Introduction

Functions of bounded variation of a single variable, or short BV-functions, were first introduced in 1881 by Camille Jordan [74]. He extended a result about the pointwise convergence of Fourier series of periodic and piecewise monotone functions proven around 50 years earlier by Johann Peter Gustav Lejeune Dirichlet [52] who gave the first rigorous proof of a conjecture on the representability of functions by means of trigonometric series originally raised in 1808 by Jean Baptiste Joseph Fourier [62]. Jordan proved that the Fourier series of any 2π-periodic function $x : \mathbb{R} \to \mathbb{R}$ of bounded variation converges at each point to the arithmetic mean of the right and left sided limits of x; in particular, if x is continuous, then its Fourier series converges even uniformly to x. This is nowadays known as the Dirichlet-Jordan-Theorem. In the same paper, Jordan also proved that any function of bounded variation may be written as a difference of two monotonically increasing functions. In this sense, the class BV of all real-valued functions of bounded variation defined on the real interval $[0, 1]$ is the linear hull of the set of monotone functions on that interval which do not form a linear space on their own.

The class BV has also been extended in many interesting directions. For instance, in the early 1920s, Norbert Wiener made the first noteworthy extension to Jordan's bounded variation concept by introducing the space WBV_2 of functions of bounded quadratic variation [154]. He proved that the Dirichlet-Jordan-Theorem still holds for functions of this type. In 1937, Laurence Chisholm Young showed that this theorem could be further extended to higher exponents and introduced the class WBV_p of functions of bounded p-variation for arbitrary $p \geq 1$ [159]. Together with Eric Russel Love, Young gave a comprehensive study of these functions [92] and finally went on to generalize Wiener's ideas by replacing the exponentiation by p by a composition with a suitable convex and increasing "gauge function" $\varphi : [0, \infty) \to [0, \infty)$ [160]. By doing so, he was hoping to extend the Dirichlet-Jordan-Theorem beyond the result he already proved for functions in WBV_p. In 1940, Raphaël Salem found a condition on φ ensuring that the Dirichlet-Jordan-Theorem holds for functions in the resulting more general space YBV_φ [139]. Moreover, in 1972, Albert Baernstein showed that among the YBV_φ-spaces, Salem's result concerning Fourier series is the best possible [19].

Also in 1972, Daniel Waterman extended the class of BV-functions in another direction by weighting the summands in Jordan's definition not by a composition but by a multiplication with a decreasing sequence Λ of positive numbers instead [151]. The resulting class ΛBV of such functions is of particular interest if one takes Λ to be the harmonic sequence $\Lambda = (1/n)_{n\in\mathbb{N}}$; in this case, ΛBV is denoted by HBV, and functions in this spaces are called "of bounded harmonic variation". Waterman showed not only that the Dirichlet-Jordan-Theorem about Fourier series holds for functions in HBV, he also pointed out that his result is best possible among all ΛBV-spaces. Moreover, he showed that for any Young function φ satisfying Salem's condition, the inclusion $YBV_\varphi \subseteq HBV$ holds. Consequently, among all the generalizations of the Dirichlet-Jordan-Theorem mentioned here, Waterman's version is the strongest.

Another notion of "bounded variation" has been introduced by Frigyes Riesz in 1910 [135, 136]. His type of variation seems very natural from a functional analytic point of view. In fact, an important result states that for fixed $p \in (1, \infty)$ a function $x : [0, 1] \to \mathbb{R}$ is of bounded p-variation in the sense of Riesz if and only if x is absolutely continuous and its derivative x' belongs to the Lebesgue space L_p. In this case, we write $x \in RBV_p$, and the Riesz variation of x may be calculated explicitly by an integral over x'. Clearly, such a formula cannot be true for $p = 1$, because functions in $RBV_1 = BV$ are in general not continuous, let alone absolutely continuous.

Remarkably, any function $x \in RBV_p$ belongs to the Sobolev space $W^{1,p}$, and any function in $W^{1,p}$ in turn agrees almost everywhere with a function in RBV_p [56]. This means that RBV_p consists precisely of the continuous representatives of $W^{1,p}$. In this sense Riesz introduced Sobolev spaces, at least in the scalar case, around 25 years prior to Sobolev.

A very comprehensive overview about properties of functions of bounded variation and their various generalizations may be found in the monograph [6].

