Chapter 1

Introduction

1.1 Motivation

Simulation of turbulent flows has gained increasing importance due to its incorporation as a powerful design tool into the industrial development process. Today and in the near future the computational power needed to model the governing equations of technical flows are not available. Modeling assumptions need to be introduced to reduce the computational effort to resolve physical effects such as turbulence which acts up to very small scales but affects the main flow properties like heat transfer or friction coefficients. Beside progress in many theoretical and experimental areas turbulence remains the most important unsolved problem in classical physics.

The importance of turbulence rests upon its significance in natural and technical flows. Examples of turbulent flows in nature are the gaseous flow in the atmosphere and the motion of water in the oceans. Also biological flows like cardiac flows at high pulse are governed by turbulence.

Turbulence is an important factor in nearly all technical flows, especially in flows for energy transformation. Mankind generates the major part of its energy for transportation and electricity from fossil energy by using combustion as the transformation process from chemical energy into heat and electrical power. These combustion processes are mixture driven and these mixing processes are controlled by turbulence.

Therefore computation and description of these important flows can only be successful if the underlying physics of turbulence is fully understood. Without the deep appreciation of turbulence it is not possible to optimize flows and energy transfer with the aim of low fuel consumption and low exhaust emissions to control pollution.

Engines and plants are complex technical systems that inherit complex geometries. Hence it is impossible in most cases to obtain analytical solutions on these applications. Here numerical simulations are of major importance for a controlled modeling of energy transformation applications. These computations are limited, due to turbulence, since turbulence's physics are ruled on a large variety of scales. And these scales must be resolved to describe the turbulence effects of the small scales on the large scale flow properties. Due to nonlinearities in the governing equations the small scales influence global flow, and are therefore needed to obtain physical reasonable solutions.

The range of scales depends on the turbulent Reynolds number which represents the ratio of the nonlinear turbulent convective term to the viscosity in the governing equations. In many technically flows this ratio is very large and causes the necessity to resolve the smallest scales of turbulence in numerical simulations of flows. Therefore it requires numerical calculations on a large number of grid points. The limiting factor in nowadays computations to compute large Reynolds number turbulent flows are memory and processor capacity of computational devices.

The main interest in computations are the behavior of flows on large scales. Hence models for turbulence are needed that represent the turbulence physics on the small scales without resolving the whole range of turbulent length scales.

1.2 Modeling of Incompressible Flows

Different turbulence modeling strategies were developed in the past. Most of them are based on phenomenological and empirical assumptions that do not hold in the full application range of turbulent flows. A large amount of very different methods has been established in the past possessing a wide range of complexity.

The oldest one is the statistical approach that is based on the statistical averaging introduced by Osborne Reynolds [68] at the end of the nineteenth century. Reynolds averaging is introduced in the Navier-Stokes equations and results in a set of equations for the mean values with new unclosed nonlinear products of fluctuating velocities which stem from the nonlinear convective terms. These so called Reynolds stresses need to be closed by modeling assumptions. The different closure models can be characterized by zero, one, two-equation models and Reynolds stress tensor models, depending on the number of equations that yield to a closure. Another group of models are the sub-grid scale models for the large eddy simulation (LES) that were developed at the end of the seventies, when computational power became accessible. Far a detailed overview of these methods see [76]. If the full range of scales is resolved we speak of direct numerical simulation (DNS) that is not based on models beyond the Navier-Stokes equations since no empirical closure is needed. First DNS calculations on turbulent flows are described by Rogallo [73]. Its limitation lies in its restriction to low Reynolds numbers [51] that are orders of magnitude below Reynolds numbers in technical applications. In addition only geometrically simple flows are considered. Therefore DNS is mostly used as an investigative research tool in turbulence theory.

The third group of turbulence modeling are the two-point or spectral methods, which are also based on Reynolds averaging. They serve in theoretical investigations of turbulence since they are usually limited to homogeneous flows and are difficult to apply to flows with walls which exist in most technical applications.

In the current work we investigate the two-point correlation equations and also refer to Fourier methods for a validation of analytical obtained models.

