Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Krein spaces, let A and D be selfadjoint operators in $\mathcal{H}$ and $\mathcal{K}$, respectively, with nonempty resolvent sets and let B be a bounded operator in $\mathcal{L}(\mathcal{H},\mathcal{K})$. For all λ in the resolvent set $\rho(D)$ we define an operator function

$$T(\lambda) = \lambda - A + B^+(D-\lambda)^{-1}B, \tag{0.1}$$

where B^+ denotes the Krein space adjoint of B. Then for all $\lambda \in \rho(D)$, for which $0 \in \rho(T(\lambda))$, the operator function $-T^{-1}$ can be represented in the form

$$-T(\lambda)^{-1} = P_1(\mathbf{M}-\lambda)^{-1}I_1 \tag{0.2}$$

where $\mathbf{M}$ is given by the operator matrix

$$\mathbf{M} = \begin{bmatrix} A & B^+ \\ B & D \end{bmatrix} \tag{0.3}$$

in $\mathcal{H}\times\mathcal{K}$, I_1 is the embedding of $\mathcal{H}$ in $\mathcal{H}\times\mathcal{K}$ and P_1 the projection on the first component in $\mathcal{H}\times\mathcal{K}$.

In this thesis we consider operator functions T which can formally be written as in (0.1). We relax the boundedness condition on B. The last term on the right of (0.1) is replaced by a term of a similar form which is a relatively compact perturbation in form sense with respect to A. This compactness assumption includes the case when A has a compact resolvent and B is bounded.

Analogously to the case of a bounded operator B, the operator function $-T^{-1}$ can then be represented as in (0.2) where the operator $\mathbf{M}$ arises from $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ by a relatively compact perturbation in form sense.

In this thesis we express relatively compact perturbations in form sense with the help of operators in riggings. In Chapter 1 we review some facts on riggings in Krein spaces. We also give a brief introduction to the theory of definitizable and locally definitizable selfadjoint operators in Krein spaces. In particular we discuss relatively form–compact perturbations of definitizable selfadjoint operators and compact perturbations of fundamentally reducible operators in Krein spaces.

Our main objective is to describe relations between spectral properties of the holomorphic operator function T and the operator $\mathbf{M}$. In Section 2.1 we introduce the notions of resolvent set, spectrum, point spectrum and Jordan chains of the operator function T. Then (cf. Section 2.3) a point λ where the function T is holomorphic, that is $\lambda \in \rho(D)$, belongs to the resolvent set of T if and only if λ belongs to the resolvent set of the operator $\mathbf{M}$. The same equivalence holds for the point spectrum. Special attention is given to the spectrum of positive and negative type of T, resp. $\mathbf{M}$. As the domain of the operator $T(\lambda)$ may depend on λ, we define the sign types of spectral points of T (i.e. spectral points of positive or negative type of T) via some rational function $f(T(\lambda))$ of $T(\lambda)$ which has values in $\mathcal{L}(\mathcal{H})$. This definition generalizes the usual one for $\mathcal{L}(\mathcal{H})$-valued functions (see [LMaM2]). It turns out that the sign types of spectral points of T can be characterized by the sign types of an extension of T to an operator of the space of positive norm to the space of negative norm of some rigging which has a domain independent of λ. It then follows that they coincide with the sign types with respect to $\mathbf{M}$ (Sections 2.1–2.3).

In Sections 2.4 and 2.5 we assume that A and D are definitizable selfadjoint operators and fulfil some further conditions such that by a perturbation result from [J3] the operator $\mathbf{M}$ is definitizable. The sign types of spectral points of T, first defined only for points λ of holomorphy of T, that is for $\lambda \in \rho(D) \cap \mathbb{R}$, can be extended to arbitrary real λ by making use of the (boundary behaviour near $\mathbb{R}$ of the) function $-T^{-1}$, which is a so–called definitizable operator function ([J4]). For points outside of $\rho(D) \cap \mathbb{R}$ the so defined sign type coincides with that of $\mathbf{M}$ if $\mathbf{M}$ satisfies some minimality condition (Proposition 2.18). Lemma 2.19 provides a simple criterion for this minimality. Similar relations hold if the sign types are replaced by the so–called intervals of type π_+ and type π_- (Proposition 2.18, Theorem 2.22).