Besides the development of the theory of Fourier series, BV-type functions have been extensively studied also in other fields of mathematics, for instance, in geometric measure theory, calculus of variations, and mathematical physics. Renato Caccioppoli and Ennio de Giorgi used them to define measures of nonsmooth boundaries of sets [34, 35, 48]. Olga Arsenievna Oleinik introduced her view of generalized solutions for nonlinear partial differential equations as functions from the space BV [125], and was able to construct a generalized solution of bounded variation of a first order partial differential equation [126]. A few years later, Edward D. Conway and Joel A. Smoller applied BV-functions to the study of a single nonlinear hyperbolic partial differential equation of first order [44], proving that the solution of the Cauchy problem for such equations is a function of bounded variation, provided the initial value belongs to the same class.

But functions of bounded variation turn out to be useful even when it comes to questions from the very foundations of analysis. For instance, it is clear that the sum of two functions with primitive again has a primitive, but this is wrong when "sum" is replaced by "product". This raises the question what the multipliers of the set Δ of functions with primitive are, that is, how the functions $g : [0, 1] \to \mathbb{R}$ look like such that xg belongs to Δ whenever x belongs to Δ. A discussion of these and more general questions will be the starting point of this thesis: We will discuss some natural "habitats" of functions of bounded variation and how they are related to other function classes.

This thesis is organized in seven chapters. The first chapter will be introductory in which we collect basic definitions, notations and function classes that we use the most. To be a little more precise we introduce in Section 1.1 the class C of continuous functions, the class B of bounded functions as well as the class D of Darboux functions (that is, functions with the intermediate value property) and discuss their relation to BV and to each other. For instance, the inclusions $BV \cap D \subseteq C \subseteq D \cap B$ hold, but none of these inclusions may be inverted. We also consider Lebesgue measurable and integrable functions, regular functions, absolutely and Lipschitz continuous functions and summarize how these classes are related to the class BV.

Section 1.2 is then devoted to functions of generalized bounded variation. We formally introduce the Wiener spaces WBV_p, the Young spaces YBV_φ, the Waterman spaces ΛBV and the Riesz spaces RBV_p. Equipped with a suitable norm building upon the corresponding type of variation, all these spaces become Banach spaces. Since functions which are zero everywhere except on a countable set become very important throughout this thesis, a major part of Section 1.2 is reserved for this kind of functions and how they behave in the various BV-type spaces. At the end of Section 1.2 we quickly discuss Helly's Selection Principle which provides a certain type of compactness in BV-spaces: Accordingly, every sequence in one of the BV-spaces that is bounded in its norm possesses a pointwise convergent subsequence.

The class Δ of derivatives to which we will give our main attention in Chapter 2 is situated between the classes C and D. From Lebesgue's and Riemann's integration theory it is well known that there are functions with primitive which are neither Lebesgue nor Riemann integrable. Consequently, in order to characterize the functions in Δ we need to pass in Section 2.1 to another notion of integration which will be functions that are integrable in the sense of Kurzweil and Henstock (KH-integrable). Every derivative is KH-integrable automatically and fulfills the Fundamental Theorem of Calculus. We also discuss another stronger form of integrability which enshrines both being KH-integrable and having a primitive. We then move on to other attempts that have been made in order to find integral free characterizations of the functions in Δ. However, it turns out that even if these attempts pretend to be integral free, they are in fact not. Nowadays, it is still not clear whether functions in Δ can be characterized without any kind of integration process; most mathematicians believe that this is impossible.

Moving on to more algebraic questions we discuss what happens when derivatives are multiplied or composed; we will do this in the Sections 2.2 and 2.3, where Section 2.2 is the largest part of this chapter. Therein we slowly approach a full discussion of the set Δ/Δ of multipliers of the class Δ as described above which simultaneously serves as a bridge to the class BV. Indeed, being continuous and of bounded variation is sufficient but not necessary to be a member of Δ/Δ, while there are functions in Δ/Δ that are bounded but neither continuous nor of bounded variation. In fact, functions in Δ/Δ turn out to be those functions that have a primitive and are in a certain sense of "local" bounded variation [59, 111].

