Introduction

A selfadjoint operator A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ is called *definitizable* if the resolvent set $\rho(A)$ is nonempty and there exists a polynomial p such that $[p(A)x, x] \geq 0$ for all $x \in \operatorname{dom}(p(A))$. It was shown in [L1] and [L5] that a definitizable operator A has a spectral function E_A which is defined for all real intervals the boundary points of which do not belong to some finite subset of the real axis. With the help of the spectral function the real points of the spectrum $\sigma(A)$ of A can be classified in points of positive and negative type and critical points: A point $\mu \in \sigma(A) \cap \mathbb{R}$ is said to be of *positive type* (*negative type*) if μ is contained in some open interval δ such that $E_A(\delta)$ is defined and $(E_A(\delta)\mathcal{K}, [\cdot,\cdot])$ (resp. $(E_A(\delta)\mathcal{K}, -[\cdot,\cdot])$) is a Hilbert space. Spectral points of A which are not of *definite type*, that is, not of positive or negative type, are called *critical points*. The set of critical points of A is finite; every critical point of A is a zero of any polynomial p with the "definitizing" property mentioned above. Spectral points of positive and negative type can also be characterized with the help of approximative eigensequences (see [LcMM], [LMM], [J6]), which allows, in a convenient way, to carry over the sign type classification of spectral points to non-definitizable selfadjoint operators and relations in Krein spaces.

In this thesis selfadjoint operators and relations in a Krein space $\mathcal{K}$ which locally have the same spectral properties as definitizable operators and relations will play an important role. More precisely, let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real line such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected. We say that a selfadjoint operator or relation A is *definitizable over* Ω if

(i) every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that the spectral points in each component of $I_\mu \backslash \{\mu\}$ are all of the same sign type and

(ii) the spectrum of A in $\Omega \backslash \overline{\mathbb{R}}$ consists of normal eigenvalues which do not accumulate to $\Omega \cap \overline{\mathbb{R}}$ and the resolvent of A is of finite order growth near to $\Omega \cap \overline{\mathbb{R}}$.

Locally definitizable operators first occur in [L2] in connection with a perturbation problem. They were introduced in a similar way as above and studied by P. Jonas in [J1], [J2], [J3], [J6]. A selfadjoint operator or relation A which is definitizable over Ω has a unique spectral function E_A on $\Omega \cap \overline{\mathbb{R}}$. This local spectral function is defined for all real intervals δ, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$, the boundary points of which are spectral points of definite type or belong to $\rho(A)$. We refer to [J6] for a detailed study of locally definitizable operators and relations, different sign type classifications of spectral points and further references.

The first of our objectives in this thesis is to prove two theorems on compact and finite rank perturbations of locally definitizable operators and relations, which will be described in the following.

We will show in Theorem 2.1 that if A is a selfadjoint relation which is definitizable over Ω and all spectral subspaces $(E_A(\delta), [\cdot,\cdot])$, $\overline{\delta} \subset \Omega \cap \overline{\mathbb{R}}$, corresponding to A are Pontryagin spaces with finite rank of negativity, then the same holds true for a selfadjoint relation B if the difference of the resolvents of A and B is compact for some $\lambda \in \rho(A) \cap \rho(B) \cap \Omega$. We allow A and B to be selfadjoint with respect to different Krein space inner products. Here it is assumed that the difference of the corresponding Gram operators is compact. Theorem 2.1 was published in a different and slightly more general form in [BJ1].

For the special case of bounded selfadjoint operators this result was shown by H. Langer, A. Markus and V. Matsaev in [LMM]. For unbounded operators a different proof of Theorem 2.1 was recently given in [AJT], where so-called spectral points of type π were studied with the help of approximative eigensequences. Our proof of Theorem 2.1 is essentially a variant of the proof of Theorem 5.1 in [LMM]. Instead of the Lyubich-Matsaev spectral subspace results here we make use of a functional calculus for unitary operators in Krein spaces with finite order growth of the resolvent in a neighbourhood of some arcs of the unit circle (cf. [J1]).

Our second perturbation result concerns finite rank perturbations of locally definitizable operators and relations. Theorem 2.3 states that a selfadjoint relation which is locally definitizable over Ω remains locally definitizable over Ω after a finite dimensional perturbation in resolvent sense if the perturbed relation is selfadjoint and the unperturbed and perturbed relation have a common point in their resolvent sets belonging to Ω. For the case of definitizable operators this result was obtained by P. Jonas and H. Langer in [JL] by constructing a definitizing polynomial for the perturbed operator. The methods used in the proof of Theorem 2.3 differ from those applied in [JL]. Our proof is based on Theorem 2.1 and a recent result from [AJ] on the spectral properties of the inverses of certain matrix-valued functions associated to locally definitizable relations.

