Chapter 1

Introduction

Hemivariational inequalities have been first introduced by P. D. Panagiotopoulos in [46, 47] as a generalization of variational inequalities to describe several problems arising in mechanics and engineering. They involve nonconvex, generally nonsmooth energy potentials called nonconvex superpotentials. Such kind of energy potentials results from the fact that nonmonotone, possibly multivalued contact laws are considered. This contact phenomena appear in several mechanical problems, such as nonmonotone friction and unilateral conditions, nonmonotone material laws, nonmonotone interface laws, etc. A large number of problems in structures and solids leading to nonconvex and nonsmooth energy potential can be found, e.g., in [41, 42, 46]. Among them we mention the adhesion problem from contact mechanics. Problems of such kind occur when two bodies are in adhesive contact, i.e., if they are glued on a surface by an adhesive material. The adhesive layer is assumed to be very thin compared with the geometry of the body, and due to damaging it exposes a nonlinear, nonsmooth behaviour. As a result a nonmonotone, multivalied adhesion law arises between the adhesion reaction force and the normal displacement on the contact boundary. Note that at some point the adhesion reaction force falls to zero. This happens when the gap between the bodies has grown too large. Another wide class of problems is related to the composite materials or sandwich structures. We point out the delamination problems, see [7], where two laminated layers under loading are considered. The binding interlayer material obeys again a nonmonotone law with complete vertical branches in the normal direction on the contact boundary.

Observe that all nonmonotone, multivalued relations mentioned above can be expressed by means of the generalized gradient of Clarke, and therefore lead to nonconvex locally Lipschitz superpotential in contrast to the Tresca friction case, where a nonsmooth, but convex functional is considered. Because of the lack of convexity of the energy superpotential, the hemivariational inequalities have generally nonunique solutions. The existence of a solution has been steadily discussed in the last years starting with Panagiotopoulos in the 1990s. For the mathematical background of hemivariational inequalities we refer also to the seminal work of Naniewicz and Panagiotopoulos [45] and the references therein. Parallel to the mathematical analysis, the framework of Haslinger et al. [33] provides a numerical solution scheme for hemivariational inequalities with finite elements, establishes a convergence analysis and presents numerical results based on bundle type methods.
In this thesis, we combine regularization techniques with the finite element method to approximate hemivariational inequalities with a superpotential expressed by a maximum or minimum function. In general, the regularization method is used to approximate a non-differentiable term by a sequence of differentiable ones. Convergence is obtained when the regularizing parameter $\varepsilon > 0$ tends to zero. A wide variety of applications of the regularization method can be found, e.g., in [5, 23, 24, 34, 37, 45, 52, 54]. In this work, we extend various forms of the regularization method in view of their application to hemivariational inequalities. First, we use a regularization procedure to smooth the nonsmooth superpotential. All regularizations are based on convolution and involve a calculation of a multivariate integral. The latter is, however, technically more demanding and not easily applicable in practice. Nevertheless, for the class of nonsmooth functions mentioned above and their compositions, like nested min-max function, the smoothing approximations can be computed explicitly. More precisely, since the nonsmooth functions we consider here can be expressed as a composition of the plus function with smooth functions, all our regularizations are based in fact on a class of smoothing approximations for the plus function [13, 21, 49, 50, 55, 57]. Secondly, we provide a finite element approach for the regularized problem and present convergence results.
The thesis is organized as follows:
In Chapter 2, we briefly summarize some important definitions and basic results from the theory of nonsmooth analysis that will be used in subse-

quent chapters. We introduce the generalized directional derivative and the Clarke's subdifferential as well as some essential properties of them, and present the notion of pseudomonotone operator that is needed for our existence result. Some basic results in function spaces are also collected.

In Chapter 3, we give an existence result and an approximation scheme for general variational inequality of the second kind with a pseudomonotone functional such that hemivariational inequalities are included. We prove convergence results for the sequence of solutions. In Chapter 4, we give a criteria for the uniqueness of a solution.

Chapter 5 concerns the formulation of hemivariational inequalities in mechanics. Next, the relation between a hemivariational inequality and the corresponding substationary problem involving nonconvex superpotential is given.

Chapter 6 to Chapter 8 deal with regularizing functions. We start with smoothing approximation defined in general via convolution. Then, we consider the class of maximum functions. Using different smoothing approximations of the plus function, several smoothing functions for the maximum function are presented. We analyze some approximability properties of the regularizing functions and their derivatives, and establish some convergence results.

