Chapter 1

Optimal distributed control problems with pointwise state constraints

A journey of a thousand miles
begins with a single step.
– Lao Tzu

1.1 Introduction

In the recent past, two regularization concepts were developed in an attempt to overcome the difficulties involved in state-constrained problems. First of all, Ito and Kunisch [78] introduced a Moreau-Yosida type regularization in the context of linear quadratic elliptic control problems. Their idea is basically to remove the inequality state constraints by including an augmented Lagrangian type penalty term. The violation of the eliminated constraints is then minimized by the penalty functional. As a result, the well-known semismooth Newton (SSN) method is applicable for solving the KKT type optimality condition associated with the penalized problem. This, certainly, represents a favorable aspect of the penalization method. Later on, Hintermüller and Kunisch [65, 66, 67] devised a path-following methodology for determining the optimal adjustment of the regularization parameter. In such a way, the penalized problem can be solved efficiently. Numerous publications related to this topic can be found, e.g., in [12, 13, 15, 42, 75, 76, 77].

Secondly, Meyer, Rösch and Tröltzsch [97] came up with a concept that incorporates a Lavrentiev type regularization into the analysis. In contrast to the first method, the pointwise inequality state constraints are approximated by mixed control-state-constraints. In some sense, they are kept as explicit constraints. The strategy suggested in [97] turns out to be competitive in some aspects. On the one hand, the Lagrange multiplier associated with the regularized problem enjoys better regularity properties than the original one. On the other hand, it has the potential to deal with ill-posedness faced in the analysis due to the compactness of the control-to-state mapping. Apart from this point of view, the SSN method applied to the regularized problem exhibits favorable numerical performances such as locally superlinear convergence and mesh-independence principles, see the recent paper [68]. Since then, the theoretical and numerical analysis of the regularization has been studied in various contexts [38, 43, 53, 70, 96, 98, 107].

This chapter is primarily devoted to the Lavrentiev type regularization applied to a class of semilinear elliptic problems. A detailed study of the penalization method will be carried out in Chapter 3. Our goal is twofold: First, we address the convergence of local solutions in the case of vanishing regularization parameter which complements the result in [68, 98]. This result is particularly important since optimization algorithms generate in general only local solutions. Secondly, following Hintermüller and the author [70], a sensitivity analysis with respect to the regularization parameter λ is introduced. More precisely, a deep insight into a specific solution structure in the linear case is provided. We study its dependence on λ. Such an issue is essential in order to have a stabilization of the numerical solution. Ignoring it would make the numerical algorithm suffer from ill-conditioning which results in large iteration numbers and reduced numerical solution accuracy. This effect has been experienced in earlier works [68, 96, 126]. It turns out that an appropriate initialization of the algorithm could significantly prevent the unstable behavior. Therefore, it becomes an important issue that has to be addressed in our present study.

We show the differentiability of the optimal solution with respect to the regularization parameter including a system of sensitivity equations characterizing uniquely the derivative. Hereafter, the theoretical results are applied to establish an extrapolation-based numerical scheme.

This chapter is organized as follows: In the upcoming section, the mathematical setting of the problem is introduced and well-known results on semilinear elliptic equations are presented. Then, we recall some results on the first- and second-order optimality conditions. In Sections 1.4-1.5, the Lavrentiev type regularization is introduced and we perform the convergence analysis. Section 1.6 contains a sensitivity analysis with respect to the regularization parameter. Thereafter, a semismooth Newton-type solver in combination with an extrapolation technique is proposed in Section 1.7. This chapter is ended with some numerical tests indicating the favorable numerical behavior of the method.

1.2 Problem formulation

Let us state the model problem that we focus on in this chapter:

$$(\mathbb{P}) \qquad \text{minimize } J(u,y) := \frac{1}{2}\int\limits_{\Omega} (y(x) - y_d(x))^2 \, dx + \frac{\alpha}{2}\int\limits_{\Omega} u(x)^2 \, dx$$

subject to the semilinear elliptic distributed value problem

$$(1.1) \qquad \begin{cases} Ay + d(\cdot, y) &= u \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma \end{cases}$$

and to the pointwise state constraints

$$(1.2) \qquad y_a(x) \le y(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega.$$

Here, Ω is a bounded domain in $\mathbb{R}^N$, $N \in \{2, 3\}$, with a Lipschitz boundary Γ. Concerning the data specified in $(\mathbb{P})$, suppose that the desired state $y_d \in L^2(\Omega)$ and the cost parameter $\alpha > 0$ are fixed. The bounds in the pointwise state constraints (1.2) are $y_a, y_b \in \mathcal{C}(\overline{\Omega})$ that satisfy $y_a(x) < y_b(x)$ for all $x \in \overline{\Omega}$. Moreover, the operator A represents a second-order elliptic partial differential operator of the form

$$(1.3) \qquad Ay(x) = -\sum_{i,j=1}^{N} D_j(a_{ij}(x) D_i y(x)).$$

The coefficient functions $a_{ij} \in L^\infty(\Omega)$ satisfy the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \theta\|\xi\|_{\mathbb{R}^N}^2 \quad \forall\ (\xi, x) \in \mathbb{R}^N \times \overline{\Omega}$$

for some constants $\theta > 0$. The function $d : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e., for every fixed $y \in \mathbb{R}$ the function $d(\cdot, y)$ is measurable and for almost all fixed $x \in \Omega$, the function $d(x, \cdot)$ is continuous. For the rest of this chapter, we impose the following assumptions on the nonlinearity d:

Assumption 1.1.

(i) *For almost all fixed $x \in \Omega$, the function $d(x, \cdot)$ is twice continuously differentiable. Furthermore*

$$d(\cdot, 0) \in L^2(\Omega) \quad \textit{and} \quad d_y(x, y) \geq 0$$

for a.a. $x \in \Omega$ and all $y \in \mathbb{R}$.