1.3 Modeling of Compressible Flows

Turbulent flows in the low Mach number regime show different behavior than in incompressible flows e.g. the decay of homogeneous turbulence is faster. Also in homogeneous shear flows a lower growth rate of turbulent kinetic energy is observed for compressible flows. The spreading rate of a compressible shear layer is decreased by compressibility. These effects need to be included into semi-empirical turbulence models that are applied to flows in the compressible regime.

Existing turbulence models for the incompressible regime do not reproduce these effects of compressibility and therefore need to be expanded to take the compressibility into account. Due to the "simplicity" of incompressible flow most theoretical and numerical investigations of turbulent flow were undertaken in the incompressible regime. The governing equations for mass and momentum decouple from the energy equation in incompressible flows, so that a solution of the flow may be obtained only by mass and momentum transfer. The next step in complexity is to investigate variable density flows that take into account a time dependent but spatially homogeneous density. A fully compressible flow is characterized by density variations in time and in space. Here we will limit ourselves to weakly compressible flows in which no shocks occur, not even locally [40]. This will limit the Mach number which is a ratio of flow velocity to the speed of sound to less then $M = 0.5$ to avoid transonic spots. Canuto *et al.* [16] gives a broad overview about spectral methods of compressible flows.

1.4 Methods Involved

A two folded approach is taken in this work. First we construct similarity laws for compressible turbulent flows using symmetry (Lie Group) methods, e.g. see [11], [61], [34] and [14]. Since the Navier-Stokes equations that describe compressible turbulent flows inherit a singularity at a Mach number $M = 0$ we need to expand the governing equations with asymptotic methods in the Mach number.

Second the governing equations are solved using DNS of incompressible and compressible flow. Here the Navier-Stokes equations are solved for simple turbulent flows without applying any closure models. For high accuracy of the results we choose a spectral numerical method. It is of high accuracy but also limited to simple boundary conditions.

1.5 Previous Work

Early analytical work on compressible turbulent flows was done by Kovasznay [43]. He introduced the decomposition into modes representing the incompressible and compressible modes and analyzed the relationships between them using a linear theory. Klein [40] used and asymptotic expansion in Mach number to separate effects that scale in different orders of Mach number.

First numerical work on DNS of homogeneous turbulence was proceeded by Rogallo [72] in the incompressible limit and by Feiereisen, Reynolds and Ferziger [23] for compressible flow, followed by the works of Passot and Pouquet [62] and Delorme [18]. Further on, the group of Erlebacher, Sarkar, Zhang and Hussaini *et al.* [80], [78], [79], [20] at ICASE carried out an asymptotic approach to model the compressible effects and compared the analytical results with DNS data. They used an expansion in Mach number, e.g. see [40], of the dependent variables to obtain an equation satisfying incompressible flow to the leading order in combination with a first order equation that fulfilled the remaining set of the Navier-Stokes equations. They obtained a correction for homogeneous isotropic flow that matched the effects of compressibility, from which they derived a model which is an expansion to the two-equation turbulence model. Similar work with a different approach was undertaken by Zeman in [97] and [98]. Further detailed DNS and analysis was proceeded by Blaisdell in [7] and [70] who also focused on the proposition of consistent initial conditions. All these research covers the low Mach number range $M = 0$ to $M < 0.5$.

In the current work we will outline a procedure to gain insight and models for compressible turbulent flows using first principles only. The results from this new method shows to be more general and covers the results of previous work in certain limits.

Chapter 2

Fundamentals of Turbulence

2.1 Navier-Stokes Equations

Fluid flows can be described by transport equations for mass, momentum and energy. The fundamental equations of fluid mechanics are named the Navier-Stokes equations, and, for compressible fluids in Cartesian coordinates and tensor notation, taking advantage of the Einstein summation, the formulation of the continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \ . \tag{2.1}$$