Besides multipliers of the class Δ we also consider multipliers in other function spaces X and Y of real-valued functions on $[0,1]$. We denote by

$$Y/X := \Big\{ g : [0,1] \to \mathbb{R} \mid xg \in Y \text{ for all } x \in X \Big\}$$

the multiplier set of Y over X. While we identify multiplier sets for some classical function spaces only in case $X = Y$ in Section 2.2 we pass in Section 3.1 to other combinations, where we also allow $X \neq Y$. Some of these combinations are easy to find. For instance, it is straightforward to show that $BV/BV = BV$, and that $D/B = C/B = C/BV$ contains only the zero function $\mathbb{0}$. However, other combinations are very difficult to find or even unknown, especially when $Y = D$. Here, the three classes D/C, D/Δ and D/D will be of particular importance for us. Some authors claim without proof that the class D/D is easily deduced from the following result due to Radakovič [133]: If a function g has the property that $x + g$ is a Darboux function whenever x is a Darboux function, then g is constant. We show in Section 3.1 that D/D may indeed be deduced from Radakovič's result, but this deduction is by far not so easy, especially when g has zeros. Moreover, since we do not know how the classes D/C and D/Δ look like, we give only partial results and show how they are related to other multiplier classes and function spaces.

Section 3.2 is then dedicated to multipliers of spaces of functions of generalized variation. Conveniently, the results are quite similar for all such spaces. Since all BV-type spaces considered in this thesis are algebras, we have $X/X = X$ whenever X is one of these spaces. On the other hand, if X and Y are two Wiener spaces, then $Y/X = Y$ for $X \subseteq Y$. If $X \not\subseteq Y$, then Y/X contains only functions from Y with countable support. The same is true if X and Y are two Young spaces or two Waterman spaces. We will also see that for two Riesz spaces X and Y the condition $X \not\subseteq Y$ yields the strong degeneracy $Y/X = \{\mathbb{0}\}$.

Especially for applications it is quite handy that many differential equations may be solved by rewriting them into integral equations. Those can then often be handled with fixed point theory, even in the space BV and its various generalizations. In order to use classical fixed point theorems like those named after Stefan Banach, Juliusz Schauder, Gabriele Darbo or Mark Alexandrovich Krasnoselskii, one has to check several sometimes complicated conditions on the linear and nonlinear operators involved. This has been done many times in the BV-type spaces mentioned above; we refer the reader to the work of the Polish mathematicians Daria Bugajewska, Dariusz Bugajewski and their colleagues [25, 26, 27, 29, 30, 31, 32, 33, 46].

However, many analytic and set theoretic properties of such operators are either extremely complicated to characterize or just unknown. While linear operators such as multiplication, substitution or integral operators are mostly relatively easy to handle, nonlinear operators like composition or superposition operators behave sometimes in a rather strange way. The aim of the Chapters 4 and 5 of this thesis is to extend the theory concerning properties of these operators in the various BV-spaces.

Here, we consider the following three linear operators in Chapter 4 on two function spaces X and Y of real-valued functions on $[0,1]$. The multiplication operator

$$M_g : X \to Y,\ M_g x(t) = x(t)g(t)$$

for a generating function $g : [0,1] \to \mathbb{R}$ in Section 4.1, the substitution operator

$$S_g : X \to Y,\ S_g x(t) = x\big(g(t)\big)$$

for a generating function $g : [0,1] \to [0,1]$ in Section 4.2, and the integral operator

$$I_g : X \to Y,\ I_g x(t) = \int_0^1 g(t,s)x(s)\,\mathrm{d}s$$

for a generating function $g : [0,1] \times [0,1] \to \mathbb{R}$ in Section 4.3. For all three operators we are particularly interested in analytic properties like acting conditions for various BV-spaces X and Y, as well as continuity (which is for linear operators equivalent to boundedness) and compactness. Especially for the multiplication operator the results of Chapter 3 will be useful: Indeed, a multiplication operator $M_g : X \to Y$ is well-defined if and only if its generator g belongs to the multiplier space Y/X. In particular, recalling the sample results from above, the operator M_g maps BV into itself if and only if $g \in BV$. Moreover, regarding compactness the operator $M_g : BV \to BV$ is compact if and only if the support of g is countable, while $S_g : BV \to BV$ is compact if and only if g has finite range. We show these and similar results for other BV-type spaces in the Sections 4.1 and 4.2 for the multiplication and substitution operator, respectively. But we also give some remarks on set theoretic properties like injectivity, surjectivity and bijectivity. For instance, $M_g : BV \to BV$ is injective, if and only if g has no zeros, while $S_g : BV \to BV$ is injective if and only if g is surjective. Thus, mapping properties of M_g may often be described in terms of the support of g, while mapping properties of S_g can often be characterized in terms of the image of g.
Especially for integral equations a comprehensive investigation of the integral operator I_g is of particular importance for us. Therefore, Section 4.3 is by far the largest section of Chapter 4. Our main concern is analytic properties, and from the aforementioned cited papers of the Polish mathematicians Bugajewska, Bugajewski and colleagues many sometimes quite technical conditions are known guaranteeing that the integral operator maps a BV-space into itself and is bounded or compact. For instance, if $g(t,\cdot) \in L_1$ for any $t \in [0,1]$ and the variation of the function $g(\cdot,s)$ is almost everywhere bounded with respect to s by some L_1-function, then I_g maps BV into itself and is bounded and compact. Similar results are known for a few other BV-spaces. We generalize the known results in two directions: The first is that we give a unified approach to tackle all BV-spaces at once. The second is that we also consider the operator I_g from L_∞ into a BV-space X and give conditions under which such operators are well-defined, bounded and compact. This will be one of our main ingredients in the investigation of integral equations.

