The universe is a gigantic system of reflexes produced by shocks.

Bernard Shaw
("The black girl in search of God")

1 Conservation laws

As introduction we give a condensed overview of the theoretical results and notions for systems of conservation laws in several space dimensions. The governing equations and their major properties will be presented.

Then we make some specifications and examples of hyperbolic conservation laws. We also introduce the solution theory following [21, 31, 32, 95, 96, 113].

1.1 Fundamental principles

In this thesis a particular class of nonlinear partial differential equations is considered, namely those of hyperbolic type. The hyperbolic nature reflects the fact that these equations to model transport or advection phenomena. In contrast to parabolic or elliptic partial differential equations which describe diffusion or equilibrium respectively, those of hyperbolic type model physical systems dominated by advection, like wave or flow phenomena. This is easily demonstrated by the most simple model for this class, the one dimensional wave equation. It is the mathematical description of the spreading of a wave-like pattern.

The mathematical models describing flow phenomena in the physical world are fully or partly hyperbolic. The Navier-Stokes equations, describing the movement of fluids under influence of body forces and friction, are partly hyperbolic due to the convective terms dominating these equations in general. However, they are also parabolic since friction and hence dissipation play an important role particularly in boundary layers.

The Euler equations are the best investigated mathematical model for a hyperbolic system of conservation laws. They describe the flow of compressible gases or liquids at high pressure where viscous effects can be neglected. From the mathematical point of view they are interesting because of their nonlinear behaviour, which implies the development of discontinuous solution from smooth initial data in finite time.

The notion conservation laws refers to the fact, that they summarise some principal physical laws, namely the conservation of mass, Newton's Second Law and the conservation of energy. The formulation in the form of conservation laws describes the conservation of the considered quantities.

Like Majda [77] pointed out, a conservation law arise from modelling physical processes in three steps:

i) It reveals a physical balance law derived from p physical quantities $u_1, \dots, u_p$ combined in a vector $\underline{u} = (u_1, \dots, u_p)^T$ with $\underline{u}(t, \underline{x})$ mapping from the space $(t, \underline{x}) = (t, x_1, \dots, x_d)^T$ spanned by one time and d space dimension into an open subset $\mathcal{S} \subset \mathbb{R}^p$, the so-called state space. The state space $\mathcal{S}$ arises from the fact, that several physical quantities, e.g. pressure or density, are nonnegative functions.

ii) The flux functions appearing in the balance law are idealised by prescribed nonlinear functions $f_j(\underline{u})$, mapping $\mathcal{S}$ into $\mathbb{R}^p$. Since we neither consider source terms $S(\underline{u}, t, \underline{x})$ like external body forces or heat sources, nor microscopic feature like diffusion and dissipation, the balanced forces are conserved. This gives rise to the conservation law.

iii) A generalised version of the principle of virtual work is applied [5].

The conservation of mass shall be treated as an example for this abstract formulation of the origin of a physical conservation law.

Conservation of mass

We consider a scalar quantity $\rho(t, \underline{x}) : \mathbb{R}^{d+1} \to \mathbb{R}$ which describes the state of the quantity in a point $\underline{x} \in \mathbb{R}^d$ at time t. We can think of ρ as the density of a streaming fluid, but this is not necessary. If we are interested in the development of the quantity, it is reasonable to ask for the change in time. For that we consider a volume V. The quantity accumulated in this volume at time t is

$$Q(t, \mathsf{V}) = \int_{\mathsf{V}} \rho(t, \underline{x}) \, d\underline{x}. \tag{1.1}$$

If we assume that volume V is impermeable and that mass is neither created nor destroyed, the mass located in V can only change by exchange through the volume faces[1].