Making an additional assumption on A and D and using a minimal representing operator for an N_κ–function we determine a minimal representing operator for $-T^{-1}$ such that this operator is unitarily equivalent to $\mathbf{M}$, if $\mathbf{M}$ is minimal (Theorem 2.17, Proposition 2.18). Here unitary equivalence is understood with respect to the inner products of the Krein spaces. For non–minimal $\mathbf{M}$ there is a local variant of this fact (Theorem 2.20).

Connections between T and $\mathbf{M}$ in the case where $\mathcal{H}$ and $-\mathcal{K}$ are Hilbert spaces have been studied in the articles [LMeM], [FM], [AL], [MS]. In these articles, in the Krein space setting, it is always assumed that $\sigma(A) \cap \sigma(D)$ is empty or a finite set and, on the other hand, that either the resolvent of A is compact or B is, in some sense, small with respect to A and D. In the publications mentioned above also completeness problems for the eigenfunctions and associated functions of T were investigated. In the present thesis we do not deal with completeness questions for T.

In [LMeM] $T(\lambda)$ is the operator in $L_2([0,1])$ corresponding to the differential expression

$$y'' + \lambda y + \frac{q}{u-\lambda} y, \tag{0.4}$$

and the boundary conditions

$$y(0) = y(1) = 0. \tag{0.5}$$

Here q and u belong to $L_\infty([0,1])$ and $\operatorname{ess\,sup} q < 0$. Then

$$\mathbf{M} := \begin{bmatrix} -\frac{d^2}{dx^2} & -\sqrt{-q} \\ \sqrt{-q} & u \end{bmatrix}$$

in $\mathcal{G} := L_2([0,1]) \times L_2([0,1])$ is of the form (0.3) and satisfies (0.2) (here $\mathcal{G}$ is considered as a Krein space with fundamental symmetry $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). In [LMeM] it is proved that $\mathbf{M}$ is a definitizable selfadjoint operator and that, if u is a step function, the eigenvectors and associated vectors of $\mathbf{M}$ form a Riesz basis.

In Chapter 3 we apply the results of Chapter 2 to Sturm–Liouville operators which are similar to (0.4). In Chapter 3 the relations between the sign types of T and $\mathbf{M}$ considered in Chapter 2 play an essential role.

In Section 3.1 we consider the case that $T(\lambda)$ is the operator in $L_2([-1,1])$ corresponding to the differential expression

$$py'' + \lambda y + \sum_{j=1}^{n_+} \frac{q_j^+}{u_j^+ - \lambda} y + \sum_{j=1}^{n_-} \frac{q_j^-}{u_j^- - \lambda} y, \tag{0.6}$$

with $\lambda \in \mathbb{C}$, on the interval $I := [-1,1]$ with boundary conditions

$$y(-1) = y(1) = 0. \tag{0.7}$$

The function p is identically equal to 1 or a simple indefinite weight. The functions $q_j^\pm$, $u_j^\pm$ are real valued measurable functions, $q_j^+ \geq 0$, $j = 1, \ldots, n_+$, $q_j^- \leq 0$, $j = 1, \ldots, n_-$, a.e. such that $q_j^\pm (1 + |u_j^\pm|)^{-1} \in L_1(I)$, $j = 1, \ldots, n_\pm$. Let D be the diagonal matrix multiplication operator

$$D = \operatorname{diag}(u_1^+, \ldots, u_{n_+}^+, u_1^-, \ldots, u_{n_-}^-),$$

in $\mathcal{K} := L_2(I)^{n_+} \times L_2(I)^{n_-}$, where $\mathcal{K}$ is considered as a Krein space with fundamental symmetry $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then the operator $\mathbf{M}$ arises from $(-p\frac{d^2}{dx^2}) \times D$ by a relatively compact perturbation in form sense and satisfies (0.2)(for a definition of $\mathbf{M}$ see page 59). It is a consequence of Section 2.3 that a point $\lambda \in \rho(D)$ belongs to the resolvent set (point spectrum, spectrum of positive type, spectrum of negative type) of T if and only if it belongs to the resolvent set (point spectrum, spectrum of positive type, spectrum of negative type) of $\mathbf{M}$.