Chapter 9 to Chapter 10 represent the main part of the thesis. Our goal is the convergence analysis for a special class of hemivariational inequalities and their numerical treatment. We begin with general coercive hemivariational inequalities defined on the boundary for which we give an existence result and for some of them we prove uniqueness. Then, we focus our attention on hemivariational inequalities with maximum (resp. minimum) superpotential. We first use a regularization method to approximate the nonsmooth functional by a sequence of differentiable ones. We discuss the convergence of the regularization method and derive some a-posteriori error estimates for solutions of the regularized problems in case of the modulus superpotential function. Then, the finite element approach using different quadrature rules for the regularized problem is analysed. We verify all the assumptions guaranteeing the convergence of the solutions of the discrete regularized problems and give the respective convergence results. In Section 10.6 we consider unique solvable continuous and descrete problems. We present a novel variant of the Céa-Falk approximation lemma applicable to unique solvable hemi-

variational inequalities. In conclusion, we discuss the convergence order of the finite element approximations under some regularity assumptions for the solution of the regularized problem, and using conforming finite element methods. This approach is extended later in Chapter 12 to coercive hemivariatioanal inequalities defined on a domain and to the case of nonquadratic growth of energy. Chapter 11 is dedicated to the more complicated semicoercive case.
Some applications in mechanics are presented in Chapter 13. In Chapter 14, we describe more specific examples and their numerical realization. More precisely, we illustrate how our idea and previous theoretical results can be used to find a numerical solution of some benchmark problems, like bilateral contact with nonmonotone friction, nonmonotone unilateral contact without friction and a delamination problem for a laminated composite structure. All examples are treated with a regularization of the nonsmooth functional using a fixed parameter ε. Then, a finite element scheme for the regularized problem is applied. Finally, we solve the involved nonlinear system of equations by minimizing the natural merit function, following [21], and using the MATLAB function *lsqnonlin*. Note that, due to the regularization techniques, we obtain optimization problems with continuously differentiable functions.

Acknowledgment

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Chapter 2

Theoretical Background

2.1 Some Elements of Nonsmooth Analysis

In this section, we provide some important definitions and basic results from the theory of the generalized directional derivative and the generalized gradient for locally Lipschitz functions which will be used throughout this work. For the properties and the calculus of the generalized gradient we refer to Clarke [17].

Let X be a Banach space with norm $\|\cdot\|$ and X^* its dual, i.e., the space of all continuous linear functionals on X. We denote the norm convergence in X and X^* by "$\rightarrow$" and the weak convergence by "$\rightharpoonup$", respectively. Moreover, $\langle\cdot,\cdot\rangle$ stands for the duality pairing between X^* and X.

A function $f : X \rightarrow \mathbb{R}$ is said to be locally Lipschitz at a given point $x \in X$ if there exists a neighborhood U_x of x and a positive constant $L_f = L_f(x)$ such that

$$|f(x_1) - f(x_2)| \leq L_f \|x_1 - x_2\| \quad \forall x_1, x_2 \in U_x.$$

First of all, we recall the definition of the generalized directional derivative in the sense of Clarke.

Definition 1 *(Generalized Directional Derivative) Let f be locally Lipschitz at a point $x \in X$. The generalized directional derivative of f at x in the direction $u \in X$, denoted by $f^0(x;u)$, is defined by*

$$f^0(x;u) = \limsup_{y \rightarrow x, t \downarrow 0} \frac{f(y+tu) - f(y)}{t}. \tag{2.1}$$

As f is locally Lipschitz, it is clear that $f^0(x;u) \in \mathbb{R}$. Some properties of f^0 are listed in the following lemma.

Lemma 1 **(i)** *The function $f^0(x;\cdot) : X \to \mathbb{R}$ is finite, positively homogeneous, subaddative, and thus convex. Moreover, it obviously satisfies the inequality*

$$|f^0(x;u)| \leq L_f \|u\| \quad \forall u \in X;$$

(ii) $f^0(x;-u) = (-f)^0(x;u) \quad \forall u \in X;$

(iii) *$f^0(x;u)$ is upper semicontinuous as a function of (x,u) and, as function of u alone, is Lipschitz with Lipschitz constant L_f.*

Note that (ii) holds, since we can write

$$f(y-tu) - f(y) = -[f(z+tu) - f(z)] \quad \text{with } z = y - tu.$$

By means of $f^0(x;u)$, we can now introduce the notion of generalized gradient or Clarke's subdifferential for locally Lipschitz functions.