Assuming that the velocity of the gas at the point $\underline{x}$ at time t is given by $\underline{v}(t, \underline{x})$ the flow rate or flux of the gas is given by $\rho(t, \underline{x})\underline{v}(t, \underline{x})$. Since we are interested in the change of the mass in the volume V in time, we have to examine the derivative with respect to time for (1.1). Due to our consideration above – or the physical principle we like to reveal – this is balanced by the flow through the surface of the volume, i.e.

$$\frac{d}{dt} \int_{\mathsf{V}} \rho(t, \underline{x}) \, d\underline{x} = - \int_{\partial \mathsf{V}} \rho(t, \underline{x}) \underline{v}(t, \underline{x}) \cdot \underline{n} \, d\underline{\sigma}, \tag{1.2}$$

where $\partial\mathsf{V}$ is the surface of the volume V. If the density is sufficiently smooth, we may interchange differentiation and integration and, applying the Gauß or divergence Theorem on the right-hand side, we derive

$$\int_{\mathsf{V}} \left[\partial_t \rho(t, \underline{x}) + \nabla \cdot (\rho(t, \underline{x}) \underline{v}(t, \underline{x})) \right] d\mathsf{V} = 0. \tag{1.3}$$

[1]Here, we already start to introduce a physical principle into our mathematical model. One can easily guess which kind of principle we like to deduce.

Since this must hold for an arbitrary control volume (1.3) holds pointwise and the integrand has to be identically zero, i.e.

$$\partial_t \rho(t, \underline{x}) + \nabla \cdot (\rho(t, \underline{x}) \underline{v}(t, \underline{x})) \quad = \quad 0. \tag{1.4}$$

This is the divergence form of the conservation of mass, while (1.3) is called the integral form. Hence, merely on the assumptions of neither creating nor destroying mass we have derived the mathematical model for conservation of mass. Later we will see that all equations which model conservation properties have the form

$$\partial_t \underline{u} + \sum_{j=1}^{d} \partial_{x_j} \underline{f}_j(\underline{u}) = 0. \tag{1.5}$$

We call this type of systems of partial differential equations systems of conservation laws. Later we will give the adequate mathematical definition for this set of equations.

1.2 Scalar conservation laws

As we have seen the hyperbolic nature of the equations considered is strongly related to the advection or wave spreading, which is modelled in this class of partial differential equations. The simplest model for this equation type is the scalar wave equation with constant speed. This equation describes the transport of a scalar quantity u depending on the direction and the velocity. This equation will be considered in a more detailed way by examining the behaviour of the solution while changing to a nonlinear form.

Transport phenomena

In our derivation of the principle of conservation of mass, we assumed that the change of mass in the control volume V only takes place by flow through the cell faces. This means that our model does not contain sinks or sources. Furthermore, if we neglect viscous phenomena, body forces etc., we have only transport or advection between the cells. If we assume the simplest form of this process, which means taking the velocity vector as constant, i.e. $\underline{v}(\underline{x}, t) = \underline{v}_{\mathsf{const}} = \mathsf{constant}$, we obtain

$$\partial_t u + \underline{v}_{\mathsf{const}} \nabla u \quad = \quad 0. \tag{1.6}$$

As one can easily see the development in time is balanced by a drift or transport with velocity $-\underline{v}_{\mathsf{const}}$. So the change of $u(t, \underline{x})$ depends on the scalar product $\langle \underline{v}_{\mathsf{const}}, \nabla u \rangle$. For simplicity we assume that initial conditions only consist of constant states u^- and u^+ separated by a discontinuity Σ. The analytic solution of the Cauchy problem (1.6) with initial condition

$$u(0, \underline{x}) \quad = \quad u_0(\underline{x}) \tag{1.7}$$

is simply

$$u(t, \underline{x}) \quad = \quad u_0(\underline{x} - \underline{v}_{\mathsf{const}} t). \tag{1.8}$$

Here, one immediately sees that the initial data propagate with velocity $\|\underline{v}_{\mathsf{const}}\|$ in space-time along the rays $\underline{x} - \underline{v}_{\mathsf{const}} t = \underline{x}_0$.