



Contents

1	Introduction	1
2	Hyperbolic Balance Laws	7
2.1	Basic Definitions and Properties	7
2.2	Weak Solutions	11
2.3	Entropies	14
3	Numerical Methods for Balance Laws and Summation-by-Parts Operators	17
3.1	Basic Semidiscretisations	17
3.1.1	Summation-by-Parts Operators	17
3.1.2	Boundary Procedures	21
3.1.3	Different Notations and Translation Rules	24
3.2	High Order Discretisations via Flux Differencing	25
3.2.1	Order of Accuracy	27
3.2.2	Conservative Form	29
3.3	Entropy Conserving Schemes	31
3.3.1	Entropy Conservative Two-Point Fluxes	31
3.3.2	Higher Order Entropy Conservative Schemes	35
3.3.3	Conservative Form	38
3.4	Entropy Stable Numerical Fluxes	39
3.4.1	Fully Discrete and Semidiscrete Entropy Stability	39
3.4.2	Godunov's Flux	41
3.4.3	Local Lax-Friedrichs/Rusanov and Harten-Lax-van-Leer Flux	42
3.4.4	Classical Fluxes: Monotone Schemes and E Schemes	44
3.4.5	Entropy Stable Fluxes via Dissipation Operators	46
4	Split Forms and General Summation-by-Parts Operators	49
4.1	Split Form of Burgers' Equation	50
4.2	General Summation-by-Parts Operators for Burgers' Equation	52
4.3	Boundary Conditions	54
4.3.1	Godunov's Flux	55
4.3.2	HLL Flux	56
4.3.3	Local Lax-Friedrichs/Rusanov Flux	57
4.3.4	Summary	58
4.4	Numerical Results	59
4.4.1	Dense Norm Operators	59



4.4.2	Curvilinear Coordinates in One Space Dimension	63
4.4.3	Boundary Conditions	65
4.5	General Polynomial Fluxes	66
4.5.1	Volume Terms	68
4.5.2	Surface Terms	68
4.5.3	Numerical Results for Conservation and Stability	80
4.5.4	Convergence Study	81
4.6	Limits of Split Forms and Generalised SBP Operators	84
4.6.1	An Entropy Conservative Flux not Corresponding to a Splitting	84
4.6.2	Numerical Results	85
4.7	Summary and Conclusions	89
5	Linear Advection with Variable Coefficients	91
5.1	Standard L^2 Estimates	92
5.1.1	Continuous Setting	92
5.1.2	Diagonal Norm Discretisations Including Boundary Nodes	93
5.1.3	General SBP Discretisations	96
5.1.4	Comparison with Results of Nordström and Ruggiu	101
5.2	Weighted L^2 Estimates	103
5.2.1	Continuous Estimates	104
5.2.2	Semidiscrete Estimates	104
5.2.3	Comparison with Results of Manzanero, Rubio, Ferrer, Valero, and Kopriva	108
5.3	Weighted L^2 Estimates for a Nonconservative Equation	108
5.3.1	Continuous Estimates	109
5.3.2	Semidiscrete Estimates	109
5.3.3	Comparison with Results of Manzanero, Rubio, Ferrer, Valero, and Kopriva	111
5.4	Numerical Results	112
5.4.1	Convergence Studies	112
5.4.2	Conservation Properties	115
5.4.3	Eigenvalues	115
5.4.4	CFL Condition	120
5.4.5	Comparison with Results of Manzanero, Rubio, Ferrer, Valero, and Kopriva	124
5.5	Summary and Conclusions	124
6	Shallow Water Equations	127
6.1	Some Properties of the Shallow Water Equations	128
6.2	Entropy Conservative Fluxes and Split Forms for Vanishing Bottom Topography	129
6.2.1	Primitive Variables h, v	130