Definition 2 *(Clarke's Subdifferential) Let f be locally Lipschitz at a point $x \in X$. The generalized gradient or the Clarke's subdifferential of f at x is the set-valued map $\partial f : X \rightrightarrows X^*$ defined by*

$$\partial f(x) = \{\xi \in X^* \, : \, \langle \xi, u \rangle \leq f^0(x;u) \quad \forall u \in X\}. \tag{2.2}$$

The Clarke's subdifferential $\partial f(x)$ is a nonempty, convex and weak*-compact subset of X^* and $\|\xi\|_{X^*} \leq L_f$ for every ξ in $\partial f(x)$.
In finite dimensional case, according to Rademacher's theorem [51], f is differentiable almost everywhere around x, and the Clarke's subdifferential $\partial f(x)$ can be computed simpler using the following equivalent construction

$$\partial f(x) = co\,\{\xi \in \mathbb{R}^m \, : \, \xi = \lim_{k\to\infty} \nabla f(x_k), x_k \to x, x_k \in D_f\}. \tag{2.3}$$

Here, D_f is the set of points where f is differentiable.

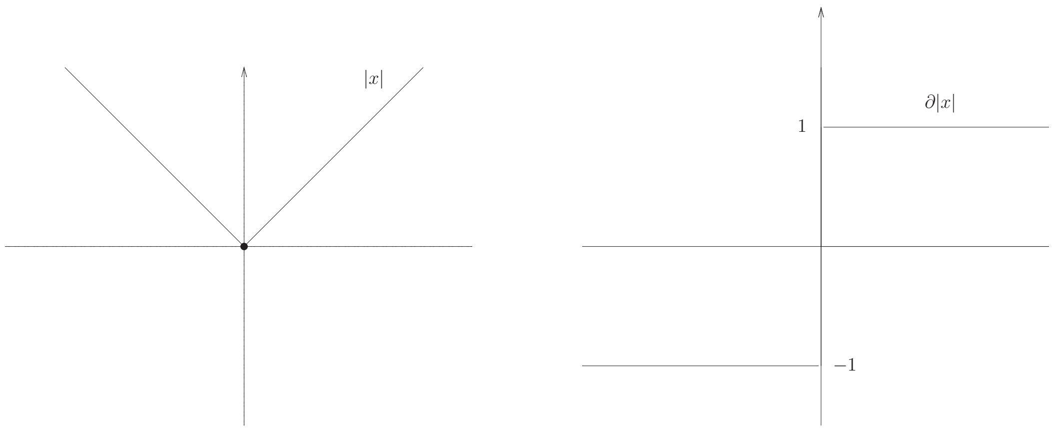

Figure 2.1: An example of a convex, globally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and its generalized gradient, $\partial|\cdot|(0) = [-1, 1]$

Further, by Hahn-Banach theorem (see also [17], Pr.2.1.2), one can check that

$$f^0(x; u) = \max_{\xi \in \partial f(x)} \langle \xi, u \rangle \quad \forall u \in X,$$

i.e., the generalized directional derivative $f^0(x; \cdot)$ is the support function of $\partial f(x)$.
The directional derivative of f at x in the direction u is defined by

$$f'(x, u) = \lim_{t \downarrow 0} \frac{f(x + tu) - f(x)}{t}$$

when this limit exists.
Now, we recall the notion of regularity in the sense of Clarke [17].

Definition 3 *(Regularity) A locally Lipschitz function $f : X \to \mathbb{R}$ is said to be regular at a point $x \in X$ if the directional derivative $f'(x, u)$ exists for every $u \in X$ and agrees with $f^0(x; u)$.*

The class of regular functions includes for example the class of convex functions and the class of maximum functions defined by

$$f(x) = \max\{g_1(x), \ldots, g_p(x)\},$$

where all $g_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1, \ldots, p$, are continuously differentiable functions.

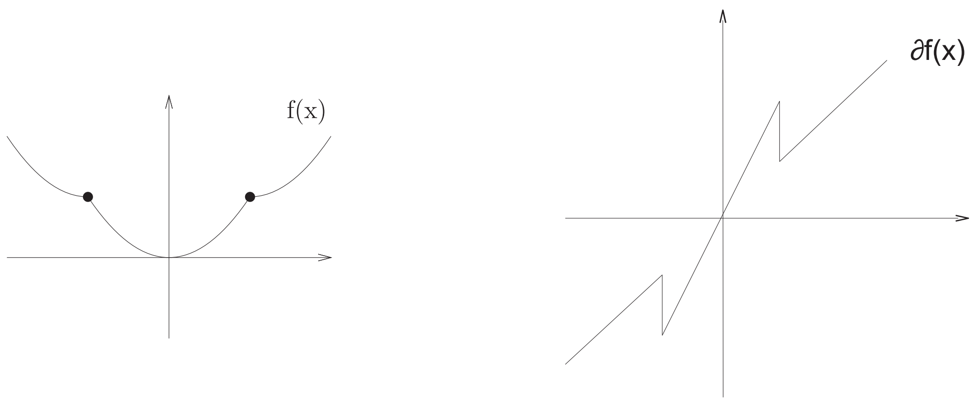

Figure 2.2: An example of a nonconvex, locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ and its generalized gradient

