1 Introduction

Section 1.1 is intended to motivate the study of the seemingly simple problem of axially moving strings as an important benchmark in modeling a wide variety of technical and scientific applications. Section 1.2 presents a brief review of the relevant literature. While emphasis is placed on the current state of research, also some remarks on the historical development in modeling slender continua are given. Finally, in Section 1.3, the objectives and structure of the thesis are outlined.

1.1 Motivation

Many objects encountered in a variety of scientific and engineering disciplines feature thread-like shapes in which one dimension is dominant over the other two. Objects that exhibit this characteristic geometry are commonly classified as one-dimensional continua. Depending on the field of application and the specific requirements, different modeling approaches have been developed over the years resulting in a wide diversity of terminology, concepts and branches of research.

In civil engineering, cables[1] in suspension bridges and cable-roof structures are typical applications taking advantage of the high tensile strength, flexibility and light-weight characteristics of thread-shaped members. Besides being economic, tensile structures feature often an aesthetic quality that fascinates both engineers and laymen. However, despite their merits, cables used as structural elements are bound to tensile stress and even then may cause stability problems due to their sensitivity to vibrations. Consequently, various models governing the vibrations of suspended cables are available.

Another, quite different field of application is found in chemistry and biology, where an increasing interest is taken in the elastic character of thread-like structures, especially in the super-coiling properties of filamentary structures such as polymers and bacterial fibers for which the theory of one-dimensional continua provides macroscopic models, as documented, for example, in [18, 23, 80]. Furthermore, cable and rod models are used in the context of studying the dynamics of rather special systems, such as mooring lines [70], whips [45], or fly lines [17].

In the examples mentioned above, the cable-like structures represent tensile-loaded, spatially fixed elements. In various fields of engineering, however, power and motion are to be transmitted using driven members. Besides conventional gear trains and tie-rod linkages, cable-driven mechanisms represent widely used power transmission systems. Perhaps the most commonly known cable-driven manipulating system is the belt drive system used in a variety of industrial machinery. The power transmission in such a closed-loop cable

[1] The term *cable* is widely used to imply tendons, tethers, ropes, wires, belts, and similar elements.

system relies on friction generated between the belt and the guiding pulleys. Toothed belts, so-called timing belts or chain-and-sprocket devices, are often used to increase the efficiency of power transmission. Belt drive systems belong to the general class of axially moving continua which encompasses such diverse systems as band conveyors, paper machines, magnetic tapes, cableways, and fluid conveying pipes.

Open-ended cable drives represent an important subclass of cable transmission systems commonly found in hoisting cranes and elevators for raising, shifting, and lowering heavy objects. In these applications, the cable-like members are designed to pull against the gravitational force. Tethered satellite systems and marine tethers are further examples of open-ended cable mechanisms.

Despite their diversity, there are common characteristics among the applications specified above, suggesting a generalized approach to the modeling and analysis of axially moving one-dimensional continua. In this connection, basically two types of models are available idealizing a slender continuum either as a string-like second-order system or as a beam-like fourth-order system, depending whether or not the effect of flexural rigidity is taken into account.

As the simplest model of a slender continuum in axial motion, the problem of a uniform string moving axially between two spatially fixed supports is frequently addressed, for example, as benchmark in stability analysis and control design. In the majority of cases, the string is assumed to travel at constant speed, while the accelerated string problem is analyzed to a significantly lesser extent. The steady-state motion might be the most important application, even though in actual operation, a motion at constant speed is always embedded between transient stages of acceleration and deceleration. Moreover, in many applications, such as belt drive systems in combustion engines, the transport speed undergoes small periodic variations, which lead to a parametric excitation of the vibration of the belt. Miranker [49] was probably the first who derived the linear equation governing the transverse vibrations of a tape moving at a time-dependent axial velocity $v(t)$ in the form[2]

$$\ddot{u} + 2v\dot{u}' + (v^2 - c_0^2)u'' + \dot{v}u' = 0 \; , \tag{1.1}$$

where u is the transverse displacement of the tape and c_0 denotes the (constant) wave speed. Here and in the following, if not stated otherwise, superposed dots and attached primes indicate partial derivatives with respect to the time and space coordinate, respectively.