Under some additional assumptions on the functions $u_j^\pm$, $j = 1, \ldots, n_\pm$ (which, in essence, imply that D is a definitizable selfadjoint operator in $\mathcal{K}$ such that D has no finite critical points), it follows that $\mathbf{M}$ is a definitizable selfadjoint operator and $T(\lambda)^{-1}$ is a definitizable operator function. In addition we prove a simple criterion for the minimality of $\mathbf{M}$ with respect to $-T^{-1}$ (cf. Theorem 3.3). If $\mathbf{M}$ is minimal with respect to $-T^{-1}$, then, by the considerations of Section 2.5, an open subset of $\overline{\mathbb{R}}$ is of positive type (negative type, type π_+, type π_-) with respect to $\mathbf{M}$ if and only if it is of the same type with respect to $-T^{-1}$. Finally, if we assume that all the functions $u_j^\pm$, $j = 1, \ldots, n_\pm$, are step functions, we can show that there exists a Riesz basis consisting of eigenvectors and associated vectors of $\mathbf{M}$.

In Section 3.2 $T(\lambda)$ is again the operator corresponding to the expression (0.6). Now we assume that $p \equiv 1$ and $I = [0, \infty)$. Instead of (0.7) we consider the boundary condition

$$y(0) = 0.$$

In this case we obtain the same relations between the various kinds of spectra of T and $\mathbf{M}$ as in Section 3.1. Moreover, under some additional assumptions on the functions $u_j^\pm$, $j = 1, \ldots, n_\pm$, the operator $\mathbf{M}$ is a definitizable operator and, again, $T(\lambda)^{-1}$ is a definitizable operator function. In Proposition 3.7 we give an example for a situation where results on the absence of positive eigenvalues for Sturm–Liouville operators can be used, in combination with the relations between the spectra of T and $\mathbf{M}$, to exclude critical points of $\mathbf{M}$ on the positive half–axis.

In Section 3.3 $T(\lambda)y$ is given by (0.4) on the interval $I = [-1, 1]$ with the boundary condition (0.7). In contrast to [LMeM], we allow q to change its sign. For simplicity, we assume that q is a real valued piecewise continuous function and that u is a real valued measurable function. Now, roughly speaking, $q(u-\lambda)^{-1}$ can be considered as a sum of two quotients $q_+(u_+-\lambda)^{-1}$ and $q_-(u_- - \lambda)^{-1}$, where the first one is defined on $\Delta_+ := \{x \in I : q(x) > 0\}$, the second one on $\Delta_- := \{x \in I : q(x) < 0\}$, and $q_\pm$ and $u_\pm$ are the restrictions of q and u to $\Delta_\pm$. Then $\mathbf{M}$ arises from $(-\frac{d^2}{dx^2}) \times u_+ \times u_-$

by a compact perturbation (in the resolvent sense). It follows that $\mathbf{M}$ is definitizable over the set

$$\overline{\mathbb{C}} \setminus ((\{\infty\} \cup \sigma_e(u_+)) \cap \sigma_e(u_-)).$$

If the functions q and u belong to $C^1(I)$ such that $u' > 0$ and q has finitely many zeros, we are able to prove that $\mathbf{M}$ is a definitizable operator in the space $L_2(I) \times L_2(\Delta_+) \times L_2(\Delta_-)$.

Finally, we consider the case of the half–axis, where $T(\lambda)$ is given by (0.4) with boundary condition $y(0) = 0$ (cf. Section 3.4). Then the operator $\mathbf{M}$ is definitizable over $\overline{\mathbb{C}} \setminus \{([0,\infty] \cup \sigma_e(u_+)) \cap \sigma_e(u_-)\}$ and, if q and u fulfil some further conditions, $\mathbf{M}$ is a definitizable operator. Moreover, the absence of eigenvalues of T can be used to locate the position of critical points of $\mathbf{M}$.