(ii) *For every $K > 0$, there exists a constant $C_d(K) > 0$ such that*

$$|d(x, y)| + |d_y(x, y)| + |d_{yy}(x, y)| \leq C_d(K)$$

$$|d_{yy}(x, y_1) - d_{yy}(x, y_2)| \leq C_d(K)|y_1 - y_2|$$

for a.a. $x \in \Omega$ and all $y, y_1, y_2 \in [-K, K]$.

Under Assumption 1.1, it is well known that for every $u \in L^2(\Omega)$, the state equation (1.1) admits a unique (weak) solution $y = y(u) \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$, cf. [29, Theorem 2.1]. Based on this, we may define the control-to-state operator associated with the state equation (1.1) $\mathcal{G} : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ that assigns to every element $u \in L^2(\Omega)$ the solution $y = y(u) \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ of (1.1). The solution operator $\mathcal{G}$ with range in $L^2(\Omega)$ is denoted by $\mathcal{S} : L^2(\Omega) \to L^2(\Omega)$. In other words, we set $\mathcal{S} = i_0\mathcal{G}$, where i_0 is the compact embedding operator from $H_0^1(\Omega)$ to $L^2(\Omega)$. With this setting at hand, the control problem ($\mathbb{P}$) can equivalently be formulated as

$$(\mathbb{P}) \qquad \begin{cases} \text{minimize } f(u) := J(u, \mathcal{S}(u)) \\ \text{over } u \in L^2(\Omega) \\ \text{subject to } y_a(x) \leq \mathcal{G}(u)(x) \leq y_b(x) \ \text{ for a.a. } x \in \Omega. \end{cases}$$

Here, $f : L^2(\Omega) \to \mathbb{R}$ is the reduced objective functional of ($\mathbb{P}$) that is given by

$$f(u) = \frac{1}{2}\|\mathcal{S}(u) - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2.$$

For the rest of this chapter, we assume that the admissible set

$$\{u \in L^2(\Omega) \mid y_a(x) \leq \mathcal{G}(u)(x) \leq y_b(x) \ \text{ for a.a. } x \in \Omega\} \tag{1.4}$$

is not empty. Thus, by classical arguments, the control problem ($\mathbb{P}$) admits an optimal solution, cf. the proof of Theorem 3.28 on page 90. Certainly, due to the nonlinearity involved in the state equation (1.1), one cannot expect the uniqueness of the optimal solution. Therefore, we concentrate in our analysis on local solutions.

Definition 1.2 (Locally optimal solution to ($\mathbb{P}$)).

(i) *A function $\bar{u} \in L^2(\Omega)$ is called a feasible control of* ($\mathbb{P}$) *if*

$$\bar{u} \in \{u \in L^2(\Omega) \mid y_a(x) \leq \mathcal{G}(u)(x) \leq y_b(x) \ \text{ for a.a. } x \in \Omega\}.$$

(ii) *A feasible control $\bar{u}$ of* ($\mathbb{P}$) *is said to be locally optimal or a local solution to* ($\mathbb{P}$) *with respect to the $L^2(\Omega)$-topology if there exists a positive real number c such that*

$$f(\bar{u}) \leq f(u)$$

for all feasible controls u of ($\mathbb{P}$) *satisfying $\|u - \bar{u}\|_{L^2(\Omega)} \leq c$.*

We close this section by recalling a standard result on differentiability of the solution operator $\mathcal{G}$. The assertion can be verified by the implicit function theorem, see [30, 34] or our argumentation in the proof of Theorem 3.19 on page 84.

Theorem 1.3 ([30, 34]). *The operator $\mathcal{G} : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is twice continuously Fréchet differentiable. The first derivative of $\mathcal{G}$ at $\bar{u} \in L^2(\Omega)$ in an arbitrary direction $u \in L^2(\Omega)$ is given by $\mathcal{G}'(\bar{u})u = y$ where $y \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is defined as the unique solution of*

$$\left\{ \begin{aligned} Ay + d_y(x, \bar{y})y = \ & u \quad \text{in} \quad \Omega \\ y = \ & 0 \quad \text{on} \quad \Gamma \end{aligned} \right. \tag{1.5}$$

with $\bar{y} = \mathcal{G}(\bar{u})$. Furthermore, the second derivative of $\mathcal{G}$ at $\bar{u}$ in arbitrary directions $u_1, u_2 \in L^2(\Omega)$ is given by $\mathcal{G}''(\bar{u})[u_1, u_2] = y$ where $y \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is defined as the unique solution of

$$\left\{ \begin{aligned} Ay + d_y(x, \bar{y})y + d_{yy}(x, \bar{y})[y_1, y_2] = \ & 0 \quad \text{in} \quad \Omega \\ y = \ & 0 \quad \text{on} \quad \Gamma \end{aligned} \right. \tag{1.6}$$

with $\bar{y} = \mathcal{G}(\bar{u})$ and $y_i = \mathcal{G}'(\bar{u})u_i$, $i = 1, 2$.

1.3 First- and second-order optimality conditions

We present in the upcoming theorem the first-order necessary condition for ($\mathbb{P}$) that is followed from Casas [25, 26], [23] and Alibert and Raymond [3]. For the proof, we refer the reader to the aforementioned references.