Most studies on accelerated continua, even in the recent years, are based on equations of the form (1.1), for example, contributions by Tabarrok et al. [83], Wickert and Mote [101], Pakdemirli and Ulsoy [63], Zhu and Guo [104], and Suweken and van Horssen [82]. It

[2] In his original paper, Miranker considers a tape moving in the opposite direction, which leads to a change of signs (of v and $\dot{v}$).

seems, however, that this equation does not correctly describe the dynamics of an axially accelerated tape or string. The term $\dot{v}u'$ appearing in (1.1) results from the questionable assumption of a constant tensile force, which does not apply to the case of an accelerated motion, as demonstrated in the course of this work.

Besides providing an elaboration of the underlying theory of one-dimensional continua, with special emphasis on the modeling and analysis of axially moving strings, the thesis at hand evaluates in particular the effect of a non-constant tensile force associated with the accelerated motion of a straight, flexible string. In this connection, the motivation for studying the seemingly simple problem of axially moving strings is well reflected by the words of Morse who wrote in 1936 [50]: "The string is the simplest case of a system with an infinite number of allowed frequencies, and it is best to discuss some of the properties common to all such systems for as simple as a system as we can find, lest the mathematical complications completely obscure the physical ideas."

1.2 Literature Review

The modeling and analysis of slender structural elements belong to the oldest topics in mechanics that can be traced back to the mid-17th century when the foundation of solid mechanics was laid by the derivation of the theory of flexible bodies, accompanied by the development of new mathematical concepts, such as differential equations and calculus of variations. Interesting accounts of the scientific achievements during the period 1638–1788 have been given, for example, by Truesdell [91]. The theory of beams developed at that time remains among the most immediately useful concepts of solid mechanics, partly because of its simplicity but also because of the pervasiveness of slender bodies in structural technology. In the course of time, the theory of one-dimensional continua has been extended in various directions resulting in a wide variety of models of different complexity and purpose. The still numerous research activities in this area are a strong indication for the richness of the involved mechanics on the one hand and their relevance from both an academic and a practical point of view on the other hand.

Any attempt to survey exhaustively all the available literature and recent research findings must be beyond the scope of this thesis. Instead, it is aimed to provide a sound overview on the subject by compiling some of the most significant studies on slender continua. The relevant literature can basically be classified according to the scheme shown in Figure 1.1. The class of continua with fixed boundaries encompasses models of axially non-moving slender structures as commonly encountered in civil engineering but also in many other fields of application. For various problems in mechanical engineering and other branches, however, models with fixed boundaries are not sufficient because of the requirement to account additionally for axial motion thus allowing a convective transport of mass across the boundaries. The corresponding class of continua with moving boundaries is further

