Introduction

Though flow is a strictly defined mathematical term, almost everyone working in this particular field of mathematics keeps in mind his own imaginary model of flow. One may think of a flow as water flowing in a river from the source to the sea. Different rivers may merge to a wider stream, or a river may branch into different creeks. In any case the paths the water may take are preshaped by the topography of the surface, and the floating water has to respect this shape except for the slow changes it effects by erosion. The flow of water is both a model and an application that shares several features with what is called a flow in discrete mathematics. Yet, the following abstraction is pivotal to the classical theory of flows: The shape of the river is the exclusive focus of mathematical reasoning.

The discrete mathematical structure in which flows are considered is a so called network, i. e., a graph with edge-capacities. A graph is a set of points, called nodes, together with a set of pairs of nodes, called edges, representing connections between two nodes. The essential property of a flow on such a structure is that at any node — except for some preselected ones called source and sink — flow may neither appear nor vanish. To be precise, a flow is described by assigning to each edge an amount — smaller than the edge-capacity — that passes along the edge in a fixed direction, such that for each node the total amount of flow entering it equals the total amount of flow leaving it.

This mathematical object of a network-flow proved to be very flexible and of great importance. As the name suggests one can immediately model problems dealing with transportation streams. But also scheduling problems or even social ones as the representation of minorities or religious belongings in groups can be modeled. But flows would never have acquired their importance in the field of discrete mathematics, if they were only a tool for comfortable problem description. Flows proved to be a story of success with respect to constructive, algorithmic tasks. At first an algorithm for the basic problem of finding a maximum or demand satisfying flow with a single source and a single sink was developed. In the following more efficient algorithms and algorithms for more complex flow problems with additional side constraints have been found. Researchers got more insights into the structure of flow problems. Consequently, new areas of applications for flows have been recognized. In the fairly short time since its introduction, several books have been written, to collect and restructure the state of the art knowledge about flows in graphs; to mention some of them we refer to [FF62, Hu69, JB80, AMO93, KV02, Sch03]. Though each of these publications was bigger than its predecessor, still they can hardly cover all facets in this field of research in detail.

Research on flows has branched into sections considering different refined definitions of flow. Still, flow is mostly consider to be defined by a local property, namely that in a normal (i. e. neither sink nor source) node no flow may appear or vanish. Therefore, one can conceive of mathematical flows as memoryless or objects without history. In contrast, the flow of each drop of water in a river does of course have a history. Actually, one could in a similar sense examine a mathematical flow together with its history: Each unit of flow has a unique path along which it floats from its source to its sink. What kind of restrictions for that kind of history are mathematically fruitful and reasonable for possible applications? It is the aim of this thesis to develop some insights and answers to this question.

Two very different types of restrictions will be considered. In the first part of this thesis we deal with a very natural constraint comparable to the age of a person. That is, we approve each unit of flow a maximal distance (life time) which it may travel from its source to its sink. In this setting it is no longer sufficient to ensure flow conservation at each non source/sink node to obtain a flow. At any arbitrary node the history of an arriving unit is of interest for the following path decisions. In other words we require from each source-sink-path along which at least some flow units travel a bounded length. Most of the useful methods developed for standard flows are breaking down by adding this slight restriction. In turn, revisiting possible application we are suddenly in a position to model transportation problems with quality management. May be, we have to distribute strawberries from different plantations to shops. Due to its perishableness no strawberry should be more than a day on transport. That is, we have a bound on each used path with regard to its time-length.

In the other part of this thesis we do not care for the individual flow units but for their common behavior. Precisely, we allow only a restricted number of different histories among all of them. In return for that, the choice of these histories is unconstrained. That is, there is only a bounded number of source-sink-paths allowed along which some flow travels. Flows of this type can be used to model transportation problems of very large sized demands. This demands may be splitted into chunks of different sizes. Still, the number of chunks should be small, e. g., cutting and agglutination may decrease the quality or is impossible. Hence, to ensure a certain quality only a bounded number of flow paths may be used.

In both parts we concentrate on single source and single sink flows. These basic problems already prove to be algorithmically hard and of an entirely different strucuture than their memoryless pendants.

Both parts will be conducted by the same pattern of analysis in order to illuminate properties and structures of the flows under consideration. In a first step we are interested in the kind of flow values that may occur: Can they be arbitrary, or will they meet some integrality or unity conditions? Next we consider complexity issues concerning decision and optimization problems including approximability questions. For a further analysis, dual problems for the different kinds of flow are developed. That amounts to taking a destructive point of view and look for a substructure, a cut, which destroys each flow of interest. Consequently, the complexity of the relevant cuts has to be scrutinized, too. Finally, we always try to find efficient, exact or approximative algorithms for the problem in question.

OVERVIEW OF THE THESIS

CHAPTER 1: The notation, terminology, and the basic definitions for the objects of interest are given. For later reference we recall some well known facts from graph theory and flows in graphs. For completeness, we describe a framework for a labeling algorithm to find multi-criteria optimal paths. Finally, we state the Chernoff bound which estimates the tail probabilities of certain random variables.

CHAPTER 2: This chapter deals with length-bounded flows and cuts and is divided into three parts. In the first part fractional flows are considered. Properties of optimal solutions like size or complexity issues are investigated: It is proved that there exists always an optimal solution of polynomial size. However, to determine whether a given value is the maximum flow value of a length-bounded single-commodity flow is NP-hard. Examples with integral edge-capacities are given such that each maximum length-bounded single-commodity flow is very fractional. Then an algorithm to find approximately maximum length-bounded flows is developed.

In the second part cuts are considered. A lower bound of order $\Omega(\sqrt{|V|})$ for the possible integrality gap between a fractional and an integral cut is given. It is shown that it is APX-hard to find a minimum length-bounded cut in arbitrary graphs with edge-capacities and that it is still NP-hard if one restricts oneself to (outer-)planar graphs. An approximation algorithm for a special case is given.

The third and last part is devoted to integral length-bounded flows. We investigate the integrality gap of the integer program for length-bounded single-commodity flows and its linear program relaxation and prove that it can be of order $\Omega(\sqrt{|V|})$. We review some hardness results to find maximum integral single-commodity flows and show strong NP-hardness even for planar graphs. Finally, outer-planar graphs are considered and NP-hardness is shown as well as a quasi-polynomial algorithm for a special case. That is, they possess some interesting properties from a complexity theoretic perspective. The chapter is closed with a short overview on results concerning the ratio of the minimum length-bounded cut value to the maximum length-bounded flow value.

CHAPTER 3: Path-flows with restrictions on the number of flow carrying paths are the topic of the last chapter. It mainly deals with single-source single-sink flows which turn out to be non-trivial.

For a special subclass of these flows, so called uniform flows, interesting structural results are shown, like a maximum-flow minimum-cut relation and a relation to general flows of bounded path number. This yields several approximation algorithms, including one for flows of bounded budget. A complexity result and a constant upper approximation bound for the general case is also shown.

At the end of the chapter the concept of a bounded number of paths is carried over to several related problems, i. e., transshipments and multi-commodity flows. The structure of solutions for this problems are discussed and some of the developed algorithmic techniques are extended to the new problems. The problem to minimize the number of used paths is briefly investigated.

Parts of this chapter have been published in [BKS02, BKS03].

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