Definition 1.4 (Linearized Slater assumption for ($\mathbb{P}$)). *We say that a control $\bar{u} \in L^2(\Omega)$ satisfies the linearized Slater assumption for* ($\mathbb{P}$) *if there exist a function $u_0 \in L^\infty(\Omega)$ and a constant $\delta > 0$ such that*

$$y_a(x) + \delta \leq \mathcal{G}(\bar{u})(x) + (\mathcal{G}'(\bar{u})u_0)(x) \leq y_b(x) - \delta \quad \forall\, x \in \overline{\Omega}. \tag{1.7}$$

Theorem 1.5 ([25, 26]). *Let $\bar{u} \in L^2(\Omega)$ be a local solution to* ($\mathbb{P}$) *with the associated state $\bar{y} \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Further, assume that $\bar{u}$ satisfies the linearized Slater* *assumption for* $(\mathbb{P})$. *Then, there exist Lagrange multipliers* $\mu_a, \mu_b \in \mathcal{M}(\overline{\Omega})$ *and an adjoint state* $\bar{p} \in W^{1,s}(\Omega)$, $1 \leq s < \frac{N}{N-1}$, *such that*

$$\begin{aligned} A\bar{y} + d(x,\bar{y}) &= \bar{u} \quad \text{in} \quad \Omega \\ \bar{y} &= 0 \quad \text{on} \quad \Gamma \end{aligned} \tag{1.8}$$

$$\begin{aligned} A^*\bar{p} + d_y(x,\bar{y})\bar{p} &= \bar{y} - y_d + (\mu_b - \mu_a)_{|\Omega} \qquad \text{in} \quad \Omega \\ \bar{p} &= (\mu_b - \mu_a)_{|\Gamma} \qquad \text{on} \quad \Gamma \end{aligned} \tag{1.9}$$

$$\bar{p} + \alpha\bar{u} = 0 \tag{1.10}$$

$$\begin{gathered} \mu_a \geq 0, \quad \mu_b \geq 0 \\ \int\limits_{\bar{\Omega}} (y_a - \bar{y})\ d\mu_a = \int\limits_{\bar{\Omega}} (\bar{y} - y_b)\ d\mu_b = 0. \end{gathered} \tag{1.11}$$

Now, we are about to present a second-order sufficient optimality condition for $(\mathbb{P})$. Undoubtedly, the concept of sufficient optimality conditions in PDE-constrained optimization was originally conceived by Goldberg and Tröltzsch [50, 51, 52]. Since then, numerous contributions towards its development for more general problems have been made. See Bonnans [21], Casas and Mateos [30], Casas and Tröltzsch [32] and Casas, Tröltzsch and Unger [35, 36]. In particular, we draw attention to Casas, de Los Reyes and Tröltzsch [29]. They recently established sufficient optimality conditions that are, in some sense, very close to the associated necessary one. In certain cases, these conditions even guarantee the existence of a local solution with $L^2(\Omega)$-quadratic growth in the $L^2(\Omega)$-topology. In other words, the two-norm discrepancy can be omitted. In the following, the result is presented in the context of (P). We will also employ the technique for the case study considered in Section 3.7 and the corresponding proof will be presented there. For further details, we refer the reader to [29].

Definition 1.6. *Let* $\bar{u} \in L^2(\Omega)$ *be a feasible control of* $(\mathbb{P})$ *with the associated state* $\bar{y} \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. *Assume that* $\mu_a, \mu_b \in \mathcal{M}(\overline{\Omega})$ *and* $\bar{p} \in W^{1,s}(\Omega)$, $1 \leq s < \frac{N}{N-1}$, *satisfy* (1.9)-(1.11).

(i) *The cone of critical directions associated with* $\bar{u}$ *is defined by*

$$\mathcal{C}_{\bar{u}} = \{h \in L^2(\Omega) \mid h \text{ satisfies (1.12) and (1.13)}\}$$

$$y_h(x) = \begin{cases} \geq 0 & \text{if} \quad \bar{y}(x) = y_a(x) \\ \leq 0 & \text{if} \quad \bar{y}(x) = y_b(x) \end{cases} \tag{1.12}$$

$$\int\limits_{\bar{\Omega}} y_h\ d\mu_a = \int\limits_{\bar{\Omega}} y_h\ d\mu_b = 0, \tag{1.13}$$

where $y_h = \mathcal{G}'(\bar{u})h$.

(ii) *The Lagrange functional* $\mathcal{L} : L^2(\Omega) \times \mathcal{M}(\overline{\Omega}) \times \mathcal{M}(\overline{\Omega}) \to \mathbb{R}$ *associated with the control problem* $(\mathbb{P})$ *is defined by*

$$\mathcal{L}(u, \mu, \xi) = f(u) + \int_{\overline{\Omega}} (y_a - \mathcal{G}(u))\, d\mu + \int_{\overline{\Omega}} (\mathcal{G}(u) - y_b)\, d\xi.$$

(iii) *We say that* $\bar{u}$ *satisfies the second order sufficient condition for* $(\mathbb{P})$ *if*

$$\text{(SSC)} \qquad \frac{\partial^2 \mathcal{L}}{\partial u^2}(\bar{u}, \mu_a, \mu_b) h^2 > 0 \quad \forall h \in \mathcal{C}_{\bar{u}} \setminus \{0\}.$$

Theorem 1.7 (Casas, de Los Reyes, Tröltzsch [29]). *Let* $N \in \{2,3\}$ *and let* $\bar{u} \in L^2(\Omega)$ *be a feasible control of* $(\mathbb{P})$ *with the associated state* $\bar{y} \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. *Assume that* $\mu_a, \mu_b \in \mathcal{M}(\overline{\Omega})$ *and* $\bar{p} \in W^{1,s}(\Omega)$, $1 \le s < \frac{N}{N-1}$, *satisfy* (1.9)-(1.11). *If* $\bar{u}$ *satisfies (SSC) in the sense of Definition 1.6, then there exist positive real numbers* ε *and* σ *such that*

$$f(\bar{u}) + \frac{\sigma}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le f(u)$$

holds for all feasible controls u *of* $(\mathbb{P})$ *satisfying* $\|u - \bar{u}\|_{L^2(\Omega)} < \varepsilon$.