In Chapter 5 we discuss mapping properties of the following two nonlinear operators on two function spaces X and Y of real-valued functions on $[0,1]$. The (autonomous) composition operator

$$C_g : X \to Y,\ C_g x(t) = g\big(x(t)\big)$$

for a generating function $g : \mathbb{R} \to \mathbb{R}$ in Section 5.1, and the (nonautonomous) superposition operator

$$N_g : X \to Y,\ N_g x(t) = g\big(t, x(t)\big)$$

for a generating function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ in Section 5.2. As for the composition operator C_g it is well known that C_g maps BV into itself if and only if g is locally Lipschitz continuous [75]. Similar results are also known for the other BV-spaces. We then give some remarks about injectivity and surjectivity in $X = Y = BV$ and other BV-spaces. Here, $C_g : BV \to BV$ is injective if and only if g is injective. However, surjectivity is not so easy to describe. We give a sufficient condition which states that $C_g : BV \to BV$ is surjective if the slope of g is at suitable points in a certain sense bounded away from zero; unfortunately, we were not able to decide whether this condition is also necessary, but we give some indication why we think that it is. We then move on to different types of continuity. In summary, one can say that the more regular g is, the more "continuous" C_g is in BV and other spaces. For instance, C_g is uniformly continuous on bounded sets if and only if g is continuously differentiable, locally Lipschitz continuous if and only if g is continuously differentiable with locally Lipschitz continuous derivative, globally uniformly continuous if and only if g is affine, and compact if and only if g is constant. Similar results hold also in other BV-spaces, where the Riesz spaces have to be treated separately. We prove all these results using a unified approach. Surprisingly, the question of whether C_g is automatically pointwise continuous in BV if g meets the acting condition has an interesting history. The first proof given in [118] is very long and complicated, the second was given only recently in [96]. We give a third proof, but for this purpose we develop some new theory in Chapter 6 and therefore present the proof there also. Nonetheless, all proofs cannot be generalized to other BV-spaces, at least to the best of our knowledge.

Section 5.2 is dedicated to the superposition operator, and we only focus on analytic properties. Although both operators C_g and N_g are defined by an outer composition, the additional dependence of t allows N_g to behave rather chaotic and complicated compared to C_g. Again, many conditions guaranteeing analytic properties are known, but the behavior of the operator N_g even in the space BV is by far not fully understood. For instance, there is no (useful) criterion for the pointwise continuity of N_g in BV. Again, we provide a unified approach to handle all BV-spaces at once. The aim of Section 5.2 is to discuss the weird properties of N_g and reveal disparities between N_g and C_g. For instance, in contrast to C_g too weak kinds of regularity of g seem not directly connected to any kind of regularity of N_g. It is possible to find a discontinuous function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ that generates a constant and therefore utmost regular operator N_g, while it is also possible to construct a globally Lipschitz continuous generator g that induces a discontinuous operator $N_g : BV \to BV$; we give a general technique on how to construct such examples. Also, in contrast to C_g, there are compact operators $N_g : BV \to BV$ generated by nonconstant functions g. Our main result, however, is

Theorem 5.2.31. It provides for the first time a sufficient condition on g guaranteeing that N_g maps any of our BV-spaces into itself and is locally Lipschitz continuous. We show that our condition also covers the corresponding results for multiplication and composition operators which can be seen as special superposition operators. Theorem 5.2.31 will also serve as one of the main ingredients in the theory of integral equations in Chapter 7.