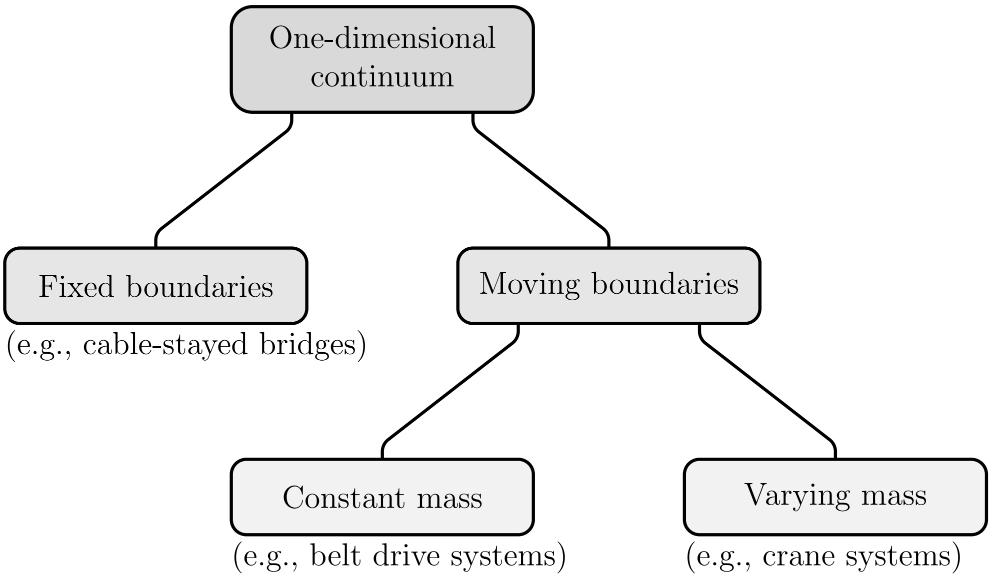

Figure 1.1: Classification of problems involving one-dimensional continua

divided into two subclasses, depending whether or not the mass of a continuum is constant with respect to the system boundaries.[3]

In the following, some significant work on the analysis of slender continua is outlined first, before addressing the special problem of modeling axially moving continua.

Advances in Modeling Slender Continua

A first approach to the analysis of slender continua is provided by the simple string model which neglects the effects of bending stiffness and self-weight, that is, equilibrium configurations are assumed to be always representable by straight lines not taking into account any sag. Obviously, such a strong idealization is only reasonable for thin, highly prestressed structures, such as strings in music instruments. In fact, the desire to understand the physics behind music gave rise to the classical problem of a vibrating string.

Mersenne is credited as being the first who published in 1625 a correct account of the vibrations of strings and their frequencies. He recognized that the frequency of vibration is inversely proportional to the length of a string and directly proportional to the square root of its cross-sectional area. At that time, especially in the second half of the 17th

[3] It is common usage in literature to refer to a potential change of length as the distinguishing feature, even though it is inexact and not equivalent to the classification stated above, since a change of length does not necessarily imply mass transport over system boundaries but may simply result from stretching of the material.

century, the mechanics of strings had been one of the most considered problems in science. Among the many contributions to this subject, the work of Noble and Pigot is mentioned exemplarily. In 1674, they found by experiment that in the mode corresponding to the kth overtone, a string possesses $k-1$ equally spaced nodes. Shortly afterwards, Sauveur established the concept of fundamental and harmonic overtones.

The first successful attempt to apply dynamical principles to a flexible body dates from 1713 when Taylor derived correctly the equation governing the balance of momentum normal to a string. However, he did not recognize it as an equation of motion and did not solve it. In 1746, D'Alembert derived the linear partial differential equation describing small vibratory motion of a taut string, nowadays known as the one-dimensional wave equation. D'Alembert also found the solution in form of the superposition of two waves of fixed shape travelling in opposite directions.

The question which functions are admissible as solutions to the wave equation was taken up by D'Alembert, Daniel Bernoulli and Euler. While the functions applied by D'Alembert all had the form of a single analytical expression, Euler allowed for arbitrary functions given by either a continuous curve or different expressions over different parts of the considered interval. In doubt of the findings of D'Alembert and Euler, Daniel Bernoulli discovered in 1753 that every motion of a taut string may be represented by the superposition of sufficiently many simple modes with suitable amplitudes. Shortly afterwards, Lagrange presented the first finite solution method when he divided a vibrating string in several equally spaced concentrated masses and showed that as many natural frequencies can be computed as masses are present in the model.

The linear theory of strings addressed so far is limited to small transversal vibrations, which also implies that the change in tensile force during vibration is small compared to the preload force. This assumption, however, is not valid in case of large amplitudes as encountered, for example, at or near resonance. Lee reported in 1957 on sudden jumps in amplitude and phase for frequencies close to the fundamental resonance frequency. Harrison and Oplinger found similar discontinuities in the response of strings. In addition, they observed, at certain critical amplitude values, motion perpendicular to the plane of the driving force. Accounting for these effects analytically leads to coupled nonlinear differential equations first derived by Murthy and Ramakrishna in 1965 [53].