1.4 Lavrentiev type regularization

To give the reader some insight into the application of a Lavrentiev type regularization to the pointwise state constraints in (1.2), let us first consider the following equation:

$$\mathcal{S}(u) = w \quad \text{in } L^2(\Omega) \tag{1.14}$$

with a given function $w \in L^2(\Omega)$. On account of the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the control-to-state mapping $\mathcal{S} : L^2(\Omega) \to L^2(\Omega)$ is compact. Hence, if $w \in L^2(\Omega)$ is given, it is known from the theory of inverse problems that the equation (1.14) belongs to the class of ill-posed problems. To overcome this, we apply the Lavrentiev type regularization

$$\lambda u + \mathcal{S}(u) = w \quad \text{with } \lambda > 0$$

and hence we obtain a well-posed equation, cf. Lavrentiev [87]. Similarly to (1.14), one is confronted with some ill-posed problems in $(\mathbb{P})$. It can be simply explained by the situation where the lower bound of the state constraints (1.2) is active almost everywhere at the optimal state, i.e., it holds that

$$\bar{y}(x) = y_a(x) \quad \text{for a.a. } x \in \Omega.$$

Then, we deal with the following ill-posed equation

$$\mathcal{S}(\bar{u}) = y_a \quad \text{in } L^2(\Omega).$$

This simple consideration was the initial motivation of approximating the state constraints (1.2) into the following mixed control-state-constraints:

$$y_a(x) \le \lambda u(x) + y(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega \tag{1.15}$$

with a constant $\lambda > 0$. Thus, we arrive at the following regularized problem:

$$(\mathbb{P}_\lambda) \qquad \begin{cases} \min\limits_{u \in L^2(\Omega)} & f(u) \\ \text{subject to} & y_a(x) \le \lambda u(x) + \mathcal{G}(u)(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega. \end{cases}$$

The above-described methodology was suggested by Meyer, Rösch and Tröltzsch [97] for the linear quadratic counterpart to $(\mathbb{P}_\lambda)$. Thereafter, Meyer and Tröltzsch [98] extended its application to the semilinear case $(\mathbb{P}_\lambda)$. We also refer to Rösch and Tröltzsch [110] for the analysis of optimal control problems involving mixed control-state-constraints and simultaneously box constraints on the control. Compared to [110], the existence of *regular* Lagrange multipliers for $(\mathbb{P}_\lambda)$ can be shown in a fairly standard way. Since there is no constraint on the control, the regularized problem $(\mathbb{P}_\lambda)$ can locally be transformed into a purely control-constrained problem, cf. Section 1.6. Hereafter, one immediately obtains the existence of *regular* Lagrange multipliers associated with the mixed control-state-constraints of $(\mathbb{P}_\lambda)$.

Definition 1.8 (Locally optimal solution to $(\mathbb{P}_\lambda)$).

(i) *A function $\bar{u}_\lambda \in L^2(\Omega)$ is called a feasible control of $(\mathbb{P}_\lambda)$ if*

$$\bar{u}_\lambda \in \{u \in L^2(\Omega) \mid y_a(x) \le \lambda u(x) + \mathcal{G}(u)(x) \le y_b(x) \text{ for a.a. } x \text{ in } \Omega\}.$$

(ii) *A feasible control $\bar{u}_\lambda$ of $(\mathbb{P}_\lambda)$ is said to be locally optimal or a local solution to $(\mathbb{P}_\lambda)$ with respect to the $L^2(\Omega)$-topology if there exists a positive real number c such that*

$$f(\bar{u}_\lambda) \le f(u)$$

for all feasible controls u of $(\mathbb{P}_\lambda)$ satisfying $\|u - \bar{u}_\lambda\|_{L^2(\Omega)} \le c$.

Theorem 1.9 ([98, Theorem 3]). *Let $\lambda > 0$ and $\bar{u}_\lambda \in L^2(\Omega)$ be a locally optimal solution to $(\mathbb{P}_\lambda)$ with the associated state $\bar{y}_\lambda \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then, there exist regular Lagrange multipliers $\mu_\lambda^a, \mu_\lambda^b \in L^2(\Omega)$ and an adjoint state $\bar{p}_\lambda \in H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ such that*

$$\begin{aligned} A\bar{y}_\lambda + d(x, \bar{y}_\lambda) &= \bar{u}_\lambda \quad \text{in } \Omega \\ \bar{y}_\lambda &= 0 \quad \text{on } \Gamma \end{aligned} \tag{1.16}$$

$$\begin{aligned} A^*\bar{p}_\lambda + d_y(x, \bar{y}_\lambda)\bar{p}_\lambda &= \bar{y}_\lambda - y_d + \mu_\lambda^b - \mu_\lambda^a \quad \text{in } \Omega \\ \bar{p}_\lambda &= 0 \quad \text{on } \Gamma \end{aligned} \tag{1.17}$$

$$\bar{p}_\lambda + \alpha \bar{u}_\lambda + \lambda(\mu_\lambda^b - \mu_\lambda^a) = 0 \tag{1.18}$$

$$\begin{gathered} \mu_\lambda^a \ge 0, \quad \mu_\lambda^b \ge 0 \\ (\mu_\lambda^a, y_a - \lambda\bar{u}_\lambda - \bar{y}_\lambda)_{L^2(\Omega)} = (\mu_\lambda^b, \lambda\bar{u}_\lambda + \bar{y}_\lambda - y_b)_{L^2(\Omega)} = 0. \end{gathered} \tag{1.19}$$

1.5 Convergence of local solutions

The following section is devoted to the convergence result of local solutions to $(\mathbb{P}_\lambda)$ in the case of $\lambda \downarrow 0$. Certainly, this is a non-trivial issue that is mainly complicated by the involved nonlinearity and the mixing of control and state variables within the explicit inequality constraints. First, we recall the convergence result of *globally* optimal solutions:

Theorem 1.10 ([68, Theorem 5.1]). *Suppose that there exists a globally optimal solution to* $(\mathbb{P})$ *satisfying the linearized Slater assumption for* $(\mathbb{P})$*. Moreover, let* $(\bar{u}_\lambda)_{\lambda>0}$ *be a sequence of globally optimal solutions to* $(\mathbb{P}_\lambda)$*. Then,* $(\bar{u}_\lambda)_{\lambda>0}$ *is uniformly bounded in* $L^2(\Omega)$ *and every weakly converging subsequence of* $(\bar{u}_\lambda)_{\lambda>0}$ *converges strongly in* $L^2(\Omega)$ *towards a global solution to the original problem* $(\mathbb{P})$ *as* $\lambda \downarrow 0$.

