C_\mathrm{J}\dot{\varphi}^2\right). \tag{1.14}$$

The Josephson coupling energy E_J is used, and its value is given by

$$E_\mathrm{J} = \Phi_0 I_\mathrm{c}/2\pi. \tag{1.15}$$

The potential $(U(\varphi))$ and kinetic $(E_\mathrm{k}(\dot{\varphi}))$ energy form the Lagrangian $\mathscr{L}(\varphi,\dot{\varphi}) = E_\mathrm{k} - U$, from which the generalized momentum can be derived as

$$p = \frac{\partial\mathscr{L}}{\partial\dot{\varphi}} = \left(\frac{\Phi_0}{2\pi}\right)^2 C_\mathrm{J}\dot{\varphi}, \tag{1.16}$$

whereas the generalized coordinate is given by the phase φ. The Hamiltonian of the system is

$$\mathscr{H} = p\dot{\varphi} - \mathscr{L} = \frac{p^2}{2m} + E_\mathrm{J}(1-\cos\varphi),$$

where the mass is defined as $m = \hbar^2 C_\mathrm{J}/4e^2$. Furthermore, with the relation for the AC-Josephson effect (1.12), one can relate the charge Q to the momentum.

$$Q = C_\mathrm{J} V = C_\mathrm{J} \frac{\Phi_0}{2\pi} \dot{\varphi} = \frac{2e}{\hbar} p \tag{1.17}$$

The quantization is done by substitution the variables with operators. In the phase basis the momentum is $\hat{p} = -\mathrm{i}\hbar\partial/\partial\hat{\varphi}$ and the coordinate $\hat{\varphi}$. Hence, the Hamiltonian reads in the flux basis

$$H = -E_\mathrm{C} \frac{\partial^2}{\partial\hat{\varphi}^2} + E_\mathrm{J}(1 - \cos\hat{\varphi}). \tag{1.18}$$

Here, the symbol is changed to simply H to denote the quantum Hamiltonian and the charging energy E_C at the junction is used as

$$E_\mathrm{C} = \frac{2e^2}{C_\mathrm{J}} \tag{1.19}$$

With the expression for the momentum the commutation relation between the charge $\hat{Q}$ and phase $\hat{\varphi}$ at the junction can be easily found

$$\left[\hat{\varphi}, \hat{Q}\right] = \frac{2e}{\hbar} \left[\hat{\varphi}, \hat{p}\right] = 2\mathrm{i}e. \tag{1.20}$$

Here, the commutation relation[6] $[\hat{\varphi}, \hat{p}] = \mathrm{i}\hbar$ as well as (1.12) and the definition of the flux quantum by (1.8) are used. Because the phase and the charge do not commute, obviously, not both can be well defined at the junction at the same time. The critical parameter is the ratio between E_J and E_C. For example, if $E_\mathrm{J} \gg E_\mathrm{C}$ the phase and therewith the current through the junction are well defined. In this case, the charge degree of freedom can couple different stable phase states as described later.

These quantum effects at the Josephson junction are sometimes called "secondary quantum effects" because superconductivity or the Josephson effect themselves are quantum effects but form the basis for the considerations above.

[6] As found for the given observables by applying the commutator to the wave function (1.1), for example in the phase basis.

2 Theoretical analysis of flux qubits and cavities

2.1 The quantum two-level system

A quantum bit, or qubit, is a physical system containing two distinguishable states. The difference to a classical bit lays in the possibility for both states to exist in a superposition, what yields a statistical probability in the measurement result. This fact is expressed by the equation for the state vector of the qubit

$$|\Psi\rangle = p_{\mathrm{g}}|g\rangle + p_{\mathrm{e}}|e\rangle. \tag{2.1}$$

In this superposition p_n^2 denotes the probability to measure state $|n\rangle$. It can take values between zero and one. The basis state vectors $|g\rangle$ and $|e\rangle$ are normalized and orthogonal. The state Ψ itself should satisfy similar normalization condition, so that $p_{\mathrm{g}}^2 + p_{\mathrm{e}}^2 = 1$. Accordingly, the total probability to measure either state $|g\rangle$ or state $|e\rangle$ is one. For illustration of the superposition of the qubit and therewith operations on the qubit the so called Bloch sphere can be used. It is sketched in Fig. 2-1. The basic states $|g\rangle$ and $|e\rangle$ are located at the poles. Each point on this unit sphere corresponds to a superposition of the basic states. For example, at the equator a perfect superposition with $p_{\mathrm{g}}^2 = p_{\mathrm{e}}^2 = 1/2$ is found. Any operation changing the qubit's state corresponds to a rotation on the Bloch sphere. Furthermore, any of this operations can be composed by rotations around the axis x, y and z, and therefore simply by a linear combination of the Pauli matrices, listed below.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.2}$$