Based on Kirchhoff's work on nonlinearity, Narashima derived in 1968 the nonlinear equations of motion of taut strings using perturbation methods [58]. As an example for recent research, the work of Molteno and Tufillaro is cited who referred to Narashima's equation when studying chaotic motion of strings [50].

Neither linear nor nonlinear string models are appropriate to represent heavy or low-tension slender continua, since in these cases, the effect of self-weight is not negligible anymore. Allowing for sagged equilibrium configurations of perfectly flexible slender continua leads to the theory of cables.

The problem of describing the curve formed by a suspended cable under the action of gravity was first studied in the early 17th century by Galileo who, however, mistook the curve for a parabola. In 1669, Jungius disproved Galilieo's claim, but the correct curve was not found until 1690 when Leibniz, John Bernoulli and Huygens solved nearly simultaneously the problem posed as challenge in a public contest. They could show that a homogeneous, perfectly flexible and inextensible cable hanging between two fixed points takes the shape of a catenary, that is, the shape of a hyperbolic cosine function [34]. The differential equations governing the equilibrium of a stretched cable satisfying Hooke's law were derived by Jacob and Johann Bernoulli, but its solution was not given before 1891 when Routh published the first closed-form solution [86].

The historical development of the vibration theory of cables is given by Irvine [27], who presented in particular a simplified approximate theory for shallow cables. More recently, the effect of non-linearities attracts increased attention. Hagedorn and Schäfer [20], for example, studied how the normal mode natural frequencies are effected when allowing for large sag. Another noteworthy study by Triantafyllou [90] deals with the dynamic response of cables under negative tension. The equations governing the dynamics of a perfectly flexible elastic or inelastic cable constitute an ill-posed problem when the tension becomes negative in any point along the cable, which causes difficulties in the numerical computation. By using Euler parameters instead of Euler angles and by including bending-stiffness terms, Tjavaras et al. [88] developed a singularity-free cable model which remains well-posed even in case of zero or negative tension.

Besides accounting for sagged equilibrium configurations, the incorporation of flexural rigidity represents another essential development in modeling slender continua. The equation governing the transverse vibrations of flexible thin beams was firstly derived in 1735 by Daniel Bernoulli. In 1744, the closed-form solutions for simply supported ends, clamped ends and free ends were found by Euler constituting what is called today Euler-Bernoulli beam theory. The inclusion of shear deformation is attributed to Timoshenko [92] whose name has established itself as synonym for this model. Extensions to further effects like rotary inertia, torsion, warping, and various other refinements follow in the 19th and 20th century.

Large displacements, stability problems, geometric and material nonlinearities, solution methods, and higher-order formulations are a few keywords for recently studied problems. While early developments in the theories of slender continua were primarily motivated by the main applications in those times, namely, buildings in architecture, nowadays the driving force for further developments is represented by applications in mechanical engineering, in vehicle and aerospace industry, in biomechanics, and so on. In this connection, a problem of particular interest is the dynamics of axially moving slender continua briefly reviewed in the sequel.

Research on Axially Moving Continua

The wide spread of axially moving systems in industrial applications has driven extensive research activity which began in the middle of the last century. The correspondingly vast literature on the dynamics of axially moving continua is comprehensively reviewed, for example, by Abrate [1], Wickert and Mote [101], and, more recently, by Chen [11]. Thus, the following discussion is restricted to the most interesting contributions, especially to those related to the present work.

One of the first documented studies on the vibrations of axially moving straight strings dates back to the late 19th century when Skutch [77] calculated the fundamental frequency of vibration for a string travelling between two spatially fixed supports. Since then, much effort has been put into the research on linear vibrations of axially moving strings basically leading to three major conclusions:

1. The natural frequencies decrease monotonically as the transport speed increases.
2. Disturbances travel at different speeds in the upstream and downstream directions.
3. Transverse vibrations feature a non-constant spatial phase.