In the upcoming result, we focus on the existence part: If a local solution $\bar{u}$ of $(\mathbb{P})$ is given, then we aim at finding a sequence of locally optimal solutions to the regularized problems $(\mathbb{P}_\lambda)$ converging strongly to $\bar{u}$ as $\lambda \downarrow 0$. Taking advantage of some results of Casas et al. [29] and Casas and Tröltzsch [33], the desired sequence can be established under certain assumptions.

Theorem 1.11. *Let* $\bar{u} \in L^2(\Omega)$ *be a local solution to* $(\mathbb{P})$ *satisfying the linearized Slater assumption for* $(\mathbb{P})$*. If* $\bar{u}$ *satisfies the second order sufficient condition (SSC) for* $(\mathbb{P})$*, then there exists a sequence of locally optimal solutions* $\{\bar{u}_\lambda\}_{\lambda>0}$ *of* $(\mathbb{P}_\lambda)$ *converging strongly in* $L^2(\Omega)$ *towards the local solution* $\bar{u}$ *as* $\lambda \downarrow 0$.

The proof of the theorem is given in the following steps:

Lemma 1.12. *Let* $v \in L^2(\Omega)$ *be a feasible control of* $(\mathbb{P})$ *satisfying the linearized Slater assumption for* $(\mathbb{P})$*, i.e., there is a* $u_0 \in L^\infty(\Omega)$ *such that*

$$y_a(x) + \delta \leq \mathcal{G}(v)(x) + (\mathcal{G}'(v)u_0)(x) \leq y_b(x) - \delta \quad \forall\, x \in \overline{\Omega} \tag{1.20}$$

with a fixed $\delta > 0$*. Then, there exists a sequence* $\{u_k^0\}_{k=1}^\infty \subset L^\infty(\Omega)$ *with the following properties:*

(i) *The sequence* $\{u_k^0\}_{k=1}^\infty$ *converges strongly in* $L^2(\Omega)$ *towards the feasible control* v *as* $k \to \infty$.

(ii) *For every* $k \in \mathbb{N}$*, there is a constant* $\lambda_k > 0$ *such that*

$$y_a(x) < \lambda u_k^0(x) + \mathcal{G}(u_k^0)(x) < y_b(x) \quad \text{for a.a. } x \in \Omega$$

for all $\lambda \leq \lambda_k$.

Proof. Since $\mathcal{C}(\overline{\Omega})$ is dense in $L^2(\Omega)$, there exists a sequence $\{a_k\}_{k=1}^\infty \subset \mathcal{C}(\overline{\Omega})$ such that

$$\|a_k - v\|_{L^2(\Omega)} \leq \frac{1}{k} \quad \forall k \in \mathbb{N}. \tag{1.21}$$

By virtue of Theorem 1.3, we find that

$$\|\mathcal{G}'(v)(a_k - v)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \leq c_0\|a_k - v\|_{L^2(\Omega)} \leq \frac{c_0}{k} \quad \forall k \in \mathbb{N} \tag{1.22}$$

with a fixed constant $c_0 > 0$ independent of k. Let us now define the sequence $\{u_k^0\}_{k=1}^\infty \subset L^\infty(\Omega)$ by

$$u_k^0 := a_k + \frac{3c_0}{\delta k} u_0. \tag{1.23}$$

Here, $u_0 \in L^\infty(\Omega)$ and $\delta > 0$ are as defined in (1.20). Our goal is to show that the sequence $\{u_k^0\}_{k=1}^\infty$ satisfies the assertion of the lemma. In view of (1.21)-(1.23), we have

$$\|u_k^0 - v\|_{L^2(\Omega)} \le \|a_k - v\|_{L^2(\Omega)} + \frac{3c_0}{\delta k}\|u_0\|_{L^\infty(\Omega)} \le (1 + c_1\|u_0\|_{L^\infty(\Omega)})\frac{1}{k}, \tag{1.24}$$

where $c_1 := 3c_0\delta^{-1}$. The latter inequality particularly implies that

$$\lim_{k\to\infty} u_k^0 = v \quad \text{in} \quad L^2(\Omega). \tag{1.25}$$

We demonstrate now that for every sufficiently large $k \in \mathbb{N}$, there exists a constant $\lambda_k > 0$ such that

$$y_a(x) < \lambda u_k^0(x) + \mathcal{G}(u_k^0)(x) < y_b(x) \quad \text{for a.a. } x \in \Omega$$

for all $\lambda \le \lambda_k$. The Taylor expansion of $\mathcal{G}$ at v implies that

$$\mathcal{G}(u_k^0) = \mathcal{G}(v) + \mathcal{G}'(v)(u_k^0 - v) + R(u_k^0), \tag{1.26}$$

where the remainder term $R : L^2(\Omega) \to H_0^1(\Omega) \cap \mathcal{C}(\overline{\Omega})$ satisfies

$$\lim_{k\to\infty} \frac{\|R(u_k^0)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}}{\|u_k^0 - v\|_{L^2(\Omega)}} = 0. \tag{1.27}$$

Further, by (1.24)

$$\begin{aligned}
\|R(u_k^0)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} &= \frac{\|R(u_k^0)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}}{\|u_k^0 - v\|_{L^2(\Omega)}}\|u_k^0 - v\|_{L^2(\Omega)} \\
&\le \frac{\|R(u_k^0)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})}}{\|u_k^0 - v\|_{L^2(\Omega)}}(1 + c_1\|u_0\|_{L^\infty(\Omega)})\frac{1}{k}.
\end{aligned}$$