As mentioned before, the aim of Chapter 6 is to provide a new proof for the fact that C_g is continuous in BV if g is locally Lipschitz continuous. In order to do that we approximate C_g by other composition operator C_{g_n} for sufficiently smooth generators $g_n : \mathbb{R} \to \mathbb{R}$, where $n \in \mathbb{N}$. This approximation has to be done in such a way that the continuity of each C_{g_n} carries over to C_g. Therefore, we investigate in Section 6.1 on the abstract level of metric spaces the following four types of convergence: Quasi uniform, semi uniform, continuously uniform and locally uniform convergence. All of these are able to transmit continuity to the limit function. Historically, quasi uniformly convergence was introduced by Cesare Arzelà [14, 15], who answered the question what on top of pointwise convergence has to be assumed in order to guarantee that the limit function of a sequence of continuous functions is again continuous. Moreover, we give criteria on such sequences and their underlying spaces under which convergent subsequences can be extracted and recall that several types of convergence can even be used to characterize compactness of the domains the functions under consideration live in. Eventually, we compare all five types of convergence (pointwise convergence included) with each other.

In Section 6.2 we then pass to the proof of the fact that C_g is continuous in BV provided that it is well-defined. For this we first develop some theory and introduce the restricted variation, another more general type of variation measuring the variation of that part of a function that falls into a given set. The main result in this section is Theorem 6.2.7. It states that a sequence (C_{g_n}) converges in BV locally semi uniformly to a given composition operator C_g if and only if the corresponding generators g_n converge in BV to g and locally have a uniformly bounded Lipschitz constant. The continuity of C_g is then a simple consequence.

As for applications Chapter 7 will probably be the most relevant. Here, we consider Hammerstein and Volterra integral equations, where the latter are only special cases of the former. A starting point of our considerations in Section 7.1 is the Hammerstein integral equation

$$x(t) = h(t) + \lambda \int_0^1 k(t,s) g\big(x(s)\big) \,\mathrm{d}s$$

and some slight modifications that have already been studied in some BV-spaces, where h, k and g are given and x is unknown. Building on our results presented in the Chapters 4 and 5 we investigate the much more general equation

$$x(t) = h\big(t, x(t)\big) + \lambda f\big(t, x(t)\big) \int_0^1 k(t,s) g\big(s, x(s)\big) \,\mathrm{d}s$$

for given data h, f, k and g and prove existence and sometimes also uniqueness for solutions in BV-spaces. Again, we use a unified approach in order to handle all our

BV-spaces simultaneously. To get uniqueness of solutions we mostly use the fixed point theorem of Banach and Caccioppoli which requires strong conditions on the data involved. We also use other fixed point theorems that require less restrictive conditions on the data for the price that they guarantee only existence of solutions. Especially for boundary and initial value problems we investigate the Hammerstein integral equation

$$x(t) = Ax(t) + \lambda \int_0^1 k(t,s) g\Big(x(s)\Big) \,\mathrm{d}s,$$

where A is a linear operator from one BV-space into itself and provide some existence results building on Schauder's fixed point theorem. In the short Section 7.2 we reformulate all results about Hammerstein integral equation to the corresponding Volterra integral equations, where the upper limit of integration is replaced by the variable t. The final Section 7.3 is then dedicated to boundary and initial value problems. In [27] the boundary value problem

$$x''(t) = -\lambda g\Big(t, x(t)\Big)$$

subject to the nonclassical boundary conditions

$$x(0) = A_0 x, \quad x(1) = A_1 x$$

are solved, where A_0 and A_1 are linear functionals on BV. Two results are presented in this paper each of which giving conditions under which the boundary value problem has a solution. We generalize the ideas, simplify the conditions and summarize everything in one stronger result that is even able to handle cases that have not been covered yet. We also give some remarks on how the theory may be applied to other similar boundary value problems. We end the section with initial value problems

$$x''(t) = -\lambda g\Big(t, x(t)\Big)$$

subject to the nonclassical initial conditions

$$x(0) = A_0 x, \quad x'(0) = A_1 x.$$

we present very similar results and conditions guaranteeing the existence of solutions.

Throughout this thesis we give a lot of estimates, proofs and results, and many of them are quite technical. Therefore, it is of particular concern to us to illustrate most results by special cases, remarks, comparisons and summaries to make the presentation as clear as possible. This will be done by a total of 14 figures, 20 tables and 166 examples and counterexamples.