The second main objective in this thesis is the investigation of a class of abstract boundary value problems with boundary conditions depending on the eigenvalue parameter. Let now A be a closed symmetric operator or relation of finite defect n in the Krein space $\mathcal{K}$ and let $\{\mathbb{C}^n, \Gamma_0, \Gamma_1\}$ be a boundary value space for the adjoint relation A^+ (see Definition 3.1). Let Ω be a domain as above and let τ be an $\mathcal{L}(\mathbb{C}^n)$-valued function which is meromorphic in $\Omega\backslash\overline{\mathbb{R}}$ and symmetric with respect to the real line, that is $\tau(\overline{\lambda}) = \tau(\lambda)^*$ holds for all λ belonging to the set $\mathfrak{h}(\tau)$ of points of holomorphy of τ. We study boundary value problems of the following form: For a given $k \in \mathcal{K}$ and $\lambda \in \mathfrak{h}(\tau) \cap \Omega$ find a vector $\hat{f} = \binom{f}{f'} \in A^+$ such that

$$(0.1) \qquad f' - \lambda f = k \qquad \text{and} \qquad \tau(\lambda)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0$$

holds. Under additional assumptions on τ and A, a solution of this problem can be obtained with the help of the compressed resolvent of a selfadjoint extension $\widetilde{A}$ of A which acts in a larger Krein space. Such a selfadjoint relation $\widetilde{A}$ is said to be a *linearization* of the boundary value problem (0.1). Based on the coupling method from [DHMS1] (see also [HKS1], [HKS2]) we construct a linearization of (0.1) and we study its local spectral properties in Ω, which are closely connected with the solvability of the boundary value problem.

In the case that A is a symmetric operator or relation in a Hilbert space, τ is a Nevanlinna function or a generalized Nevanlinna function and Ω coincides with $\overline{\mathbb{C}}$, boundary value problems of the form (0.1) have extensively been studied in a more or less abstract framework in the last decades (see e.g. [DHMS1], [DL], [DLS1], [DLS2], [DLS3], [E], [LM], [R]). Problems of the type (0.1) with symmetric operators and relations of defect one in Krein spaces and special classes of scalar functions in the boundary condition were considered in [B], [BJ2] and [BT]. In [D1] and [D2] symmetric operators or relations of infinite defect and operator functions in the boundary condition were allowed. Very general classes of locally holomorphic functions in the boundary condition can be found in e.g. [D2], [DL] and [DLS2].

Here we will assume that τ is a locally definitizable function in Ω, that is, for every domain Ω', $\overline{\Omega'} \subset \Omega$, τ can be written as the sum of a definitizable function (cf. [J4], [J5]) and a function holomorphic on Ω'. Similarly to selfadjoint operators and relations definitizable over Ω the points in $\Omega \cap \overline{\mathbb{R}}$ can be classified in points of positive and negative type and critical points of τ. The well-known representation of Nevanlinna functions and generalized Nevanlinna functions with the help of resolvents of selfadjoint operators and relations in Hilbert and Pontryagin spaces (see e.g. [KL3]) was generalized to locally definitizable functions in [J7]. More precisely, the locally definitizable function τ can be minimally represented with a selfadjoint relation T_0 definitizable over Ω', $\overline{\Omega'} \subset \Omega$, in some Krein space $\mathcal{H}$ such that the sign types of τ and T_0 coincide in $\Omega' \cap \overline{\mathbb{R}}$.

In order to apply the coupling method for the construction of the linearization $\widetilde{A}$ of (0.1) we have to realize the function τ in the boundary condition of (0.1) as the Weyl function corresponding to a symmetric operator