Thus, (1.27) ensures the existence of an index number k_0 such that

$$\|R(u_k^0)\|_{H_0^1(\Omega)\cap\mathcal{C}(\overline{\Omega})} \le \frac{c_0}{k} \quad \forall k \ge k_0. \tag{1.28}$$

Now, let $k \in \mathbb{N}$ be arbitrarily fixed with $k \ge \max\{k_0, c_1\}$ and we rewrite (1.26) as

$$\begin{aligned}
\mathcal{G}(u_k^0) &= \mathcal{G}(v) + \mathcal{G}'(v)(u_k^0 - v) + R(u_k^0) \\
&= \mathcal{G}(v) + \mathcal{G}'(v)(a_k + \tfrac{c_1}{k}u_0 - v) + R(u_k^0) \\
&= (1 - \tfrac{c_1}{k})\mathcal{G}(v) + \mathcal{G}'(v)(a_k - v) + \tfrac{c_1}{k}(\mathcal{G}(v) + \mathcal{G}'(v)u_0) + R(u_k^0).
\end{aligned}$$

Since v is a feasible control of ($\mathbb{P}$) and due to (1.22), (1.20) and (1.28), it immediately follows that

$$\mathcal{G}(u_k^0) \le (1 - \frac{c_1}{k})y_b + \frac{c_0}{k} + \frac{c_1}{k}(y_b - \delta) + \frac{c_0}{k} = y_b - \frac{c_0}{k}, \tag{1.29}$$

where we have used $c_1 = 3c_0\delta^{-1}$. Thus

$$\lambda u_k^0(x) + \mathcal{G}(u_k^0)(x) \le \lambda \|u_k^0\|_{L^\infty(\Omega)} + y_b(x) - \frac{c_0}{k} \quad \text{for a.a. } x \in \Omega.$$

We choose now a constant $\lambda_k > 0$ such that

$$\lambda \|u_k^0\|_{L^\infty(\Omega)} < \frac{c_0}{k} \quad \forall \lambda \le \lambda_k.$$

Therefore

$$\lambda u_k^0(x) + \mathcal{G}(u_k^0)(x) < y_b(x) \quad \text{for a.a. } x \in \Omega$$

for all $\lambda \le \lambda_k$. By analogous arguments, we find for all sufficiently small λ that

$$\lambda_n u_k^0(x) + \mathcal{G}(u_k^0)(x) > y_a(x) \ \text{ for a.a. } x \in \Omega.$$

Thus, we end up with the conclusion that for every sufficiently large k, there is a constant $\lambda_k > 0$ such that

$$y_a(x) < \lambda u_k^0(x) + \mathcal{G}(u_k^0)(x) < y_b(x) \quad \text{for a.a. } x \in \Omega$$

for all $\lambda \le \lambda_k$. Hence, the assertion is immediately verified. $\square$

In the sequel, let $\bar{u}$ be a *local solution* to $(\mathbb{P})$ satisfying the linearized Slater assumption for $(\mathbb{P})$. Moreover, assume that $\bar{u}$ satisfies (SSC) for $(\mathbb{P})$. By virtue of Theorem 1.7, there exist positive real numbers $\widetilde{\varepsilon}$ and $\widetilde{\sigma}$ such that

$$f(\bar{u}) + \frac{\widetilde{\sigma}}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \le f(u) \tag{1.30}$$

is satisfied for all feasible controls u of $(\mathbb{P})$ with $\|u - \bar{u}\|_{L^2(\Omega)} < \widetilde{\varepsilon}$. Next, let us introduce the following auxiliary problem:

$$(\widetilde{\mathbb{P}}_\lambda) \qquad \begin{cases} \min & f(u) \\ \text{subject to} & u \in U_{\lambda,\widetilde{\varepsilon}} \end{cases}$$

where

$$\begin{aligned} U_{\lambda,\widetilde{\varepsilon}} := \{u \in L^2(\Omega) \mid \|u - \bar{u}\|_{L^2(\Omega)} \le \widetilde{\varepsilon} \ \text{ and} \\ y_a(x) \le \lambda u(x) + \mathcal{G}(u)(x) \le y_b(x) \text{ for a.a. } x \in \Omega\}. \end{aligned} \tag{1.31}$$

It should be emphasized that the idea of considering the particular form $(\widetilde{\mathbb{P}}_\lambda)$ is adapted from Casas and Tröltzsch [33]. Now, according to Lemma 1.12, one finds a $\hat{u} \in L^2(\Omega)$ and a constant $\hat{\lambda} > 0$ such that $\hat{u}$ is a feasible control of $(\widetilde{\mathbb{P}}_\lambda)$ for all $\lambda \le \hat{\lambda}$, i.e., it holds that

$$\hat{u} \in U_{\lambda,\widetilde{\varepsilon}} \quad \forall \lambda \le \hat{\lambda}.$$

Thus, for all $\lambda \le \hat{\lambda}$, $(\widetilde{\mathbb{P}}_\lambda)$ admits at least one global solution in $U_{\lambda,\widetilde{\varepsilon}}$. For the rest of this section, let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} \lambda_n = 0 \ \text{ and } \ \lambda_n \le \hat{\lambda} \quad \forall n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, let $\widetilde{u}_n \in L^2(\Omega)$ be a (global) solution to $(\widetilde{\mathbb{P}}_{\lambda_n})$ and our goal now is to prove that $\widetilde{u}_n \to \bar{u}$ strongly in $L^2(\Omega)$.