The conservation of momentum is described by

$$\frac{\partial \rho u_j}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_i} + \frac{\partial p}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_i} \ , \tag{2.2}$$

and the transport equation of internal energy e is given by

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \rho u_i e}{\partial x_i} + p\frac{\partial u_i}{\partial x_i} = \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \Phi \tag{2.3}$$

with the stress tensor described by the Stokes relation

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\mu\frac{\partial u_k}{\partial x_k}\delta_{ij} \tag{2.4}$$

and the dissipation

$$\Phi = \tau_{ij}\frac{\partial u_i}{\partial x_j} \ . \tag{2.5}$$

This system of partial differential equations is closed by the equation of state for an ideal gas

$$\rho = \frac{p}{RT} \quad . \tag{2.6}$$

The equation of in its conservation form (2.3) can be rewritten in the primitive variable p using $e = c_v T$, where c_v is the heat capacity at constant volume (and c_p is the heat capacity at constant pressure) and the equation of state. Then we obtain

$$\frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \gamma p \frac{\partial u_i}{\partial x_i} = (\gamma - 1)\left[\frac{\partial}{\partial x_i}\left(k \frac{\partial T}{\partial x_i}\right) + \Phi\right] . \tag{2.7}$$

Here γ is the ratio of the heat capacities at constant volume and pressure

$$\gamma = \frac{c_p}{c_v} \quad . \tag{2.8}$$

The relation between the heat capacities and the ideal gas constant R is described by

$$R = c_p - c_v \quad . \tag{2.9}$$

2.2 Non Dimensional Form

After normalization of the independent and dependent variables by their reference values, ρ_R for the reference density, u_R representing the reference velocity and T_R for the reference temperature, the non-dimensional values are obtained and denoted by a superscript

$$l^* = \frac{l}{l_R} \quad , \tag{2.10}$$

$$t^* = \frac{t \cdot u_R}{l_R} \quad , \tag{2.11}$$

$$\rho^* = \frac{\rho}{\rho_R} \quad , \tag{2.12}$$

$$u_i^* = \frac{u_i}{u_R} \quad , \tag{2.13}$$

$$p^* = \frac{p}{p_R} \qquad \text{with} \quad p_R = \rho_R u_R^2 \quad , \tag{2.14}$$

$$T^* = \frac{T}{T_R} \quad . \tag{2.15}$$

We introduce the non-dimensional Reynolds, Prandtl and Mach number, which are defined as

$$Re = \frac{u_R L_R}{\nu_R} \ , \tag{2.16}$$

$$Pr = \frac{c_{pR}\mu_R}{\lambda_R} \ , \tag{2.17}$$

$$M = \frac{u_R}{c_R} \ , \tag{2.18}$$

where c_R denotes the reference speed of sound, L_R a reference length, ν_R the reference dynamic viscosity, c_{pR} the reference specific heat capacity at constant pressure, μ_R the reference dynamic viscosity and λ_R the reference heat conductivity. The conservation equations of mass, momentum and energy (2.1)-(2.3) can be written in their non-dimensional form. After dropping the superscripts the following set of equations is obtained:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0 \ , \tag{2.19}$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} = \frac{1}{Re}\frac{\partial \tau_{ij}}{\partial x_j} \ , \tag{2.20}$$

$$\frac{\partial p}{\partial t} + u_i \frac{\partial p}{\partial x_i} + \gamma p \frac{\partial u_i}{\partial x_i} = \frac{1}{RePrM^2}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right) + \frac{(\gamma - 1)}{Re}\Phi \ . \tag{2.21}$$

This set of equations cannot be solved analytically for the dependent variables but similarity laws can be derived and therefore the equations (2.19)-(2.21) are investigated by symmetry methods in chapter 3. These analytic results contain constants, ratios of group parameters and constants from integration, which cannot be fixed by the method of symmetries alone and therefore in chapter 4 numerical methods are used from which these constants are determined by comparison.

2.3 Statistical Averaging

In turbulent flow fluctuations of the dependent variables u_i, p and ρ are observed. Depending on Reynolds number these fluctuations act on rather small length scales and cannot be resolved in numerical computations for most engineering problems. Since in technical applications engineers are mostly interested in the flow effects on large scales like drag or shear stresses on a body or averaged heat transfer coefficients, Reynolds [67] introduced a statistical averaging. This Reynolds averaging will exemplarily be shown on an arbitrary variable Z representing the dependent flow variables such as u_i, p and ρ.