Moreover, the natural frequencies reach zero when the transport speed equals the wave speed. The corresponding velocity is termed critical speed. It is interesting to note that axially moving strings and fluid conveying pipes can be described by the same equation, as firstly recognized by Archibalt and Emslie in 1958.

Another fundamental feature of the free transverse vibrations of axially moving strings concerns the total mechanical energy which, in contrast to the case of a non-moving string, is non-constant, as discovered by Lee and Mote [36]. Hence, a string in axial motion represents a non-conservative system.

As for the stability, axially moving continua may be subject to the following types of instabilities [31]:

- divergence instability at the critical speed,
- flutter instability at speeds above the critical speed, and/or
- Mathieu-type instability due to parametric excitation.

Parametric vibration may occur in axially moving strings due to tension variation and axial transport acceleration. Mote [52] showed that deceleration introduces a destabilizing effect in transverse vibration, while acceleration introduces a stabilizing or damping effect. Furthermore, Pakdemirli and Batan [62] found that a constant acceleration has a destabilizing effect compared with a harmonic speed variation. Öz et al. [59] studied the transition behavior from string to beam for an axially accelerating material. It could be shown that the incorporation of bending stiffness causes the stability boundaries to shift to higher frequency values. The stability of axially accelerating strings subject to frictional guiding forces was analyzed, amongst others, by Zen and Müftü [103]. All the studies mentioned above are based on Miranker's equation (1.1) whose validity is challenged by the thesis at hand.

As for fourth-order beam-like models, a careful definition of appropriate boundary conditions is required. In these systems, an initial curvature arises from the bending about the guiding pulleys or wheels leading to a single non-trivial curved equilibrium at subcritical speeds which bifurcates to multiple curved equilibrium states at supercritical speeds, which has been studied, for example, by Hwang and Perkins [24, 25]. While the natural frequencies associated with an axially moving string vanish at the same critical speed, the natural frequencies associated with an axially moving beam vanish at distinct critical speeds due to the dispersive nature of the system.

Assuming the beam effect to be small, Özkaya and Pakdemirli [61] applied perturbation methods to derive approximate solutions of the boundary-layer problem representing an axially accelerated beam.

The linear theory of axially moving strings and beams is essentially limited to small-amplitude motions of materially linear continua. Linear models are neither capable to account for tension variations due to transverse displacements nor do they incorporate the effect of speed variations due to deformations. Close to critical speeds, however, those nonlinear effects may dominate the dynamic response. As a rule of thumb, the linear theory underestimates stability for subcritical speeds, overestimates it for supercritical speeds and is virtually inapplicable in the close-up range of critical speeds [100].

Besides including certain nonlinearities and allowing for flexural rigidity, the incorporation of sagged equilibrium configurations caused by gravity represents another important model variant associated with the modeling of slack cables. It can be shown that a curved equilibrium state does not possess a critical speed as the taut string model would predict. Furthermore, there exists another arch-like equilibrium state which can be stabilized at sufficiently high transport speeds [31]. In horizontally travelling cables, the phenomena of frequency coalescence and mode reversion occur, that is, modes of higher order may revert to the shape of modes of lower order. Triantafyllou [89] studied the effect of various parameters, such as the elastic stiffness, inclination angle, sag-to-span ratio and speed variation, on the dynamics of inclined travelling cables. His research revealed in particular that the phenomenon of frequency coalescence never occurs in inclined cables which feature instead regions of strong mode interaction with significant amplification of the dynamic tension.

As indicated in Figure 1.1, an important branch of research on axially moving one-dimensional continua involves the modeling of varying-mass systems. The first studies on this field date back to the mid of the 20th century. Stühler and Kießling [81, 28], for example, derived the solution of a vertically hanging cable which is drawn through a fixed eyelet at constant speed. Unfortunately, the proposed approach proved to be too involved and specialized inhibiting a straightforward adaption to other problems. One of the first generally applicable models of stretched continua of varying length is due to Wauer [95]