Lemma 1.13. *Every weak limit $\tilde{u} \in L^2(\Omega)$ of any subsequence of $\{\widetilde{u}_n\}_{n=1}^\infty$ is a feasible control of* $(\mathbb{P})$ *or equivalently*

$$y_a(x) \le \mathcal{G}(\tilde{u})(x) \le y_b(x) \quad \textit{for a.a. } x \in \Omega.$$

Proof. Assume that a subsequence of $\{\widetilde{u}_n\}_{n=1}^\infty$ denoted w.l.o.g. again by $\{\widetilde{u}_n\}_{n=1}^\infty$ converges weakly to a $\tilde{u} \in L^2(\Omega)$. In particular, $\{\widetilde{u}_n\}_{n=1}^\infty$ is uniformly bounded in $L^2(\Omega)$ and hence

$$\lim_{n\to\infty} \lambda_n \widetilde{u}_n = 0 \quad \text{in } L^2(\Omega).$$

Consequently, we can extract a subsequence, w.l.o.g $\{\lambda_n \widetilde{u}_n\}_{n=1}^\infty$, converging to zero almost everywhere in Ω:

$$\lim_{n\to\infty} \lambda_n \widetilde{u}_n(x) = 0 \quad \text{a.e. in } \Omega. \tag{1.32}$$

By standard arguments, cf. [118], the weak convergence $\widetilde{u}_n \rightharpoonup \tilde{u}$ in $L^2(\Omega)$ yields

$$\mathcal{G}(\widetilde{u}_n) \rightharpoonup \mathcal{G}(\tilde{u}) \quad \text{weakly in } H_0^1(\Omega).$$

Thus, invoking the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$

$$\lim_{n\to\infty} \mathcal{G}(\widetilde{u}_n) = \mathcal{G}(\tilde{u}) \quad \text{in } L^2(\Omega). \tag{1.33}$$

Since $\widetilde{u}_n$ is a feasible control of $(\widetilde{\mathbb{P}}_{\lambda_n})$ for all $n \in \mathbb{N}$, we have

$$y_a(x) \le \lambda_n \widetilde{u}_n(x) + \mathcal{G}(\widetilde{u}_n)(x) \le y_b(x) \text{ for a.a. } x \in \Omega \quad \forall n \in \mathbb{N}.$$

Hence, in view of (1.32)-(1.33), the assertion of the lemma is verified. □

Lemma 1.14. *The sequence $\{\widetilde{u}_n\}_{n=1}^\infty$ converges strongly in $L^2(\Omega)$ towards the local solution $\bar{u}$.*

Proof. We have already mentioned that $\hat{u}$ is a feasible control of $(\mathbb{P}_{\lambda_n})$ for all $n \in \mathbb{N}$. Consequently

$$f(\hat{u}) \ge f(\widetilde{u}_n) \ge \frac{\alpha}{2}\|\widetilde{u}_n\|_{L^2(\Omega)}^2 \quad \forall n \in \mathbb{N}.$$

Particularly, the sequence $\{\widetilde{u}_n\}_{n=1}^\infty$ is uniformly bounded in $L^2(\Omega)$. Thus, there exists a subsequence of $\{\widetilde{u}_n\}_{n=1}^\infty$ denoted w.l.o.g. by $\{\widetilde{u}_n\}_{n=1}^\infty$ such that $\widetilde{u}_n \rightharpoonup \tilde{u}$ weakly in $L^2(\Omega)$. Based on Lemma 1.13, this weak limit $\tilde{u}$ is a feasible control of $(\mathbb{P})$. Furthermore, since the set

$$\{u \in L^2(\Omega) \mid \|u - \bar{u}\|_{L^2(\Omega)} \le \widetilde{\varepsilon}\}$$

is weakly closed, it satisfies

$$\|\tilde{u} - \bar{u}\|_{L^2(\Omega)} \le \widetilde{\varepsilon}. \tag{1.34}$$

According to Lemma 1.12, there exists a sequence $\{u_k^0\}_{k=1}^\infty \subset L^\infty(\Omega)$ such that

(i) The sequence $\{u_k^0\}_{k=1}^\infty$ converging strongly in $L^2(\Omega)$ to $\bar{u}$ as $k \to \infty$ and it holds that $\|u_k^0 - \bar{u}\|_{L^2(\Omega)} \le \widetilde{\varepsilon}$ for all $k \in \mathbb{N}$.

(ii) For each $k \in \mathbb{N}$, there exists an index number $n_k \in \mathbb{N}$ such that u_k^0 is feasible for $(\mathbb{P}_{\lambda_n})$ for all $n \geq n_k$, i.e.

$$y_a(x) \leq \lambda_n u_k^0(x) + \mathcal{G}(u_k^0)(x) \leq y_b(x) \quad \text{for a.a. } x \in \Omega \quad \forall n \geq n_k.$$

By the definition of the admissible set $U_{\lambda_n,\widetilde{\varepsilon}}$ in (1.31), (i)-(ii) imply particularly that for every k, u_k^0 is feasible for $(\widetilde{\mathbb{P}}_{\lambda_n})$ for all $n \geq n_k$. Consequently

$$f(\widetilde{u}_n) \leq f(u_k^0) \quad \forall n \geq n_k.$$

Passing to the limit $n \to \infty$, it follows from the lower semicontinuity of f that

$$f(\tilde{u}) \leq \liminf_{n\to\infty} f(\widetilde{u}_n) \leq \limsup_{n\to\infty} f(\widetilde{u}_n) \leq f(u_k^0). \tag{1.35}$$

Since (1.35) holds true for every arbitrary $k \in \mathbb{N}$, passing to the limit $k \to \infty$, the continuity of f together with (i) imply that

$$f(\tilde{u}) \leq \liminf_{n\to\infty} f(\widetilde{u}_n) \leq \limsup_{n\to\infty} f(\widetilde{u}_n) \leq \lim_{k\to\infty} f(u_k^0) = f(\bar{u}). \tag{1.36}$$

In addition, taking account of (1.34) and since $\tilde{u}$ is a feasible control of $(\mathbb{P})$, (1.30) ensures that

$$f(\bar{u}) + \frac{\widetilde{\sigma}}{2}\|\tilde{u} - \bar{u}\|^2_{L^2(\Omega)} \leq f(\tilde{u}).$$