$T \subset T_0$ and a boundary value space for T^+. We show in Theorem 3.9 that this is possible for an $\mathcal{L}(\mathbb{C}^n)$-valued locally definitizable function τ which is *strict*, that is,

$$\bigcap_{\lambda \in \Omega \cap \mathfrak{h}(\tau)} \ker \frac{\tau(\lambda) - \tau(\mu_0)^*}{\lambda - \overline{\mu}_0} = \{0\}$$

holds for some $\mu_0 \in \mathfrak{h}(\tau) \cap \Omega$. For matrix-valued generalized Nevanlinna functions this fact can be found in [DHS1] and for scalar local generalized Nevanlinna functions a proof was given in [BJ2]. We emphasize, that a Weyl function corresponding to a symmetric relation of finite defect and a boundary value space is in general not strict (see Example 3.15). In the case that τ is a non-strict locally definitizable function we show in Theorem 3.12 that τ can be written in the form

$$\lambda \mapsto \tau(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & \tau_s(\lambda) \end{pmatrix} + S, \quad S = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

where τ_s is a strict $\mathcal{L}(\mathbb{C}^s)$-valued locally definitizable function which is also minimally represented by the relation T_0, $s < n$, and S is a symmetric matrix constant. With the help of a suitable $(n-s)$-dimensional extension B of A and a boundary value space $\{\mathbb{C}^s, \Gamma_0^s, \Gamma_1^s\}$ for B^+ we rewrite the boundary value problem (0.1) in the form

$$f' - \lambda f = k, \quad \tau_s(\lambda)\Gamma_0^s \hat{f} + \Gamma_1^s \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in B^+. \tag{0.2}$$

A basic assumption will be that the selfadjoint extension $A_0 := \ker \Gamma_0$ of A in the Krein space $\mathcal{K}$ is locally definitizable over Ω and that the sign types of A_0 and τ are d-compatible in $\Omega \cap \overline{\mathbb{R}}$ (see Definition 3.16). Then the selfadjoint relation $A_0 \times T_0$ in the Krein space $\mathcal{K} \times \mathcal{H}$ is locally definitizable. The linearization $\widetilde{A}$ of the boundary value problem (0.1), (0.2) is a finite dimensional perturbation in resolvent sense of $A_0 \times T_0$. Therefore, by Theorem 2.3, $\widetilde{A}$ is also locally definitizable and its sign types are d-compatible with the sign types of A_0 and τ in $\Omega \cap \overline{\mathbb{R}}$.

This thesis is organized as follows. In Section 1.1 and Section 1.2 we provide some basic definitions and we introduce the spectral points of positive and negative type with the help of approximative eigensequences. We recall the definitions of locally definitizable selfadjoint relations and selfadjoint relations locally of type π_+ in Section 1.3 and we introduce the local spectral function. Section 1.4 is devoted to matrix-valued locally definitizable functions and matrix-valued local generalized Nevanlinna functions. In particular Theorem 1.12 on minimal operator representations of these functions from [J7] and [J8] will be used in the proofs of Theorem 2.3 and Theorem 3.9.

Section 2 consists of the two results on compact and finite rank perturbations of locally definitizable selfadjoint relations described above.

In Section 3 we investigate boundary value problems of the form (0.1). First we recall the concepts of boundary value spaces and corresponding Weyl functions for closed symmetric relations in Krein spaces. If the symmetric relation A in the Krein space $\mathcal{K}$ is of finite defect and has a selfadjoint extension which is definitizable over some domain Ω we conclude from the well-known resolvent formula (Theorem 3.3) and Theorem 2.3 on finite rank perturbations that all selfadjoint extensions A_Θ of A in $\mathcal{K}$ with $\rho(A_\Theta) \cap \Omega \neq \emptyset$ are definitizable over Ω (see Theorem 3.4).

Section 3.2 deals with boundary value spaces and Weyl functions of direct products of closed symmetric relations. Theorem 3.5 and Theorem 3.6 are essentially a consequence of the general transformation properties of boundary value spaces and can be found in a slightly different form in [DHMS1].

In Section 3.3 we show how strict matrix-valued locally definitizable functions can be realized as Weyl functions corresponding to symmetric operators of finite defect and suitable boundary value spaces. Theorem 3.12 deals with the non-strict case. A simple example of a symmetric operator of defect one in $(\mathbb{C}^2, [\cdot,\cdot])$ and a boundary value space where the corresponding Weyl function is identically equal to zero will be given at the end of Section 3.3.

The λ-dependent boundary value problem (0.1) is studied in Section 3.4. First we introduce the notion of d-compatibility of sign types of locally definitizable functions and locally definitizable selfadjoint relations in Definition 3.16. The main result in Section 3.4 is Theorem 3.18. Here we construct a minimal linearization $\widetilde{A}$ of the boundary value problem (0.1), (0.2) such that the compressed resolvent of $\widetilde{A}$ onto the basic space yields the unique solution of (0.1), (0.2). In Theorem 3.20 we show that the eigenvectors corresponding to an eigenvalue μ of $\widetilde{A}$ yield solutions of the "homogeneous" boundary value problem

$$f' - \mu f = 0 \quad \text{and} \quad \tau(\mu)\Gamma_0 \hat{f} + \Gamma_1 \hat{f} = 0, \quad \hat{f} = \begin{pmatrix} f \\ f' \end{pmatrix} \in A^+.$$

We finish Section 3.4 with some special cases of Theorem 3.18.