6.2.2	Entropy Variables w	131
6.3	Adding Well-Balanced Source Discretisations	136
6.3.1	Extended Numerical Fluxes and Entropy Conservation	136
6.3.2	Well-Balancedness: Preserving the Lake-at-Rest Steady State	139
6.4	Extension to General SBP Operators	142
6.5	Numerical Surface Fluxes and Positivity Preservation	153
6.5.1	Positivity Preservation	153
6.5.2	Entropy Conservative Fluxes for Constant Bottom Topography b	155
6.5.3	Adding Dissipation for Constant Bottom Topography b	156
6.5.4	Classical Numerical Fluxes for Constant Bottom Topography b	159
6.5.5	Suliciu Relaxation Solver for Constant Bottom Topography b	160
6.5.6	Kinetic Solver for Constant Bottom Topography b	161
6.5.7	Hydrostatic Reconstruction Approach for General Bottom Topography b	163
6.6	Finite Volume Subcells	164
6.7	Numerical Tests	165
6.7.1	Well-Balancedness and Entropy Conservation	165
6.7.2	Lake-at-Rest with Emerged Bump	167
6.7.3	Moving Water Equilibrium with Varying Bottom b	169
6.7.4	Dam Break	171
6.8	Summary and Conclusions	174
7	Euler Equations	175
7.1	Some Properties of the Euler Equations	176
7.2	Entropy Conservative Fluxes	177
7.2.1	Using $\sqrt{\frac{\rho}{p}}, \sqrt{\frac{\rho}{p}}v, \sqrt{\rho p}$ as Variables	177
7.2.2	Using ρ, v, β as Variables	178
7.2.3	Using $\rho, v, \frac{1}{p}$ as Variables	180
7.2.4	Using ρ, v, p as Variables	181
7.2.5	Using ρ, v, T as Variables	183
7.2.6	Using Other Variables	185
7.3	Reversing the Role of Energy and Entropy	185
7.3.1	Using ρ, v, T as Variables	187
7.3.2	Using Other Variables	188
7.4	Kinetic Energy Preservation	189
7.4.1	New Approach to Kinetic Energy Preservation	190
7.4.2	A Kinetic Energy Preserving and Entropy Conservative Numerical Flux	193
7.5	Numerical Surface Fluxes/Riemann Solvers	195
7.5.1	Preserving Positivity of the Density	195
7.5.2	Preserving Positivity of the Pressure	197



7.5.3	Suliciu Relaxation Solver	198
7.6	Numerical Results	199
7.6.1	Modified Version of Sod's Shock Tube: Subcell Flux Differencing	199
7.6.2	A Blast Wave Problem	201
7.6.3	Kinetic Energy Preservation	201
7.6.4	Summary of the Numerical Results	203
7.7	Summary and Conclusions	203
8	Stability of Time Integration Methods	209
8.1	Linear Ordinary Differential Equations and Explicit Runge-Kutta Methods	210
8.1.1	General Runge-Kutta Methods	211
8.1.2	Explicit Runge-Kutta Methods	213
8.2	Third Order Methods Using Three Stages	213
8.3	SSPRK(10,4)	215
8.4	Other Third and Fourth Order Methods	222
8.5	Numerical Comparison of SSPRK(3,3) and SSPRK(10,4)	228
8.6	Summary and Conclusions	228
9	Entropy Variational Ideas	231
9.1	Variational Principle for Godunov's Flux	233
9.1.1	Scalar Conservation Laws	233
9.1.2	Hyperbolic Systems of Conservation Laws	235
9.2	Variational Principle in ENO Type Methods	238
9.2.1	Minimising the Entropy Rate per Boundary	240
9.2.2	Minimising the Total Entropy per Cell	245
9.3	Choice of Stencils in ENO Type Methods	251
9.3.1	Only One Stencil Choice per Time Step	251
9.3.2	Stencil Oscillations for few Cells	253
9.3.3	Deterministic Stencil Switching	256
9.3.4	Further Approaches	257
9.4	Summary and Conclusions	258
10	Summary and Conclusions	261
	Bibliography	263