Applying the latter inequality to (1.36)

$$f(\bar{u}) + \frac{\widetilde{\sigma}}{2}\|\tilde{u} - \bar{u}\|^2_{L^2(\Omega)} \leq f(\bar{u}).$$

Consequently, $\bar{u} = \tilde{u}$. From the latter equality together with (1.36), it follows that

$$\lim_{n\to\infty} f(\widetilde{u}_n) = f(\bar{u}).$$

Hence, invoking again the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we arrive at

$$\lim_{n\to\infty} \|\widetilde{u}_n\|_{L^2(\Omega)} = \|\bar{u}\|_{L^2(\Omega)}.$$

Consequently, by virtue of the weak convergence $\widetilde{u}_n \rightharpoonup \bar{u}$, the assertion is verified. □

It should be pointed out that the global solution $\widetilde{u}_n$ of $(\widetilde{\mathbb{P}}_{\lambda_n})$ could possibly be located at the *boundary* of the ball $B_{\widetilde{\varepsilon}}(\bar{u}) = \{u \in L^2(\Omega) \mid \|u - \bar{u}\|_{L^2(\Omega)} \leq \widetilde{\varepsilon}\}$. In such a case, $\widetilde{u}_n$ is not a local solution to $(\mathbb{P}_{\lambda_n})$. Nevertheless, by the convergence $\widetilde{u}_n \to \bar{u}$ in $L^2(\Omega)$, one can show that, for all sufficiently large n, $\widetilde{u}_n$ is a local solution of $(\mathbb{P}_{\lambda_n})$. Therefore, it cannot be located at the boundary of $B_{\widetilde{\varepsilon}}(\bar{u})$.

Lemma 1.15. *For every sufficiently large n, $\widetilde{u}_n$ is a local solution to $(\mathbb{P}_{\lambda_n})$.*

Proof. Let u be a feasible control of $(\mathbb{P}_{\lambda_n})$ satisfying $\|u - \widetilde{u}_n\|_{L^2(\Omega)} \leq \frac{\widetilde{\varepsilon}}{2}$. Then, for all sufficient large n, the strong convergence $\widetilde{u}_n \to \bar{u}$ implies that

$$\|u - \bar{u}\|_{L^2(\Omega)} \leq \|u - \widetilde{u}_n\|_{L^2(\Omega)} + \|\widetilde{u}_n - \bar{u}\|_{L^2(\Omega)} \leq \frac{\widetilde{\varepsilon}}{2} + \frac{\widetilde{\varepsilon}}{2} = \widetilde{\varepsilon}. \tag{1.37}$$

Consequently, we have $u \in U_{\lambda_n,\tilde{\varepsilon}}$ and hence since $\widetilde{u}_n$ is an optimal solution to $(\widetilde{\mathbb{P}}_{\lambda_n})$, we infer

$$f(\widetilde{u}_n) \le f(u).$$

Altogether, for all sufficiently large n

$$f(\widetilde{u}_n) \le f(u)$$

holds for all feasible controls u of $(\mathbb{P}_{\lambda_n})$ satisfying $\|u - \widetilde{u}_n\|_{L^2(\Omega)} \le \frac{\tilde{\varepsilon}}{2}$. Thus, $\widetilde{u}_n$ is a local solution to $(\mathbb{P}_{\lambda_n})$ for every sufficiently large n. □

Finally, collecting the results above, the assertion of Theorem 1.11 is verified.

1.6 Sensitivity analysis of the linear quadratic counterpart to $(\mathbb{P})$

We continue our study by performing a sensitivity analysis with respect to the regularization parameter λ. Our main goal is to establish the local Lipschitz-continuity and the differentiability of the mapping $\lambda \mapsto \bar{y}_\lambda$. As pointed out in the introduction, such an issue is useful for devising stable numerical algorithms associated with $(\mathbb{P}_\lambda)$. The corresponding analysis is performed for the linear quadratic counterpart to $(\mathbb{P})$, i.e., the case where $d(\cdot, y) \equiv 0$. In other words, we consider the following problem:

$$(\mathbb{P}) \qquad \begin{cases} \text{minimize } J(u,y) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to } Ay = u \quad \text{in } \Omega \\ \qquad\qquad\quad y = 0 \quad \text{on } \Gamma \\ \qquad\qquad y_a(x) \le y(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega. \end{cases}$$

Before going into the details, let us underline again that the results presented in the following have been published in [70]. For the convenience of the reader, the linear quadratic problem is denoted again by $(\mathbb{P})$ and we use the same notation as before. Since $d(\cdot, y) \equiv 0$, the solution operator

$$\mathcal{G} : L^2(\Omega) \to H^1_0(\Omega) \cap \mathcal{C}(\overline{\Omega})$$

is now linear. Recall that the solution operator with range in $L^2(\Omega)$ is denoted by $\mathcal{S} : L^2(\Omega) \to L^2(\Omega)$, see page 3. Thanks to the linearity and continuity of $\mathcal{S}$, the reduced objective functional of $(\mathbb{P})$ that is given by

$$(1.38) \qquad f : L^2(\Omega) \to \mathbb{R},\ f(u) = J(u, \mathcal{S}u) = \frac{1}{2}\|\mathcal{S}u - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)},$$

is strictly convex. Thus, $(\mathbb{P})$ admits a unique solution and the first-order optimality condition for $(\mathbb{P})$ is sufficient. Similarly to the semilinear case, the Lavrentiev type regularization approximates the pointwise state constraints in $(\mathbb{P})$ by mixed control-state-constraints:

$$(\mathbb{P}_\lambda) \qquad \begin{cases} \text{minimize } J(u,y) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \\ \text{subject to } Ay = u \quad \text{in } \Omega \\ \qquad\qquad\quad y = 0 \quad \text{on } \Gamma \\ \qquad\qquad y_a(x) \le \lambda u(x) + y(x) \le y_b(x) \quad \text{for a.a. } x \in \Omega. \end{cases}$$