In Section 3.5 we formulate the main result from Section 3.4 for the case that A has a selfadjoint extension which is locally of type π_+ and the function τ is a strict local generalized Nevanlinna function. Finally, in Theorem 3.27, we consider the "global" case, that is, we assume that A is a densely defined operator in a Pontryagin space and τ is a not necessarily strict matrix-valued generalized Nevanlinna function.

1. Locally Definitizable Selfadjoint Relations and Locally Definitizable Functions

In this section we introduce a class of selfadjoint operators and relations which admit a spectral decomposition into two relations one of which is definitizable. Moreover, we define a class of functions which correspond to these operators and relations. For a detailed study of locally definitizable selfadjoint relations and locally definitizable functions we refer to the recent papers [J6] and [J7] of P. Jonas.

1.1. Spectral Points of Positive and Negative Type

The linear space of bounded linear operators defined on a Krein space $\mathcal{K}_1$ with values in a Krein space $\mathcal{K}_2$ is denoted by $\mathcal{L}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} := \mathcal{K}_1 = \mathcal{K}_2$ we simply write $\mathcal{L}(\mathcal{K})$. We study linear relations from $\mathcal{K}_1$ to $\mathcal{K}_2$, that is, linear subspaces of $\mathcal{K}_1 \times \mathcal{K}_2$. The set of all closed linear relations from $\mathcal{K}_1$ to $\mathcal{K}_2$ is denoted by $\widetilde{\mathcal{C}}(\mathcal{K}_1, \mathcal{K}_2)$. If $\mathcal{K} = \mathcal{K}_1 = \mathcal{K}_2$ we write $\widetilde{\mathcal{C}}(\mathcal{K})$. Linear operators from $\mathcal{K}_1$ into $\mathcal{K}_2$ are viewed as linear relations via their graphs. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [DS1]. The sum and the direct sum of subspaces in $\mathcal{K}_1 \times \mathcal{K}_2$ will be denoted by $\boldsymbol{+}$ and $\dot{\boldsymbol{+}}$.

In the following let $(\mathcal{K}, [\cdot,\cdot])$ be a separable Krein space and let S be a closed linear relation in $\mathcal{K}$. The *resolvent set* $\rho(S)$ of S is the set of all $\lambda \in \mathbb{C}$ such that $(S-\lambda)^{-1} \in \mathcal{L}(\mathcal{K})$, the *spectrum* $\sigma(S)$ of S is the complement of $\rho(S)$ in $\mathbb{C}$. The *extended spectrum* $\widetilde{\sigma}(S)$ of S is defined by $\widetilde{\sigma}(S) = \sigma(S)$ if $S \in \mathcal{L}(\mathcal{K})$ and $\widetilde{\sigma}(S) = \sigma(S) \cup \{\infty\}$ otherwise. The *extended resolvent set* $\widetilde{\rho}(S)$ of S is defined by $\widetilde{\rho}(S) = \overline{\mathbb{C}} \backslash \widetilde{\sigma}(S)$.

A point $\lambda \in \mathbb{C}$ is an *eigenvalue* of S if $\ker(S-\lambda) \neq \{0\}$; we write $\lambda \in \sigma_p(S)$. We say that $\lambda \in \mathbb{C}$ belongs to the *continuous spectrum* $\sigma_c(S)$ (the *residual spectrum* $\sigma_r(S)$) of S if $\ker(S-\lambda) = \{0\}$ and $\operatorname{ran}(S-\lambda)$ is dense in $\mathcal{K}$ (resp. if $\ker(S-\lambda) = \{0\}$ and $\operatorname{ran}(S-\lambda)$ is not dense in $\mathcal{K}$). An eigenvalue $\lambda \in \mathbb{C}$ of a closed linear relation S is called *normal* if the root manifold

$$\mathcal{L}_\lambda(S) := \bigcup_{k=0}^{\infty} \ker\big((S-\lambda)^k\big)$$

corresponding to λ is finite-dimensional and there exists a projection P with $P\mathcal{K} = \mathcal{L}_\lambda(S)$ such that

$$S = S \cap \big(P\mathcal{K}\big)^2 \,\dot{\boldsymbol{+}}\, S \cap \big((I-P)\mathcal{K}\big)^2$$