Moreover, denoting by $I(x)$ the set of indices at which the maximum of f is attained, i.e.,

$$I(x) = \{i \, : \, g_i(x) = f(x)\},$$

the following presentataion for the Clarke's subdifferential holds

$$\partial f(x) = \mathrm{co}\,\{\nabla g_i(x) \, : \, i \in I(x)\}.$$

Another example of regular functions is a weakly convex function, namely, a function $f : \mathbb{R}^m \to \mathbb{R}$ that can be represented as $f(x) = g(x) - \rho\|x\|^2$, where g is a convex function and ρ is a positive constant.

2.2 Some Inequalities and Preliminary Results in Function Spaces

In this section, we recall some well-known inequalities and results for function spaces which we shall frequently use in the subsequent analysis. For more information in the field we refer, e.g., to [1, 2, 19].

Lemma 2 *Let $a, b \geq 0$ and $p \in [1, \infty)$. Then*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p). \tag{2.4}$$

Let Ω be a domain (non-empty open set) in $\mathbb{R}^N$, $N \geq 1$, and $p \in [1, \infty]$ be a real number.

Hölder's Inequality

If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $p \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $uv \in L^1(\Omega)$ and

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

Minkowski's Inequality

Let $p \in [1, \infty]$ and $u, v \in L^p(\Omega)$. Then

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Definition 4 (equiintegrability) *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain and $\mathcal{U} \subset L^1(\Omega)$ a family of integrable functions. We say that $\mathcal{U}$ is an equiintegrable family if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set E with Lebesgue measure $|E| < \delta$ there holds*

$$\int_E |u| dx < \varepsilon$$

for all $u \in \mathcal{U}$.

According to [3] the sequence $\{u_n\}_{n \in \mathbb{N}} \subset L^1(\Omega)$ is equiintegrable iff

$$\lim_{a \to \infty} \left(\sup_{n \in \mathbb{N}} \int_{\{|u_n| > a\}} |u_n(x)| \, dx \right) = 0. \tag{2.5}$$

Lemma 3 *If for some $\theta > 0$*

$$\sup_{n \in \mathbb{N}} \int_\Omega |u_n|^{1+\theta} \, dx < \infty, \tag{2.6}$$

then the sequence $\{u_n\}_{n \in \mathbb{N}} \subset L^1(\Omega)$ is equiintegrable.

Proof Indeed, we have

$$\int_{\{|u_n| > a\}} |u_n(x)| \, dx = \int_{\{|u_n| > a\}} |u_n(x)| a^\theta a^{-\theta} \, dx \leq a^{-\theta} \int_{\{|u_n| > a\}} |u_n(x)|^{1+\theta} \, dx$$

$$\leq a^{-\theta} \int_\Omega |u_n(x)|^{1+\theta} \, dx \leq a^{-\theta} \sup_{n \in \mathbb{N}} \left(\int_\Omega |u_n(x)|^{1+\theta} \, dx \right) \overset{(2.6)}{\leq} C a^{-\theta}.$$

Hence,

$$\sup_{n\in\mathbb{N}} \int_{\{|u_n|>a\}} |u_n(x)|\,dx \le Ca^{-\theta} \to 0 \quad \text{as } a \to \infty$$

and according to (2.5) the sequence $\{u_n\}$ is equiintegrable. □

Now, we introduce the Lemma of Vitali.

Lemma 4 (Vitali) *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a bounded domain and $\{u_n\}$ a sequence in $L^1(\Omega)$. Suppose*

(i) $u_n \to u$ *a.e.* $\in \Omega$;

(ii) *the sequence $\{u_n\} \subset L^1(\Omega)$ is equiintegrable.*

Then

$$\lim_{n\to\infty} \int_\Omega u_n(x)\,dx = \int_\Omega u\,dx$$

In general, the following theorem holds.

Theorem 1 (Vitali) *Suppose that the sequence $\{u_n\}$ is equiintegrable in $L^p(\Omega)$, $1 \le p < \infty$, and $u_n \to u$ a.e. in Ω. Then $u_n \to u$ in $L^p(\Omega)$.*

The following lemma is an application of Vitali's Lemma.

