Introduction

This thesis deals with domain decomposition methods for advection-diffusion equations. In the last years domain decomposition methods have become a very active research area in the field of the numerical approximation of partial differential equations. The key idea of domain decomposition methods is simple to explain: The global boundary value problem, given in a domain Ω, is divided into local boundary value problems in subdomains Ω_i, the union of which gives Ω (cf. Figure 1). The local problems are linked together by suitable coupling terms or transmission conditions. This leads to discrete schemes like the *mortar method* (cf. [BMP94, Bel99, Woh99]) or the *three-field formulation* (cf. [BM94, BM92]). The more general latter approach is presented here.

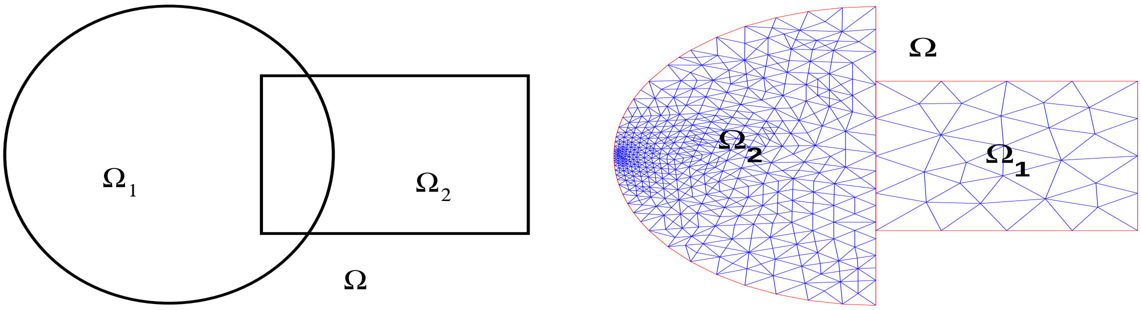
Domain decomposition methods allow to couple different models, i.e. different partial differential equations or different discretization methods on local subdomains. In this work we concentrate on finite element discretizations given in local subdomains. Each discretization can be independent of the remaining ones (cf. Figure 1 (b)). Therefore we are interested in the case of nonmatching grids, which causes nonmatching ansatz functions on the boundaries of the subdomains. By virtue of our approach it is possible to apply different software tools for specific geometries on complex domains by dividing the domain into subdomains with these specific geometries.

Having such a multi-domain formulation there are several strategies to split the global problem into a sequence of local problems by iterative decoupling. Assigning the local problems to different processors we get a very intrinsic way to solve our numerical problems in parallel. In complex three-dimensional domains the use of parallel methods is mandatory.

The resulting methods can be classified into several groups. First it can be differentiated between *nonoverlapping* and *overlapping methods*. In the overlapping case the domain Ω is divided into overlapping subdomains Ω_i. The *alternating Schwarz method*, introduced by H.A. SCHWARZ in 1869[1], was probably the first example of a domain decomposition method. Starting with a decomposition into two overlapping subdomains Ω_1 and Ω_2 (cf. Figure 1 (a)) the equations are solved iteratively on the subdomains using Dirichlet values of the neighbor domains computed in the previous step. In this way H.A. SCHWARZ could show the existence of a solution of the Poisson problem for a domain with nonsmooth boundary.

Moreover one can distinguish between *additive* and *multiplicative Schwarz methods*. Denoting the solution of iteration step i in subdomain Ω_j by u_j^i for the two-domain case the multiplicative variant can be described as follows: Starting with an initial guess, first a new solution in Ω_1 is computed. Then, already using this solution, the solution in Ω_2 is solved, and so on. In contrast the additive algorithm uses the solution of the previous step instead of the current solution (cf. Figure 2). The second method has got the advantage that the solution of all subdomain problems

[1] cf. O.B. WIDLUND [Wid90] for a short history of domain decomposition methods



(a) The original example of H.A. SCHWARZ (b) Decomposition into simple domains

Figure 1: *The figure shows two simple decompositions. (a) is an overlapping decomposition. In (b) the meshes of Ω_1 and Ω_2 are nonmatching at the interface.*

can be completely done in parallel. In the multi-domain case the multiplicative variant requires a coloring of the subdomains.
In this thesis we focus on nonoverlapping methods. Overlapping methods have the drawback of some overlap of data and very often the partitions are much harder to generate. Moreover, different models in different subdomains require the nonoverlapping approach.
A direct analogue of the Schwarz algorithm to the nonoverlapping case is not possible, because in general the iterative scheme does not converge, if Dirichlet data of the subdomain boundaries is interchanged. But if we replace the Dirichlet-condition by other transmission conditions like Neumann- or Robin-conditions, we get further classes of methods, sometimes called *iteration-by-subdomain* methods. This leads to schemes like the Robin-Robin (cf. [LMO00, NR95]), the Dirichlet-Neumann (cf. [GGQ96]) or the Robin-Dirichlet (cf. [ATV98]) method. To demonstrate these methods the interchanging of Robin conditions across the interface is discussed in this work.