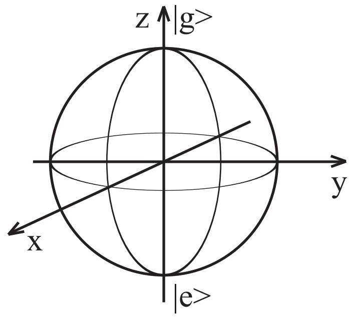

Figure 2-1: Sketch of the Bloch sphere. Each quantum state of the qubit corresponds to a single point on a unit sphere. The basis states $|g\rangle$ and $|e\rangle$ are located on the poles.

Two important examples of such linear combinations are the raising and lowering operators defined by

$$\sigma_{\pm} = \frac{1}{2}\left(\sigma_x \pm \mathrm{i}\sigma_y\right), \tag{2.3}$$

that only transform the ground to excited state and vice versa, respectively. Natural candidates for qubits are trapped ions, nuclear or electronic spins and quantum dots. In contrast, in this work the qubit is formed by a superconducting circuit.

The superconductivity ensures the coherence needed for a quantum system. Furthermore, as shown in chapter 1.4 Josephson junctions can be described using the laws of quantum mechanics. Following the statement in the mentioned chapter the solid-state qubits are distinguished depending on the well-defined quantum variable. For $E_\mathrm{J} \approx 20E_\mathrm{C}$ they are usually called flux qubits [23]. Other types include phase ($E_\mathrm{J} \approx 200E_\mathrm{C}$) and charge qubits [22] ($4E_\mathrm{J} \approx E_\mathrm{C}$) as well as transmons [29] ($E_\mathrm{J} \approx 100E_\mathrm{C}$)[1].

2.2 The superconducting flux qubit

The flux qubit consists of a superconducting loop interrupted by at least one Josephson junction. The Josephson junction needs to be considered when calculating the conditions of the flux quantization in the loop. Namely, the phase difference on the junctions is added to the integration over the gradient of the superconducting phase (first term in (1.8))

[1]Note that definition of the charging energy E_C differs in some works to the one given in this work. For example it can be defined concerning only a single electron instead of a cooper pair.

$$\begin{aligned}\frac{\hbar}{2e}(\varphi+2\pi n) &= \Phi \\ \frac{\varphi}{2\pi} &= \frac{\Phi}{\Phi_0}-n.\end{aligned} \tag{2.4}$$

Therefore, the flux in the loop Φ is directly related to the phase difference φ at the junction. If, for a moment the inductance L_q of the qubit loop is considered, it is easy to find that

$$\Phi = \Phi_\mathrm{e} - L_\mathrm{q} I. \tag{2.5}$$

Here, Φ_e is the externally applied flux and I the current in the qubit loop. Equation (2.5) implies that the external flux is partly compensated by the flux created due to the circulating current I and it follows

$$\frac{\varphi}{2\pi} = \frac{\Phi_\mathrm{e}}{\Phi_0} - \frac{L_\mathrm{q} I}{\Phi_0} - n. \tag{2.6}$$

The effective flux in the loop may be defined as the difference between external flux and the one compensated by the current flowing through the loop inductance $\Phi = \Phi_\mathrm{e} - L_\mathrm{q} I$. Introducing more junctions with phase differences $\tilde{\varphi}_m$ to (2.4) gives

$$\frac{\varphi}{2\pi} = \frac{\Phi}{\Phi_0} - n - \sum_m \frac{\tilde{\varphi}_m}{2\pi}. \tag{2.7}$$

When comparing this equation to (2.6) it is obvious that the additional Josephson junctions have the same influence as the loop inductance [23]. Furthermore, smaller inductances are preferable, since they provide less coupling to the noisy environment (compare section 2.4). Therefore, usually three junctions are fabricated to a low inductance qubit loop. The typical shape of a flux qubit is sketched in Fig. 2-2.

To understand the quantum behavior of a flux qubit, it is necessary to find the corresponding Hamilton operator and therewith the energy level structure. As seen in Fig. 2-2 the standard flux qubit consists of a loop containing three Josephson junctions. Two junctions are designed to have identical size while the one of the third is scaled by a factor $\alpha < 1$. If the inductance of the loop is neglected the potential energy is given by the sum of the Josephson energies (1.14),

$$U(\varphi_1,\varphi_2) = E_\mathrm{J}\left(2+\alpha-\cos\varphi_1-\cos\varphi_2-\alpha\cos\left[2\pi f+\varphi_1-\varphi_2\right]\right). \tag{2.8}$$