Lemma 5 *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be a bounded domain and $\{u_n\}$ a sequence in $L^1(\Omega)$. Suppose*

(i) $u_n \to u$ *a.e. in* Ω;

(ii) *the sequence $\{u_n\}$ is bounded in $L^p(\Omega)$ for some $p > 1$.*

Then

$$u_n \to u \;\text{ in }\; L^r(\Omega) \;\text{ for all }\; 1 \le r < p.$$

Proof Invoking Corollary 2.2.21 in [19], it follows from assumptions (i) and (ii) that $u_n \to u$ in L^p, and consequently $u \in L^p$.

Define $v_n := |u_n - u|^r$ for some $r < p$. In view of assumption (i), $v_n \to 0$ a.e. in Ω.

Then, since

$$\|v_n\|_{L^{\frac{p}{r}}}^{\frac{p}{r}} = \int_\Omega |v_n|^{\frac{p}{r}}\,dx = \int_\Omega |u_n - u|^p\,dx < C$$

one can conclude that $\{v_n\}$ is bounded in $L^{\frac{p}{r}}(\Omega)$ with $\frac{p}{r} > 1$. Hence, by (2.6) the sequence $\{v_n\} \subset L^1(\Omega)$ is equiintegrable, and thus by Vitali's lemma

$$\lim_{n\to\infty} \int_\Omega v_n(x)\,dx = \int_\Omega \lim_{n\to\infty} v_n(x)\,dx = 0.$$

Further,

$$\|u_n - u\|^r_{L^r} = \int_\Omega |u_n - u|^r\,dx = \int_\Omega v_n(x)\,dx \to 0$$

and consequently $u_n \to u$ in $L^r(\Omega)$. □

2.3 Pseudomonotone Operators

In this section, we present some of the basic results on pseudomonotone operators from a real, reflexive Banach space X into its dual space X^*. This notion was invented by H. Brézis [9] in view of applications to nonlinear partial differential equations, see Minty [39, 40] and Browder [10]. It combines monotonicity of the leading part in divergence form with compactness for lower order terms.

Definition 5 *(see [53]) The operator $T : X \to X^*$ is (topologically) pseudomonotone iff, for each $u \in X$ and each sequence $\{u_n\}$ in X,*

$$u_n \rightharpoonup u \quad \textit{and} \quad \limsup_{n\to\infty} \langle Tu_n, u_n - u\rangle \le 0$$

imply

$$\langle Tu, u - w\rangle \le \liminf_{n\to\infty} \langle Tu_n, u_n - w\rangle \quad \textit{for all} \quad w \in X.$$

More generally, pseudomonotone bifunctions can be defined, see [27, 29].

Definition 6 *Let K be a weakly closed subset of X. Then a bifunction $\varphi : K \times K \to \mathbb{R}$ is called pseudomonotone on K, if for any sequence $\{u_n\}$ in K,*

$$u_n \rightharpoonup u \quad \textit{and} \quad \liminf_{n\to\infty} \varphi(u_n, u) \ge 0$$

imply

$$\varphi(u, v) \ge \limsup_{n\to\infty} \varphi(u_n, v) \quad \forall v \in K.$$

One can prove, see [53], that monotone and hemicontinuous or strongly continuous operators (bifunctions) give rise to pseudomonotone operators (bifunctions). So, one can say that the theory of topologically pseudomonotone operators unifies both monotonicity and compactness arguments.
Furthermore, in view of applications, it is very important that the sum of two pseudomonotone operators (bifunctions) is again a pseudomonotone operator (bifunction). For further properties of the pseudomonotone operators we refer to [53].
Recall also that $T : X \to X^*$ is

(a) strongly continuous iff $u_n \rightharpoonup u$ in X implies $Tu_n \to Tu$ in X^*;

(b) hemicontinuous iff T is continuous on line segments in K, i.e., for every pair of points $x, y \in X$, the following function

$$t \to \langle T(tx + (1-t)y), x - y\rangle, \quad t \in [0,1]$$

is continuous.

A simple example for a pseudomonotone function is $\varphi(x, y) := f(y) - f(x)$, where $f : K \to \mathbb{R}$ is weakly lower semicontinuous function. A compact, not necessarily linear operator, $T : X \to X^*$ gives also rise to a pseudomonotone function $\psi(x, y) := \langle T(x), y - x\rangle$. In general, $\psi(x, y)$ is pseudomonotone, if and only if, T is a pseudomonotone operator. More generally,

$$\psi(x, y) := max\{\langle u, y - x\rangle \ : \ u \in T(x)\},$$

where $T : X \rightrightarrows X^*$ is a multivalued operator with nonempty convex closed bounded values.