A second well established class of methods, called *iterative substructuring methods*, is given by a linear system for the interface degrees of freedom, which is constructed by eliminating the unknowns inside the subdomains. On the discrete level the resulting equation is called the *Schur complement equation*; on the continuous level the equation depends on the *Steklov–Poincaré operator*. Applying the Steklov–Poincaré operator resp. the Schur complement matrix corresponds to the solution of local problems with Dirichlet conditions on the interface. Normally the discrete equation is solved by an iterative algorithm. Especially Krylov methods like CG or GMRES methods are used, where each step requires the solution of local boundary value problems. Since the interface equation is poorly conditioned, preconditioning is essential for an efficient implementation. The construction of good preconditioners for the Schur complement equation is a very active research area. In order to be able to parallelize the solution procedure, the preconditioners are built by local problems. So we get for example the BPS-preconditioner (cf. [BPS86]), the Neumann-Neumann preconditioner (cf. [DW95, DSW94]) or the Robin-Robin preconditioner (cf. [AJT$^+$99, ATNV00]). A variant of the latter preconditioner is presented in chapter 6.

In this work we try to give a unified presentation of some nonoverlapping domain decomposition methods for the stationary advection–diffusion–reaction equation

$$-\epsilon\triangle u + \mathbf{b}\cdot\nabla u + cu = f$$

additive Schwarz algorithm	**multiplicative Schwarz algorithm**
1. initial guess u_1^0, u_2^0 2. $i = 0$ 3. **until convergence** 4. $\quad i = i + 1$ 5. $\quad$ Compute u_j^i using u_j^{i-1}, $j = 1, 2$ 6. **end**	1. initial guess u_2^0 2. $i = 0$ 3. **until convergence** 4. $\quad i = i + 1$ 5. $\quad$ Compute u_1^i using u_2^{i-1} 6. $\quad$ Compute u_2^i using u_1^i 7. **end**

Figure 2: *Additive and multiplicative Schwarz algorithm for two subdomains.*

in a bounded domain Ω. The starting point of the analysis is a variant of the *three-field formulation* of F. BREZZI and D. MARINI (cf. [BM94, BM92]). Given a partition of Ω into subdomains Ω_i three different classes of function spaces are defined. The first one lives on the local subdomains, the second one is a space of Lagrange multipliers defined on the local boundaries of the subdomains and the third one is given on the union of the local subdomains, called (global) interface. If these spaces are coupled by specific terms, we get an alternative, well posed, hybrid problem. This formulation is treated in chapter 3.
A direct discretization of this scheme requires, that two conditions, called Babuška-Brezzi conditions, are satisfied. The mathematical treatment of the arising saddle point problems is briefly discussed in the appendix. The first one demands that the function space of the local functions is sufficiently 'rich' compared to the space of Lagrange multipliers. In contrast the second inf-sup condition requires the same relation between the space of Lagrange multipliers and the third class of functions. But because the discrete ansatz spaces should be chosen completely independent, in this work the Babuška-Brezzi conditions are circumvented by adding some stabilization terms.
Further difficulties arise in the singularly perturbed case, the case of $\epsilon << 1$. Therefore we introduce the SUPG-method in the local subdomains in order to suspend oscillations in streamwise direction. Together with the above stabilization terms we get a new stabilized three-field formulation. Its analysis is discussed in chapter 4. An a-priori result is derived in special consideration of the singularly perturbed case and is used to determine certain stabilization parameters. Our results are optimal compared to the standard SUPG-method.
When using this approach on the local subdomains local Dirichlet problems arise in an intrinsic way. The boundary conditions are worked in with the help of Lagrange multipliers. Since the arising local systems are interesting by themselves (fictious domain approaches, wavelet discretizations), we investigate them in detail in chapter 2 and derive a priori and a posteriori estimates. So far in the literature these schemes have not been extended to the nonsymmetric case nor extensive numerical studies have been carried out. Here, we will close this gap.

In a next step it is shown, that the stabilized three-field formulation is a proper basis for a unified presentation of nonoverlapping methods. This is demonstrated on the continuous and the discrete level for two typical algorithms in part III of the thesis.
In chapter 6 the Schur complement equation is derived from the three-field formulation. As a preconditioner we use a proposal of Y. ACHDOU ET AL. (cf. [AJT$^+$99, ATNV00]). The preconditioner is built up by solving local boundary value problems with Robin conditions on the interface. Unfortunately the analysis of this method is not complete. Because of the nonsymmetric structure

of the problem the standard techniques for symmetric problems cannot be applied.
In chapter 7 an iteration-by-subdomain algorithm is derived following a technique of R. GLOWINSKI and P. LE TALLEC (cf. [GT89, GT90]). So we get an algorithm, where in each iteration step Robin conditions at the local interfaces are interchanged. Finally, both methods will be compared by some numerical experiments and by a Fourier analysis for the case of two subdomains and constant coefficients (cf. G. RAPIN, G. LUBE [RL01]).
Moreover in chapter 5 it is explained, how the three-field formulation can be extended to the Oseen equations. The presence of the pressure and the divergence-free constraint cause additional difficulties. This is the first attempt of such an extension. The Oseen equations arise in many linearization strategies of the Navier–Stokes equations. Therefore, normally a huge amount of degrees of freedom is used in order to resolve the finer scales of the solution. Hence, parallel methods for the Oseen equations are very important.

The thesis is split into four parts. In the first part we introduce the advection-diffusion-reaction equation and discuss weakly enforced Dirichlet conditions for a single domain. The second part is dedicated to the three-field formulation and includes the chapters about the stabilization and the extension to the Oseen equations. In part III we show, how the three-field formulation can be solved efficiently by iterative decoupling. We present two different algorithms and compare their performance.
We complete this work by an appendix, where the functional setting and some auxiliary results are presented: In appendix A some basic results of functional analysis are cited and the theory of saddle point problems is developed. Then the definitions and properties of different function spaces, which are used, are shortly reported in appendix B. Finally, we give a brief introduction to the theory of finite element methods.

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